Hsiang algebras of cubic minimal cones

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Zürich, ICM94

Minimal cones and singular solutions

- Evans, Crandall, Lions, Jensen: Given uniformly elliptic operator F, the Dirichlet problem

$$F(D^2u)=0$$

  has a unique **viscosity solution**.
- Trudinger, Caffarelli, early 80's: the solution is always $C^{1,\varepsilon}$
- Nirenberg, 50's: if $n=2$ then u is classical (C^2) solution
- Nadirashvili, Vlăduț, 2007: there are solutions which are not C^2 for $n=24$.

Theorem (Nadirashvili, Vlăduț, V.T., 2012)

The function $w(x):=\frac{u_1(x)}{|x|}$ *where*

$$u_1(x)=x_5^3+\tfrac{3}{2}x_5(x_1^2+x_2^2-2x_3^2-2x_4^2)+\tfrac{3\sqrt{3}}{2}x_4(x_2^2-x_1^2)+3\sqrt{3}x_1x_2x_3,$$

is a singular viscosity solution of the uniformly elliptic Hessian equation

$$(\Delta w)^5+2^8 3^2(\Delta w)^3+2^{12}3^5\Delta w+2^{15}\det D^2(w)=0.$$

Hsiang's Problems

W.-Y. Hsiang (*J. Diff. Geometry*, **1**, 1967): Let u be a homogeneous polynomial in $\mathbb{R}^n$. Then $u^{-1}(0)$ is a minimal cone iff

$$\Delta_1 u := |Du|^2\Delta u - \tfrac{1}{2}\langle Du, D|Du|^2\rangle \equiv 0 \mod u.$$

- In $\deg = 2$: $\{(x,y) \in \mathbb{R}^{k+m} : (m-1)|x|^2 = (k-1)|y|^2\}$
- The first non-trivial case: $\deg u = 3$ and then

$$\Delta_1 u = \text{a quadratic form} \cdot u(x) \tag{1}$$

In fact, all known irreducible cubic minimal cones satisfy very special equation:

$$\Delta_1 u = \lambda |x|^2 \cdot u(x) \tag{2}$$

Problem 1: Classify all cubic minimal cones, or at least all solutions of (2).

Problem 2: Are there irreducible minimal cones in $\mathbb{R}^m$ of arbitrary high degree?

Problem 3: Prove that any closed minimal submanifold $M^{n-1} \subset \mathbb{S}^m$ is algebraic.

Hsiang's Problems

(ii) Partly due to the lack of "canonical" normal forms for $r < 2$ and partly due to the rapid rate of increase of the dimension of $\mathfrak{H}_n^r$ with respect to r, the little help obtained from the normal forms is not enough to solve the problem of classifying minimal algebraic cones of higher degrees. For example, it is very difficult to solve even the following very special equation: $F(x) = 0$, where $F(x)$ is an irreducible cubic form in n variables such that

$$(\Delta F)\cdot|\nabla F|^2 - \nabla F\cdot HF\cdot\nabla F^t = \pm(x_1^2 + \cdots + x_n^2)\cdot F\,.$$

Since the above equation is invariant with respect to the orthogonal linear substitutions, we may assume that F is given in some kind of "normal form" which amounts to reduce the number of indeterminant coefficients by $n(n-1)/2$. A systematic attempt to solve the above equation will involve the job of solving over-determined simultaneous algebraic equations of many variables. So far, we have only four non-trivial solutions (cf. §§ 1, 2), but there is no reason why there should be no others.

A homogeneous cubic form $u(x)$ is called a *Hsiang cubic* if

$$\Delta_1 u = \lambda |x|^2 u(x), \quad \lambda \in \mathbb{R}.$$

What about "the four nontrivial solutions"?

Hsiang's trick: let $X \in \mathscr{H}_k'(\mathbb{A})$ = trace free hermitian $k \times k$-matrices over $\mathbb{A} = \mathbb{R}$ or $\mathbb{C}$

- Δ_1 is an $O(n)$-invariant $\Rightarrow \Delta_1(\operatorname{tr} X^3) =$ is a polynomial in $\operatorname{tr} X^2, \ldots, \operatorname{tr} X^k$
- $\deg(\Delta_1 \operatorname{tr} X^3) = 5$
- if $3 \le k \le 4$ then $\Delta_1 u(X) = c_1 \operatorname{tr} X^2 \operatorname{tr} X^3 = c_1 |X|^2 u(X)$.

Thus $u = \operatorname{tr} X^3$ is a Hsiang cubic. This yields the four Hsiang examples w in

$$\mathscr{H}_3'(\mathbb{R}) \cong \mathbb{R}^5, \quad \mathscr{H}_3'(\mathbb{C}) \cong \mathbb{R}^8, \quad \mathscr{H}_4'(\mathbb{R}) \cong \mathbb{R}^9, \quad \mathscr{H}_4'(\mathbb{C}) \cong \mathbb{R}^{15}$$

An important observation: $\deg w = 3$ implies

$\operatorname{tr}(D^2u) = 0$	the harmonicity of u
$\operatorname{tr}(D^2u)^2 = C_1 \lvert x \rvert^2$	the quadratic trace identity
$\operatorname{tr}(D^2u)^3 = C_2 u$	the cubic trace identity

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How do nonassociative algebras enter?

- $u = \mathrm{Re}(z_1 z_2) z_3$, $z_i \in \mathbb{A}_d$, $d = 1, 2, 4, 8$, the triality polynomials in $\mathbb{R}^{3d}$ where

$$\mathbb{A}_1 = \mathbb{R},\ \ \mathbb{A}_2 = \mathbb{C},\ \ \mathbb{A}_4 = \mathbb{H},\ \ \mathbb{A}_8 = \mathbb{O}$$

  are the classical Hurwitz algebras.

- $u(x) = \begin{vmatrix} \frac{1}{\sqrt{3}}x_1 + x_2 & x_3 & x_4 \\ x_2 & \frac{-2}{\sqrt{3}}x_1 & x_5 \\ x_4 & x_5 & \frac{1}{\sqrt{3}}x_1 - x_2 \end{vmatrix}$ = a Cartan isoparametric cubic in $\mathbb{R}^5$

- $u(x) = \begin{vmatrix} x_1 & x_2 & x_3 \\ x_4 & x_5 & x_6 \\ x_7 & x_8 & x_9 \end{vmatrix}$

So, Hsiang minimal cubics can be interpreted very nicely certain algebraic structures.

Which ones and how?

Hsiang cubics of Clifford type

Example. The **Lawson cubic cone** in $\mathbb{R}^4$ with the defining polynomial

$$u(z) = (x_1^2 - x_2^2)y_1 + 2x_1x_2y_2 = \langle x, A_1x\rangle y_1 + \langle x, A_2x\rangle y_1, \qquad z = (x,y) \in \mathbb{R}^4$$

$$A_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Theorem (V.T., 2010) Let $\{A_i\}_{1\le 1\le q}$ be a symmetric Clifford system, i.e.

$$A_i^2 = I \quad \text{and} \quad A_iA_j + A_jA_i = 0, \ \ \forall i \ne j.$$

Then

$$u_A(z) = \sum\nolimits_{i=1}^{q} \langle x, A_ix\rangle y_i, \quad z = (x,y) \in \mathbb{R}^{2p} \times \mathbb{R}^q$$

is a Hsiang cubic.

The existence of a symmetric Clifford system in $\mathbb{R}^{2p}$ is equivalent to

$$q - 1 \le \rho(p),$$

$\rho(p)$ = Hurwitz-Radon function = $1 + \#$(of independent vector fields on $\mathbb{S}^{p-1}$)

The dichotomy of Hisang cubics

A Hsiang cubic u is of **Clifford type** if $u \cong u_A$, otherwise it is called **exceptional**.

Theorem (V.T., 2010) *Hsiang cubics of Clifford type are congruent if and only if the corresponding symmetric Clifford systems are geometrically equivalent.*

$\Rightarrow$ representation theory of Clifford algebras yields a complete classification of Hsiang cubics of Clifford type.

Main Problem: How to determine all exceptional Hsiang cubics?

Exceptional Hsiang cubics

The Hsiang examples u in $\mathbb{R}^5, \mathbb{R}^8, \mathbb{R}^9, \mathbb{R}^{15}$ are exceptional Hsiang cubics.

Proof. By contradiction: is u is the Clifford type then

$$u \cong u_A = \sum_{i=1}^{q} \langle x, A_i x\rangle y_i, \quad \Rightarrow \quad \mathrm{tr}(D^2 u_A)^2 = 2q|x|^2 + 2p|y|^2.$$

But the **quadratic trace identity** yields

$$\mathrm{tr}(D^2 u)^2 = C_1(|x|^2 + |y|^2),$$

implying by the $O(n)$-invariance of Δ_1 that $q = p$.

Since $q - 1 \le \rho(p) \Rightarrow p \in \{1, 2, 4, 8\}$, thus $n = q + 2p \in \{3, 6, 12, 24\}$, a contradiction.

□

The nonassociative algebra approach

Given a cubic form u on an inner product vector space $(V, \langle\cdot,\cdot\rangle)$ define a composition law $(x, y) \mapsto xy$ as the unique element satisfying

$$u(x, y, z) = \langle xy, z\rangle, \qquad \forall z \in V.$$

This defined algebra is commutative, nonassociative and metrized and in this setting,

- $u(x) = \frac{1}{6}\langle x, x^2\rangle$

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- L_x is self-adjoint: $\langle L_x y, z\rangle = \langle y, L_x z\rangle$

(cf. the Freudenthal-Tits-Springer construction of exceptional Jordan algebras)

The nonassociative algebra approach

a cubic form u satisfying a PDE	$\leftrightsquigarrow$	a metrized algebra $V(u)$ with an identity

Examples:

- The Laplace equation:

$$\Delta u(x) = 0 \quad \Longrightarrow \quad \operatorname{tr} L_x = 0$$

- The eiconal (Cartan-Münzner) equation:

$$|Du(x)|^2 = 9|x|^4 \quad \Longrightarrow \quad |x^2|^2 = 36|x|^4$$

- The third fundamental form $\nabla_X A(Y,Z)$ of an isoparametric hypersurface in a space form

Key steps of the proof

- The set of idempotents $\mathrm{Ide}(V)$ is nonempty: any stationary point of $u(x)$ on $\mathbb{S}^{n-1}$ gives rise to an idempotent.
- Given $c \in \mathrm{Ide}(V)$, L_c is a self-adjoint $\Rightarrow$ the **Peirce decomposition**

$$V = \bigoplus_{\alpha=1}^{k} V_c(t_\alpha), \qquad V_c(t_\alpha) := \ker(L_c - t_\alpha)$$

- Use the defining PDE to capture the "multiplication table":

$$V_c(t_\alpha) V_c(t_\beta) \subset \bigoplus_{\gamma} V_c(t_\gamma)$$

- If the PDE is 'good enough', there are some natural (Clifford or Jordan) algebra structures hidden inside V.
- The "tetrad" decomposition

The nonassociative algebra approach

Now, let $u(x)$ be a Hsiang cubic, i.e.

$$|Du(x)|^2\Delta u(x) - \tfrac{1}{2}\langle Du(x), D|Du(x)|^2\rangle = \lambda|x|^2u(x).$$

Then the corresponding Freudenthal-Springer algebra satisfies

$$\boxed{\langle x^2, x^2\rangle \operatorname{tr} L_x - \langle x^2, x^3\rangle = \tfrac{2}{3}\lambda\langle x, x\rangle\langle x^2, x\rangle}$$

Definition. A commutative metrized algebra is called Hsiang if the latter identity satisfied.

Hsiang cubics	$\leftrightsquigarrow$	Hsiang algebras

Key steps of the proof

A Hsiang algebra := **trivial** if $\dim VV = 1$ (or $u \cong \langle a, x\rangle^3$).

Theorem A (The Harmonicity)

Any nontrivial Hsiang algebra is harmonic: $\operatorname{tr} L_x = 0$ and $\lambda \neq 0$. In particular,

$$\langle x^2, x^3\rangle = \frac{4}{3}\langle x^2, x\rangle |x|^2.$$

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What about Hsiang cubics of Clifford type?

Theorem B

u is a Hsiang cubic of Clifford type iff $V(u)$ admits a non-trivial $\mathbb{Z}_2$-grading

$$V = V_0 \oplus V_1, \quad V_0V_0 = 0$$

and $\forall x \in V_0$: $L_x^2 = |x|^2$ on V_1.

Key steps of the proof

Theorem C (The hidden Clifford algebra structure)

Let V be a Hisang algebra. Then

(i) $\forall c \in \mathrm{Ide}(V)$, the associated Peirce decomposition is

$$V = V_c(1) \oplus V_c(-1) \oplus V_c(-\tfrac{1}{2}) \oplus V_c(\tfrac{1}{2}) \quad \text{and} \quad \dim V_c(1) = 1;$$

(ii) The Peirce dimensions $n_1 = \dim V_c(-1)$, $n_2 = \dim V_c(-\frac{1}{2})$ and $n_3 = \dim V_c(\frac{1}{2})$ do not depend on a particular choice of c and

$$n_3 = 2n_1 + n_2 - 2;$$

(iii) If ρ is the Hurwitz-Radon function then

$$n_1 - 1 \leq \rho(n_1 + n_2 - 1).$$

In particular, for each n_2 there exist only finitely many possible values of n_1.

Key steps of the proof

Theorem D (The Multiplication Table)

If $V_0 = V_c(1)$, $V_1 = V_c(-1)$, $V_2 = V_c(-\frac{1}{2})$, $V_3 = V_c(\frac{1}{2})$ then

	V_0	V_1	V_2	V_3
V_0	V_0	V_1	V_2	V_3
V_1	V_1	V_0	V_3	$V_2 \oplus V_3$
V_2	V_2	V_3	$V_0 \oplus V_2$	$V_1 \oplus V_2$
V_3	V_3	$V_2 \oplus V_3$	$V_1 \oplus V_2$	$V_0 \oplus V_1 \oplus V_2$

In particular, $V_0 \oplus V_1$ and $V_0 \oplus V_2$ are subalgebras of V.

Jordan algebras

An algebra V with a **commutative** product $\bullet$ is called Jordan if

$$[L_x, L_{x^2}] = 0 \qquad \forall x \in V.$$

Main examples

1) The Jordan algebra $\mathscr{H}_n(\mathbb{A}_d)$ of Hermitian matrices of order n, $d = 1, 2, 4$ with

$$x \bullet y = \tfrac{1}{2}(xy + yx)$$

2) The spin factor $\mathscr{S}(\mathbb{R}^{n+1})$ with $(x_0, x) \bullet (y_0, y) = (x_0 y_0 + \langle x, y\rangle;\ x_0 y + y_0 x)$

Theorem (Jordan-von Neumann-Wigner, 1934)

Any finite-dimensional *formally real* Jordan algebra is a direct sum of the simple ones:

- the spin factors $\mathscr{S}(\mathbb{R}^{n+1})$;
- the Jordan algebras $\mathscr{H}_n(\mathbb{A}_d)$, $n \geq 3$, $d = 1, 2, 4$;
- the Albert algebra $\mathscr{H}_3(\mathbb{A}_8)$.

Key steps of the proof

Theorem E (The hidden Jordan algebra structure)

Let V be a Hisang algebra and $c \in \mathrm{Ide}(V)$. Then the subspace

$$J_c := V_c(1) \oplus V_c(-\tfrac{1}{2})$$

carries a structure of a formally real rank 3 Jordan algebra, and the following conditions are equivalent:

(i) the Hsiang algebra V is *exceptional*;

(ii) J_c is a *simple* Jordan algebra;

(iii) $n_2 \neq 2$ and the *quadratic trace identity* $\operatorname{tr} L_x^2 = k|x|^2$ holds for some $k \in \mathbb{R}$.

The proof of the first part of the theorem is heavily based on the McCrimmon-Springer construction of a cubic Jordan algebra.

The Finiteness of Exceptional Hsiang Algebras

- If V is an exceptional Hsiang algebra then

$$J_c = V_c(1) \oplus V_c(-\tfrac{1}{2})$$

  is a simple formally real Jordan algebra of rank≤ 3 and $\dim J_c = 1 + n_2$.

- The Jordan-von Neumann-Wigner classification implies that

  either $\dim J_c = 1$ or $\dim J_c = 3d + 3$, where $d \in \{1, 2, 4, 8\}$.

  Thus, $n_2 = 0$ or $n_2 = 3d + 2$.

- Using the obstruction

$$n_1 - 1 \leq \rho(n_1 + n_2 - 1)$$

  implies the finiteness.

The Finiteness of Exceptional Hsiang Algebras

Theorem B

There exists finitely many isomorphy classes of exceptional Hsiang algebras.

n	2	5	8	14	26	9	12	15	21	15	18	21	24	30	42	27	30	33	36	51	54	57	60	72
n_1	1	2	3	5	9	0	1	2	4	0	1	2	3	5	9	0	1	2	3	0	1	2	3	7
n_2	0	0	0	0	0	5	5	5	5	8	8	8	8	8	8	14	14	14	14	26	26	26	26	26

In the realizable cases (uncolored):

- If $n_2 = 0$ then $u = \frac{1}{6}\langle z, z^2\rangle$, $z \in \mathscr{H}_3'(\mathbb{A}_d)$, $d = 0, 1, 2, 4, 8$.
- If $n_1 = 0$ then $u(z) = \frac{1}{12}\langle z^2, 3\bar{z} - z\rangle$, $z \in \mathscr{H}_3(\mathbb{A}_d)$, $d = 2, 4, 8$.
- If $n_1 = 1$ then $u(z) = \mathrm{Re}\langle z, z^2\rangle$, $z \in \mathscr{H}_3(\mathbb{A}_d) \otimes \mathbb{C}$, $d = 1, 2, 4, 8$.
- If $(n_1, n_2) = (4, 5)$ then $u = \frac{1}{6}\langle z, z^2\rangle$, $z \in \mathscr{H}_3(\mathbb{O}) \ominus \mathscr{H}_3(\mathbb{R})$

$\mathscr{H}_3(\mathbb{A}_d)$ is the Jordan algebra of 3×3-hermitian matrices over the Hurwitz algebra $\mathbb{A}_d$

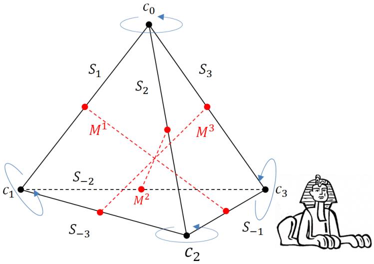
Towards a finer classification

The Tetrad Decomposition

Let V be an exceptional Hsiang algebra, $n_2 = 3d + 2$. Then

$$V = S^1 \oplus S^2 \oplus S^3 \oplus M^1 \oplus M^2 \oplus M^3, \quad S^\alpha = S_\alpha \oplus S_{-\alpha},$$

- M^α are nilpotent;
- each S_α is a real division algebra isomorphic to $\mathbb{A}_d$;
- Any 'vertex-adjacent' triple $(S_\alpha, S_\beta, S_\gamma)$ is a triality



Some analytical corollaries

Theorem F

Let u be a nontrivial Hsiang cubic. Then

- $\Delta u(x) = 0$
- the cubic trace identity holds:

$$\operatorname{tr}(D^2u)^3 = 3\lambda(n_1 - 1)u, \qquad n_1 \in \mathbb{Z}^+$$

- $n_2 = \frac{1}{2}(n + 1 - 3n_1) \in \mathbb{Z}^+$
- $u(x)$ is exceptional Hsiang cubic iff $n_2 \neq 2$ and the quadratic trace identity holds

$$\operatorname{tr}(D^2u)^2 = k|x|^2, \quad k \in \mathbb{R}$$

Remark

Peng Chia-Kuei and Xiao Liang (On the Classification of cubic minimal cones, J. Grad. School, Academ. Sinica, Vol.10, n.1, 1993): under assumption that $\Delta u = 0$ proved some particular results.

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THANK YOU FOR YOUR ATTENTION!