Quotients

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Geometry and Lie theory.
Dedicated to Eldar Straume on his 70th birthday.
Trondheim, 03.11.16

Plan of the talk

- Some observations and quasi historical remarks

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    - binary forms and *n*-ary forms
    - invariants of irreducible representations of semisimple Lie groups

The General Moduli Problem

"Describe" the orbit space Ω/G of a group G -action on a space Ω.

Main collection:

- Ω -is a smooth manifold, G - is a Lie group, $G \times \Omega \to \Omega$ proper and free action $\Longrightarrow \Omega \diagup G$ smooth manifold and $\Omega \to \Omega \diagup G$ is a principal G - bundle (J.L. Koszul and R. Palais).
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  Regular orbits are separated by *polynomial invariants*.
- Ω- is an algebraic manifold, G -is an algebraic Lie group, $G \times \Omega \to \Omega$ -algebraic action $\Longrightarrow$ regular orbits are separated by *rational invariants* (M. Rosenlicht).
  There is no Hilbert's 14th problem!

The Differential Moduli Problem

"Describe" the orbit space Ω/G, where Ω is a solution space of a differential equation and G is a symmetry pseudogroup.

Jet level:

PDEs system $\mathcal{E}^{\cdot} \subset \mathbf{J}^{\infty}$, G-symmetry Lie pseudogroup.

- $\mathcal{E} = \mathbf{J}^{\infty}$.
  Lie-Tresse Theorem: Microlocally (i.e. in a neighborhood of $\mathbf{J}^{\infty}$) algebra differential G-invariants is generated by a number of basic differential invariants and G-invariant total derivations.(S. Lie, A. Tresse, A. Kumpera for Lie pseudogroups and L. Ovsyannikov and P. Olver for Lie groups).

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- Lie-Tresse Theorem for PDEs: $\mathcal{E} \subset \mathbf{J}^{\infty}$is a formally integrable PDEs system , G-Lie pseudogroup of symmetries. (B.Kruglikov & VL).

Basic Setup

- M is a smooth manifold, $\mathbf{J}_0^k(M,M)$ - the manifold of k -jets of diffeomorphisms of M, $\mathbf{J}^k(M,n)$ - the manifold of k-jets of of submanifolds in M, having dimension n.
  Affine and algebraic structures:

$$\cdots \longrightarrow \mathbf{J}^3(M,n) \xrightarrow{\mathbf{S}^3\tau^*\otimes\nu} \mathbf{J}^2(M,n) \xrightarrow{\mathbf{S}^2\tau^*\otimes\nu} \mathbf{J}^1(M,n) \xrightarrow{\mathrm{Gr}_n(\mathbf{T}M)} M$$

  and

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- $\mathcal{E}^{\cdot} \subset \mathbf{J}^\infty(M, n)$, $\{\mathcal{E}^k \subset \mathbf{J}^k(M, n)\}$, is a PDEs system on submanifolds (of dimension n) of M.
- Given point $a \in M$, by $\mathbf{J}_a^k$ and $\mathcal{E}_a^k$ we denote the fibres of projections $\mathbf{J}^k(M, n) \to M$ and $\mathcal{E}^k \to M$ at the point a, G_a^k are stabilizers G^k of the point.

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- The prolongated actions of G on $\mathbf{J}^k(M,n)$, $k = 1, 2, \ldots$ are *algebraic.* That is, G_a^k are algebraic groups acting algebraically on algebraic manifolds $\mathbf{J}_a^k$.
- $\mathcal{E}^{\cdot}$ is G -invariant formally integrable PDE system and $\mathcal{E}_a^k \subset \mathbf{J}_a^k$ are irreducible algebraic submanifolds.

Differential Invariants

- "Common sense": A smooth function I densely defined on equation $\mathcal{E}^l \subset \mathbf{J}^l(M, n)$ and invariant with respect to the prolongated G-action is called a *differential G -invariant* of order $\leq l$.

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- "Good sense": By a *differential G -invariant* of order $\leq l$ *we mean* a differential G -invariant which is rational along fibres $\pi_{s,0} : \mathcal{E}^s \rightarrow M$, for some $s \leq l$, and polynomial along fibres $\pi_{l,s} : \mathcal{E}^l \rightarrow \mathcal{E}^s$.

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- Similar to the case of differential invariants we'll play with coefficients of the total vector fields in order to get invariant derivations in "good sense".
- In reality, one needs derivations but not only total vector fields!

Cartan-Kuranishi type theorem for singularities

We say that a closed subset $S \subset \mathcal{E}^k$ is *Zariski closed* if all its intersections $S_a = S \cap \mathbf{J}_a^k$ are Zariski closed.

Theorem

There exists a number l and a Zariski closed invariant proper subset $\Sigma_l \subset \mathcal{E}^l$ such that the action is regular in $\mathcal{E}^{\cdot} \setminus \pi_{\infty,l}^{-1}(\Sigma_l)$ i.e. for any $k \geq l$, the orbits of G on $\mathcal{E}^k \setminus \pi_{k,l}^{-1}(\Sigma_l)$ are closed, have the same dimension and seperated by "good" differential invariants.

Lie-Tresse theorem

Our second result gives finiteness for differential invariants.

Theorem

There exists a number l and a Zariski closed invariant proper subset $\Sigma_l \subset \mathcal{E}^l$ such that the algebra of good differential invariants separates the regular orbits and is finitely generated in the following sense.
There exists a finite number of good differential invariants $I_1, ..., I_n$ and of good invariant derivations $\nabla_1, .., \nabla_s$ such that any good differential invariant is a polynomial of $\nabla_J(I_i)$, where $\nabla_J(I_i) = \nabla_{j_1} \circ \cdots \circ \nabla_{j_r}$, for some multi-indices J, with coefficients being rational functions of I.

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- An important issue (not appearing micro-locally) is that some of the derivations ∇_j may not be represented by total vector fields.
- Finiteness theorem valid invariant differential forms, tensors and other natural geometric objects.

Arnold conjecture on Poincarè function

V. Arnold (1994) made a conjecture that Poincarè functions in differential moduli problems are rational.

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- $P_{G,\mathcal{E}}(z) = \sum_k d_k \; z^k$ - the Poincaré function of the pseudogroup action.
- Then:
  1. The Hilbert function H_G is a polynomial for large k.
  2. The Poincaré function equals

$$P_G(z) = \frac{p(z)}{(1-z)^d},$$

for some polynomial $p(z)$ and integer $d > 0$.

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  are rational differential invariants of order 2.
- We say that metric g is *Ricci regular* at a point $a \in M$ if total differentials $\widehat{d}J_1, \widehat{d}J_2, ..., \widehat{d}J_n$ are linear independent at the point $a_3 = [g]_a^3 \in \mathbf{J}^3\left(\mathbf{S}^2\mathbf{T}^*\right)$.
  Here $[g]_a^3$ is the 3 - jet of g at the point, $\mathbf{J}^3\left(\mathbf{S}^2\mathbf{T}^*\right)$ the manifold of 3 -jets of the bundle $\mathbf{S}^2\mathbf{T}^*M \to M$.

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- Denote by
  $$\Sigma_3 = \{\widehat{d}J_1 \wedge \widehat{d}J_2 \wedge \cdots \wedge \widehat{d}J_n = 0\} \subset \mathbf{J}^3\left(\mathbf{S}^2\mathbf{T}^*\right)$$
  be the set of Ricci singular points.

At Ricci regular points we represent the metric in the following form

$$g = \sum_{i \leq j} G_{ij} \ \widehat{d} J_i \cdot \ \widehat{d} J_j .$$

Then G_{ij} are rational differential invariants of order 3 defined at Ricci regular points $\mathbf{J}^3 \left(\mathbf{S}^2 \mathbf{T}^* \right) \setminus \Sigma_3$.

The Tresse derivations

$$\nabla_1 = \frac{D}{DJ_j}, ..., \nabla_n = \frac{D}{DJ_n}$$

are rational, defined and linear independent at Ricci regular points. The basic frame formed by gradients of basic invariants:

$$E_1 = \operatorname{grad}_g J_1, ..., E_n = \operatorname{grad}_g J_n.$$

Differential invariant algebra

By differential invariant of order k we mean an invariant which is rational along fibres $\mathbf{J}^k\left(\mathbf{S}^2\mathbf{T}^*\right) \to M$ and polynomial along fibres $\mathbf{J}^k\left(\mathbf{S}^2\mathbf{T}^*\right) \to \mathbf{J}^3\left(\mathbf{S}^2\mathbf{T}^*\right)$.

Theorem

Algebra metric differential invariants generated by basic invariants G_{ij} and Tresse derivatives $\nabla_1, ..., \nabla_n$.

Metric differential invariants

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\begin{aligned}
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J &= (J_1(g), ..., J_n(g)),
\end{aligned}
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where $J_i(g)$ are the values of J_i at metric g, Σ_g - Ricci singular points, $\mathbf{D}_g = \mathrm{Im}(J)$.

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  where $J_i(g)$ are the values of J_i at metric g, Σ_g - Ricci singular points, $\mathbf{D}_g = \operatorname{Im}(J)$.
- Factor equation $\mathcal{E}_{\mathrm{metric}}$ for metrics h on $\mathbb{R}^n$:

$$\operatorname{Tr} R_h = x_1, \ \operatorname{Tr} R_h^2 = x_2, \cdots, \ \operatorname{Tr} R_h^n = x_n.$$

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- Basic invariants: If

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Theorem

Algebra differential invariants for Einstein metrics is generated by basic invariants G_{ij} and Tresse derivatives $\nabla_1, ..., \nabla_4$.

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Theorem

Two regular binary forms are $\mathbf{Gl}(2, \mathbb{C})$ *-equivalent iff their invariants coincide (up to multiplier).*

Remarks and generalizations

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Thank you for your attention