Symmetry gaps for geometric structures

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Geometry and Lie theory, Trondheim

(based on joint work with Boris Kruglikov)

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Fix a geometry for which the maximal (infinitesimal) sym dim is known.

Q: What is the next possible ("submaximal") symmetry dimension $\mathfrak{S}$?

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Warning: A priori, submax sym models may not be homogeneous!

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   - (Q: Fixing Petrov type, what is the max (conformal) sym dim?)

Cartan geometries

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Example (Riemannian metrics (M^2, g))

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- Have H-principal bundle with $\omega \in \Omega^1(\mathcal{G}; \mathfrak{g})$ a coframing.
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"Underlying structure $\leftrightarrow$ Cartan geometry with normalization on K"

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Parabolic geometries

Focus on Cartan geometries of type (G, P), where G = semisimple Lie group, P = parabolic subgroup.

Theorem (Tanaka, Morimoto, Čap–Schichl)

Regular, normal parabolic geometries are equivalent to underlying geometric structures.

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Example

Given $y'' = f(x, y, y')$, have a 3-mfld (x, y, p) with contact distribution $C = (dy - pdx)^{\perp}$ with a splitting $C = E \oplus V$. These are spanned by $\partial_x + p\partial_y + f(x, y, p)\partial_p$ and ∂_p. This underlies a $(\mathrm{SL}_3, P_{1,2})$ geometry.

Parabolic subalgebras and gradings

$(\mathfrak{g}, \mathfrak{p}) \rightsquigarrow \mathbb{Z}$-grading: $\quad \mathfrak{g} = \mathfrak{g}_- \oplus \overbrace{\mathfrak{g}_0 \oplus \mathfrak{g}_+}^{\mathfrak{p}}$. Reductive part is $\mathfrak{g}_0 = \mathfrak{z}(\mathfrak{g}_0) \oplus \mathfrak{g}_0^{ss}$ and have a unique grading element $Z \in \mathfrak{z}(\mathfrak{g}_0)$.

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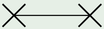
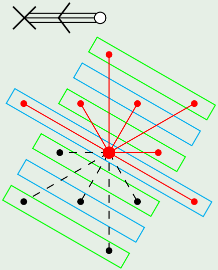
Example ($\mathrm{SL}_3/P_{1,2}$ and G_2/P_1)



$$\mathfrak{sl}_3 = \begin{pmatrix} 0 & 1 & 2 \\ -1 & 0 & 1 \\ -2 & -1 & 0 \end{pmatrix} \quad ; \quad G_2 \quad :$$



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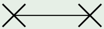

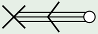

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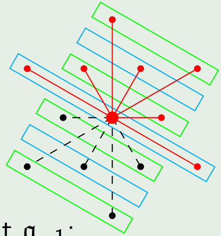

G-invariant structure on $T(G/P) \rightsquigarrow$ look at $\mathfrak{g}_{-1}$:

- $\mathrm{SL}_3/P_{1,2}$: $C = E \oplus V$.
- G_2/P_1: $(2, 3, 5)$ distribution.

Sample gap results for parabolic geometries

Geometry	Range	Model	Max	$\mathfrak{S}$
Sig. (p,q) conformal geometry in dim. $n = p+q$	$p, q \geq 2$	$\mathrm{SO}_{p+1,q+1}/P_1$	$\binom{n+2}{2}$	$\binom{n-1}{2} + 6$
Systems of 2nd order ODE in m dependent variables	$m \geq 2$	$\mathrm{SL}_{m+2}(\mathbb{R})/P_{1,2}$	$(m+2)^2 - 1$	$m^2 + 5$
Generic rank ℓ distributions on $\frac{1}{2}\ell(\ell+1)$-dim. manifolds	$\ell \geq 3$	$\mathrm{SO}_{\ell,\ell+1}/P_\ell$	$\binom{2\ell+1}{2}$	$\begin{cases} \frac{\ell(3\ell-7)}{2} + 10, & \ell \geq 4; \\ 11, & \ell = 3 \end{cases}$
Lagrangean contact structures	$\ell \geq 3$	$\mathrm{SL}_{\ell+1}(\mathbb{R})/P_{1,\ell}$	$\ell^2 + 2\ell$	$(\ell-1)^2 + 4$
Contact projective structures	$\ell \geq 2$	$\mathrm{Sp}_{2\ell}(\mathbb{R})/P_1$	$\ell(2\ell+1)$	$\begin{cases} 2\ell^2 - 5\ell + 8, & \ell \geq 3; \\ 5, & \ell = 2 \end{cases}$
Contact path geometries	$\ell \geq 3$	$\mathrm{Sp}_{2\ell}(\mathbb{R})/P_{1,2}$	$\ell(2\ell+1)$	$2\ell^2 - 5\ell + 9$
Exotic parabolic contact structure of type E_8	-	E_8/P_8	248	147

Table: Kruglikov–The (2013): sample new results

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Doubrov–The (2013): (i) conf. Riem: $\mathfrak{S} = \begin{cases} \binom{n-1}{2} + 3, & 5 \leq n \neq 6; \\ \frac{n^2}{4} + n, & n = 4, 6. \end{cases}$

(ii) conf. Lor: $\mathfrak{S} = \binom{n-1}{2} + 4, \quad n \geq 4.$

Two key ingredients

For (regular, normal) parabolic geometries, there are two key ingredients for studying the gap problem in a uniform way:

1. harmonic curvature κ_H. $\boxed{\text{Geometry is flat iff } \kappa_H = 0}$.
2. Tanaka prolongation.

Harmonic curvature

Curvature: $K = d\omega + \frac{1}{2}[\omega, \omega] \quad \Leftrightarrow \quad \kappa : \mathcal{G} \to \bigwedge^2 (\mathfrak{g}/\mathfrak{p})^* \otimes \mathfrak{g} \cong \bigwedge^2 \mathfrak{g}_+ \otimes \mathfrak{g}$.

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- Regular: κ takes values in positive Z-eigenspaces.
- Normal: $\partial^* \kappa = 0$, where ∂^* is the homology differential.

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Examples (Harmonic curvature)

- conformal geometry: Weyl ($n \geq 4$) or Cotton ($n = 3$);
- scalar 2nd order ODE: Tresse (relative) invariants I_1, I_2.

Tanaka prolongation

Given $(\mathfrak{g}, \mathfrak{p})$, we have $\mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_+$.

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$$\mathrm{pr}_{\mathfrak{g}}(\mathfrak{g}_-, \mathfrak{a}_0) = \mathfrak{g}_- \oplus \mathfrak{a}_0 \oplus \mathfrak{a}_1 \oplus \dots$$

by defining for $k \geq 1$ that

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$$\mathfrak{a}^\phi = \left(\begin{array}{c|c|c} c & 0 & 0 \\ \hline * & 4c & 0 \\ \hline * & * & -5c \end{array}\right) \quad \Rightarrow \quad \mathfrak{a}^\phi_+ = 0, \quad \dim(\mathfrak{a}^\phi) = 4.$$

Results of Kruglikov–The (2013)

Fix (G, P). Among regular, normal G/P geometries $(\mathcal{G} \to M, \omega)$,

$$\begin{aligned} \mathfrak{S} &:= \text{submaximal sym. dim.} \\ &:= \max\{\dim(\mathfrak{inf}(\mathcal{G}, \omega)) \mid \kappa_H \not\equiv 0\} \\ \mathfrak{U} &:= \max\{\dim(\mathfrak{a}^\phi) \mid 0 \neq \phi \in H^2_+(\mathfrak{g}_-, \mathfrak{g})\} \end{aligned}$$

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General real case: Have $\mathfrak{S} \leq \mathfrak{U} \leq \mathfrak{U}^{\mathbb{C}}$, where $\mathfrak{U}^{\mathbb{C}}$ is easily computable.

Maximizing the Tanaka prolongation

$H^2_+ = 0 \;\Rightarrow$ locally flat. Otw, $H^2_+ = \bigoplus_i \mathbb{V}_i$ (as $\mathfrak{g}_0$-irreps). We have $\mathfrak{U} = \max_i \mathfrak{U}_i$, where $\mathfrak{U}_i = \max\{\dim(\mathfrak{a}^\phi) \mid 0 \neq \phi \in \mathbb{V}_i\}$.

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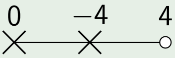
Example (pair of 2nd order ODE: $\mathrm{SL}_4/P_{1,2}$-geometry)

$\mathbb{V} =$  $\subset H^2_+(\mathfrak{g}_-, \mathfrak{g})$. Let $\phi_0 \in \mathbb{V}$ be a l.w. vector.

$$\mathfrak{a}^{\phi_0} = \left(\begin{array}{c|c|cc} c_1 & * & 0 & 0 \\ \hline * & c_2 & 0 & 0 \\ \hline * & * & 0 & 0 \\ * & * & * & -c_1 - c_2 \end{array}\right) \Rightarrow \begin{array}{l} \dim(\mathfrak{a}^{\phi_0}_+) = 1, \\ \dim(\mathfrak{a}^{\phi_0}) = 9. \end{array}$$

4-dim Lorentzian conformal geometry

$SO(2,4)/P_1$: $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$, with $\mathfrak{g}_0 = \mathbb{R} \oplus \mathfrak{so}(1,3) = \mathbb{R} \oplus \mathfrak{sl}(2,\mathbb{C})_{\mathbb{R}}$,

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In terms of std $\mathbb{C}$-basis $\{H, X, Y\}$ of $\mathfrak{sl}(2,\mathbb{C})$:

Petrov type	Normal form in $S^4\mathbb{C}^2$	Annihilator $\mathfrak{a}_0$	$\dim(\mathfrak{a})$	sharp?
N	x^4	$X, iX, 2Z - H$	7	✓
III	x^3y	$Z - 2H$	5	×
D	x^2y^2	H, iH	6	✓
II	$x^2y(x-y)$	0	4	✓
I	$xy(x-y)(x-ky)$	0	4	✓

Get bounds for constant Petrov type structures. In particular, $\boxed{\mathfrak{S} \leq 7}$.

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Proof outline:

(1) Prop: At regular points, $(*)$ is true.

(2) Lemma: The set of regular points is open and dense in M.

(3) Any nbd of a non-flat point contains a non-flat regular pt. $\square$

Realizability - proof outline

Via $H^2 \cong \ker(\Box)$, get an explicit l.w. vector $\phi_0 \in \mathbb{V} \subset H^2_+ \subset \bigwedge^2 \mathfrak{g}_-^* \otimes \mathfrak{g}$ from Kostant.

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Non-exceptions: $\mathfrak{f}/\mathfrak{f}^0 \rightsquigarrow$ non-flat model, $\dim(\mathfrak{f}) = \mathfrak{U}$, so $\mathfrak{S} = \mathfrak{U}$.

Realizability - proof outline

Via $H^2 \cong \ker(\Box)$, get an explicit l.w. vector $\phi_0 \in \mathbb{V} \subset H^2_+ \subset \bigwedge^2 \mathfrak{g}_-^* \otimes \mathfrak{g}$ from Kostant. IDEA: Produce sym. alg $\mathfrak{f}$ (filtered) from $\mathfrak{a} = \mathfrak{a}^{\phi_0}$ (graded). Define $\mathfrak{f} = \mathfrak{a}$ as *vector spaces*, but with deformed bracket

$$[X, Y]_{\mathfrak{f}} := [X, Y]_{\mathfrak{a}} - \phi_0(X, Y).$$

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Have algorithm for constructing an explicit submax. sym. model.

Concluding remarks

General real (non split) cases are difficult to treat in a uniform way.

- $\mathfrak{S} \leq \mathfrak{U} \leq \mathfrak{U}^{\mathbb{C}}$ always. Computing $\mathfrak{U}$ may not be easy.
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2. IDEA: If $\kappa_H(u) \neq 0$, $\mathfrak{a}^{\kappa_H(u)}$ does not reach the top-slot $\mathfrak{g}_\nu$ of the grading on $\mathfrak{g}$. (On the flat model, $\mathfrak{g}_\nu \leftrightarrow$ 2-jet det syms.)