Noncommutative Algebraic Geometry, Topology, and Physics

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The relationship between algebraic geometry, topology, and physics, is well documented, and the field is very popular. I shall, in my talk (do my best to) introduce an extension of the methods used up to now, to include my version of non-commutative algebraic geometry.

The interest would be, to explain the relationship between notions like Ghost fields, Chern-Simons classes, Maxwell and Bloch’s equations, for electromagnetism, resp. time development of spin structures and entanglement, and Seiberg-Witten’s monopole equation. I shall, maybe, just vaguely touch upon the beautiful results in low-dimension algebraic topology, based upon these methods, due to Donaldson and his followers.
CONTENT OF TALK

• Mathematical models in physics, an Introduction
• Time and Dynamics
• Gauge Groups and Measurements
• Geometry of Representations
• Dynamics and the Dirac Derivation
• I Phase Spaces of Associative Algebras
• The Kodaira-Spencer class
• Hamiltonians and Connections
• Chern-Simons classes and Chern characters
• The iterated Phase Space functor as a co-simplicial resolution
• Relations to the de Rham complex
• Dynamics and the Dirac Derivation
• Preparations and Time-Evolution of Representations.
• Finite dimensional representations of $Ph^\infty(A)$. 
• II The generic dynamical structure, $C(\sigma_g)$, of a polynomial $k$-algebra $C$, induced by a metric $g$
• The Dirac Derivation in $C(\sigma_g)$
• Force Laws in $C(\sigma_g)$, General Relativity
• Classical Connections
• Chern Characters and Chern-Simons Classes
• III Quotients in Geometry

• Quotients in non-commutative algebraic geometry

• Gauge Groups and quotients in physics

• The classical case
- IV Local Gauge Groups and their Actions
- Lie-Cartan Pairs and Lie Algebroids
- The role of the Cartan subalgebra of a local Gauge Group in physics
• VI-VII Deformations of Associative Algebras
• The case of U, a singular point with a 3-dimensional tangent space
• The Versal family, \( U \) of U, the Toy Model, \( \tilde{H} \)
• Metric and Time, a Big Bang Model, Kepler and Newton
- VIII The Universal Local Gauge Group of the Standard Model
- Spin and Isospin
- Local Gauge Groups and their Actions
- Photones and Quarks
• **IX-X-XI** Elementary Particles, Weyl, Pauli and Dirac spinors, Chirality

• The **Generic Time-Action**

• Maxwell-, Bloch’s and Seiberg-Witten’s Equations.

• The **Generalized Dirac Equation**
• XII The Standard Model
If we want to study a natural phenomenon, called $\mathbf{P}$, we must in the present scientific situation, describe $\mathbf{P}$ in some mathematical terms, say as a mathematical object, $X$, depending upon some parameters, in such a way that the changing aspects of $\mathbf{P}$ would correspond to altered parameter-values for $X$. This object would be a *model for $\mathbf{P}$* if, moreover, $X$ with any choice of parameter-values, would correspond to some, possibly occurring, aspect of $\mathbf{P}$.

Two mathematical objects $X(1)$, and $X(2)$, corresponding to the same aspect of $\mathbf{P}$, would be called equivalent, and the set, $\mathcal{P}$, of equivalence classes of the objects $\mathbf{P}$, would correspond to (possibly a quotient of) the *moduli space*, $\mathbf{M}$, of the models, $X$. The study of the natural phenomena $\mathbf{P}$, and its changing aspects, would then be equivalent to the study of the *structure* of $\mathcal{P}$, and therefore to the study of the geometry of the moduli space $\mathbf{M}$. 
In particular, the notion of time would, in agreement with Aristotle and St. Augustin, correspond to some metric on this space. It turns out that to obtain a complete theoretical framework for studying the phenomenon \( P \), or the model \( X \), together with its dynamics, we should introduce the notion of dynamical structure, defined for the space, \( \mathcal{M} \). This is done via the construction of a universal non-commutative Phase Space-functor, \( \text{Ph}(\cdot) : \text{Alg}_k \to \text{Alg}_k \). It extends to the category of schemes, and its infinite iteration \( \text{Ph}^\infty(\cdot) \), is outfitted with a universal Dirac derivation, \( \delta \in \text{Der}_k(\text{Ph}^\infty(\cdot), \text{Ph}^\infty(\cdot)) \).

A dynamical structure defined on an associative \( k \)-algebra \( A \in \text{Alg}_k \) is now a \( \delta \)-stable ideal \( \sigma \subset \text{Ph}^\infty(A) \), and its quotient \( A(\sigma) := \text{Ph}^\infty(A)/(\sigma) \), with its Dirac derivation. The structure we are interested in, is the space \( U := \text{Ph}^\infty(\mathcal{M})/\sigma \), corresponding to an open affine covering by algebras of the type, \( A(\sigma) \).
But now we observe that there may be an action of a Lie algebra $\mathfrak{g}$, on $\mathbf{U}$, such that the dynamics of $\mathcal{P}$, really corresponds to that of the quotient $\mathbf{U}/\mathfrak{g}_0$.

To any open subset $U$, of $\mathbf{U}$, there would be associated a, not necessarily commutative, affine $k$-algebra, $A(\sigma) := \mathcal{O}_U(U)$, with an action of the Lie algebra $\mathfrak{g}_0$, such that the non-commutative quotient $\mathbf{U}/\mathfrak{g}_0$, represented by the system $A(\sigma)/\mathfrak{g}_0$ of simple representations of the algebra, $A(\sigma)$, with a $\mathfrak{g}$-connection. This system contains all the available information about the structure of $\mathbf{U}$, restricted to $U$. An element of $A(\sigma)$ would be called a \textit{local observable}, and wishing to measure the \textit{value} of an observable, leads to the study of the eigenvectors, and their eigenvalues, of the $\mathfrak{g}_0$-invariant representations of this algebra, which as we shall see, is the same as the representations of the system $A(\sigma)/\mathfrak{g}_0$. 
We may show that any (geometric: finitely generated) $k$-algebra $A$, may be recovered from its finite dimensional simple representations, and that there is an underlying quasi-affine (commutative) scheme-structure on each component $\text{Simp}_n(A)$, parametrizing the simple representations of dimension $n$. In fact, we have shown the following,

There is a commutative $k$-algebra $C(n)$ with an open subvariety $U(n) \subseteq \text{Simp}_1(C(n))$, an étale covering of $\text{Simp}_n(A)$, over which there exists a versal representation $\tilde{V} \simeq C(n) \otimes_k V$, a vector bundle of rank $n$ defined on $\text{Simp}_1(C(n))$, and a versal family, i.e. a morphism of algebras,

$$\tilde{\rho} : A \longrightarrow \text{End}_{C(n)}(\tilde{V}) \rightarrow \text{End}_{U(n)}(\tilde{V}),$$

inducing all isoclasses of simple $n$-dimensional $A$-modules.
Dynamics and the Dirac Derivation.

\( \text{End}_{\mathbb{C}(n)}(\tilde{V}) \) induces also a bundle, of operators, on the étale covering \( U(n) \) of \( \text{Simp}_n(A) \). Assume given a universal derivation, \( \delta \in \text{Der}_k(A) \).

Pick any \( \nu \in \text{Simp}_n(A) \) corresponding to the right \( A \)-module \( V \), with structure homomorphism \( \rho_{\nu} : A \to \text{End}_k(V) \), then \( \delta \) composed with \( \rho_{\nu} \), gives us an element,

\[
\delta_{\nu} \in \text{Ext}^1_A(V, V).
\]

Therefore, \( \delta \) defines a unique one-dimensional distribution in \( \Theta_{\text{Simp}_n(A)} \), which, once we have fixed a versal family, defines a vector field, \([\delta] \in \Theta_{\text{Simp}_n(A)}\), and in good cases, a (rational) derivation, the \textbf{Dirac Derivation}:

\[
[\delta] \in \text{Der}_k(\mathbb{C}(n)),
\]

governing the dynamics of our model.

Problem: How do we find this universal derivation?
Given an associative $k$-algebra $A$, denote by $A/k - alg$ the category where the objects are homomorphisms of $k$-algebras $\kappa : A \to R$, and the morphisms, $\psi : \kappa \to \kappa'$ are commutative diagrams,

\[
\begin{array}{ccc}
A & \xrightarrow{\kappa} & \kappa' \\
\kappa & \downarrow & \downarrow \\
R & \xrightarrow{\psi} & R'
\end{array}
\]

and consider the functor,

\[\text{Der}_k(A, -) : A/k - alg \longrightarrow \text{Sets}.\]

It is representable by a $k$-algebra-morphism, $\iota : A \longrightarrow Ph(A)$, with a universal family, i.e. a universal derivation,

\[d : A \longrightarrow Ph(A).\]
A universal derivation associated to an $A$-module

Clearly we have the identities,

$$d_* : \text{Der}_k(A, A) = \text{Mor}_A(Ph(A), A),$$

and,

$$d^* : \text{Der}_k(A, Ph(A)) = \text{End}_A(Ph(A)),$$

the last one associating $d$ to the identity endomorphism of $Ph$. Let now $V$ be a right $A$-module, with structure morphism

$$\rho : A \to \text{End}_k(V).$$

We obtain another universal derivation,

$$c : A \to \text{Hom}_k(V, V \otimes_A Ph(A)),$$

defined by, $c(a)(v) = v \otimes d(a)$. 
The Kodaira-Spencer class

Using the long exact sequence, of Hochschild cohomology,

\[
0 \to \text{Hom}_A(V, V \otimes_A \text{Ph}(A)) \to \text{Hom}_k(V, V \otimes_A \text{Ph}(A)) \\
\to \text{Der}_k(A, \text{Hom}_k(V, V \otimes_A \text{Ph}(A))) \xrightarrow{\kappa} \text{Ext}^1_A(V, V \otimes_A \text{Ph}(A)) \to 0,
\]

and,

\[
c \in \text{Der}_k(A, \text{Hom}_k(V, V \otimes_A \text{Ph}(A))),
\]

we obtain the non-commutative Kodaira-Spencer class,

\[
c(V) := \kappa(c) \in \text{Ext}^1_A(V, V \otimes_A \text{Ph}(A)),
\]

inducing, via the identity \(d_*\), the Kodaira-Spencer morphism,

\[
g : \Theta_A := \text{Der}_k(A, A) \to \text{Ext}^1_A(V, V).
\]
Using again the long exact sequence, of Hochschild cohomology,

\[ 0 \to \text{Hom}_A(V, V) \to \text{Hom}_k(V, V) \to ^l \text{Der}_k(A, \text{Hom}_k(V, V)) \xrightarrow{\kappa} \text{Ext}^1_A(V, V) \to 0, \]

we prove,

**Theorem**

Let \( \rho : A \to \text{End}_k(V) \), be an A-module, and let \( \delta \in \text{Der}_k(A, \text{Hom}_k(V, V)) \), map to 0 in \( \text{Ext}^1_A(V, V) \), i.e. assume \( \kappa(\delta) = 0 \), then there exist an element, \( Q_\delta \in \text{Hom}_k(V, V) \), the Hamiltonian, such that for all \( a \in A \),

\[ \rho(\delta(a)) = [Q_\delta, \tilde{\rho}(a)]. \]

If \( V \) is a simple A-module, \( \text{ad}(Q_\delta) \) is unique.
As is well known, in the commutative case, the Kodaira-Spencer class gives rise to a Chern character by putting,

$$ch^i(V) := 1/i! \ c^i(V) \in Ext^i_A(V, V \otimes_A Ph(A)),$$

and if $c(V) = 0$, the curvature $R(V)$ of $\nabla$, induces a curvature class,

$$R_{\nabla} \in H^2(k, A; \Theta_A, End_A(V)).$$
The iterated Phase Space functor, $Ph^*$

The phase-space construction may be iterated. Given the $k$-algebra $A$ we may form the sequence, $\{Ph^n(A)\}_{0 \leq n}$, defined inductively by

$$Ph^0(A) = A, \quad Ph^1(A) = Ph(A), \ldots, \quad Ph^{n+1}(A) := Ph(Ph^n(A)).$$

Let $i_0^n : Ph^n(A) \to Ph^{n+1}(A)$ be the canonical imbedding, and let $d_n : Ph^n(A) \to Ph^{n+1}(A)$ be the corresponding derivation. Since the composition of $i_0^n$ and the derivation $d_{n+1}$ is a derivation $Ph^n(A) \to Ph^{n+2}(A)$, corresponding to the homomorphism,

$$Ph^n(A) \to i_0^n Ph^{n+1}(A) \to i_0^{n+1} Ph^{n+2}(A)$$

there exist by universality a homomorphism $i_1^{n+1} : Ph^{n+1}(A) \to Ph^{n+2}(A)$, such that,

$$i_0^n \circ i_1^{n+1} = i_0^n \circ i_0^{n+1}$$

and such that,

$$d_n \circ i_1^{n+1} = i_0^n \circ d_{n+1}.$$
Clearly we may continue this process constructing new homomorphisms,

\[ \{ i_j^n : Ph^n(A) \to Ph^{n+1}(A) \}_{0 \leq j \leq n}, \]

such that,

\[ i^n_p \circ i^{n+1}_0 = i^n_0 \circ i^{n+1}_p \]

with the property,

\[ d_n \circ i^{n+1}_{j+1} = i^n_j \circ d_{n+1}. \]

We find the following identities,

\[ i^n_p i^{n+1}_q = i^n_{q-1} i^{n+1}_p, \quad p < q \]
\[ i^n_p i^{n+1}_p = i^n_p i^{n+1}_p \]
\[ i^n_p i^{n+1}_q = i^n_i^{n+1}_p, \quad q < p. \]
To see this, compose with $i_0^{n-1}$ and $d_{n-1}$, and use induction. Thus, the $Ph^*(A)$ is a semi-co-simplicial $k$-algebra with a co-section $h_0$, onto $A$. And it is easy to see that $h_0$ together with the corresponding co-sections $h_p : Ph^{p+1}(A) \to Ph^p(A)$, for $Ph^p(A)$ replacing $A$, form a trivialising homotopy for $Ph^*(A)$. Thus, we have,

$$H^n(Ph^*(A)) = 0, n \geq 0,$$

i.e.

- $Ph^*(A)$ is a cosimplicial resolution of the algebra $A$.

Therefore, for any object,

$$\kappa : A \to R \in A/k - alg$$

the co-simplicial algebra above induces simplicial sets,

$$Mor_k(Ph^*(A), R), \ Mor_A(Ph^*(A), R),$$

and one should be interested in the homotopy.
See also that this generalizes to a canonical functor,

\[ \text{Spec} : (k \text{-alg})^\Delta \op \rightarrow SPr(k) \]

where \((k \text{-alg})^\Delta\) is the category of co-simplicial \(k\)-algebras, and \(SPr(k)\) is the category of simplicial presheaves on the category of \(k\)-schemes enriched by some Grothendieck topology.

As usual, the imbedding of the category of \(k\)-algebras in the category of co-simplicial algebras is defined simply by giving any \(k\)-algebra a constant co-simplicial structure. The fact that \(Ph^*(A)\) is a resolution of \(A\), is therefore simply saying that,

\[ \text{Spec}(Ph^*((A))) \rightarrow \text{Spec}(A), \]

is a week equivalence in \(SPr(k)\).

This might be a starting point for a theory of homotopy for (non-commutative) \(k\)-schemes.
We may also consider, for any $k$-algebra $R$, the simplicial $k$-vectorspace,

$$\text{Der}_k(\Phi^*(A), R),$$

Consider this complex for $R = A$, i.e. $\text{Mor}_A(\Phi^*(A), A)$. Clearly,

$$\text{Mor}_A(\Phi^{n+1}(A), A) = \text{Der}_k(\Phi^n(A), A), n \geq 0,$$

and we have,

$$\text{Mor}_A(\Phi^n(A), A) = \{\xi_0 \circ \xi_1 \circ \ldots \circ \xi_r \mid 0 \leq i_l \leq i_{l+1} \leq n \mid \xi_0 = \text{id}_A, \xi_i \in \text{Der}_k(A), i \geq 1\}.$$

Since $\Phi$ is a functor, and $\Phi^{*+1}$ is a co-simplicial resolution of $A$, we may apply this to any scheme $X$, given in terms of an affine covering $\mathbf{U}$, and obtain an algebraic homology (or cohomology), with converging spectral sequences,

$$E_{pq}^1 = H_p(\mathbf{U}^{-q}(\text{Der}_k(\Phi^*(A), A))), E_{q,p} = H_{\mathbf{U}}^{-q}(H_p(\text{Der}_k(\Phi^*(A), A))).$$

If we, in $\text{Mor}_A(\Phi^n(A), A)$, identify $\xi \sim \alpha \xi, \alpha \in k^*$, we obtain a rational cohomology with converging spectral sequences,

$$E_{pq}^1 = H_p(H_{\mathbf{U}}^q(\text{Mor}_A(\Phi^n(A), A), \mathbf{Q})), E_{q,p}^2 = H_{\mathbf{U}}^q(H^p(\text{Mor}_A(\Phi^n(A), A), \mathbf{Q})).$$
Consider now the diagram,

\[
\begin{array}{c}
A \xrightarrow{i_0^0} Ph(A) \xrightarrow{i_1^1} Ph^2(A) \xrightarrow{i_2^2} Ph^3(A) \xrightarrow{i_3^3} \\
\uparrow \quad \uparrow \quad \uparrow \\
m_1^1 \xrightarrow{i_p^1} m_2^1 \xrightarrow{i_p^2} m_3^1 \xrightarrow{i_p^3}
\end{array}
\]

where, for each integer \( n \), the symbol \( i_n^p \), for \( p = 0, 1, \ldots, n \) signify the family of \( A \)-morphisms between \( Ph^n(A) \) and \( Ph^{n+1}(A) \) defined above, and where \( m_n^1 \) is the ideal of \( Ph^n(A) \) generated by \( im(d) \), which is the same as the ideal generated by the family, \( \{i_p^{n-1}(i_p^{n-2}(...(i_p^1(d(A))...))\} \), for all possible \( p \). And, inductively, let \( m_n^m \) be the ideal generated by \( m_n^1 m_n^{m-1} \).
We find an extended diagram,

\[
\begin{aligned}
A & \xrightarrow{i_0^0} Ph(A) \xrightarrow{i_p^1} Ph^2(A) \xrightarrow{i_p^2} Ph^3(A) \xrightarrow{i_p^3} \cdots \\
A & \xrightarrow{i_0^0} A \xrightarrow{i_p^1} A \xrightarrow{i_p^2} A \xrightarrow{i_p^3} \cdots \\
& \downarrow d \quad \downarrow d \quad \downarrow d \quad \downarrow d \\
& \downarrow d \\
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& \downarrow d \\
& m_1^1/m_1^2 \xrightarrow{i_p^1} m_2^1/m_2^2 \xrightarrow{i_p^2} m_3^1/m_3^2 \xrightarrow{i_p^3} \cdots \\
& m_1^2/m_1^3 \xrightarrow{i_p^1} m_2^2/m_2^3 \xrightarrow{i_p^2} m_3^2/m_3^3 \xrightarrow{i_p^3} \cdots \\
\end{aligned}
\]

The diagonals are not necessarily complexes, but to kill all \(d^n, \ n \geq 2\), it suffices to kill \(d^2\), and for this it suffices to kill \(d_1d_0\), as one easily see, applying the edge homomorphisms to, \(d_1(d_0(a))\) for all \(a \in A\).
Definition

The curvature $R(A)$ of the associative $k$ algebra, $A$, is the $k$-linear map composition of $d_0$ and $d_1$,

$$R(A) = d_0 d_1 : A \rightarrow m_2^2/m_2^3.$$

Now, kill the curvature $R(A)$, and all the terms under the first diagonal, beginning with $m_1^2/m_1^3$, together with all terms generated by the actions of the edge homomorphisms on these terms, and let, $\Omega^m_n$ be the quotient of $m_n^m/m_n^{m+1}$, for $n \geq 0$. Clearly, $\Omega^0_n = A$ for all $n \geq 0$, and we have got a graded semi co-simplicial $A$-module, with a $k$-differential $d$, such that $d^2 = 0$. 
Generalized de Rham complex.

The diagram is now looking like,

\[ A \xrightarrow{i_0^0} Ph(A) \xrightarrow{i_1^1} Ph^2(A) \xrightarrow{i_2^2} Ph^3(A) \xrightarrow{i_3^3} \]

\[ \xrightarrow{i_0^0} A \xrightarrow{i_0^1} A \xrightarrow{i_1^2} A \xrightarrow{i_2^3} A \]

\[ \begin{array}{c}
\Omega_1^1 \xrightarrow{i_0^1} \Omega_2^1 \xrightarrow{i_1^2} \Omega_3^1 \\
\Omega_2^2 \xrightarrow{i_2^3} \Omega_3^2
\end{array} \]

It is therefore a graded complex, in two ways. First as a complex induced from the semi-cosimplicial structure, with differential of bidegree (1,0), and second, as complex with differential \( d \), of bidegree (1,1).
The commutative case

Consider now the complex,

\[ A \to^d \Omega^1_1 \to^d \Omega^2_2 \to^d \Omega^3_3 \to^d \ldots \]

Theorem

Suppose \( A \) is commutative, then there is a natural morphism of complexes of \( A \)-modules,

\[ \Omega^* \subset \Omega^*_\ast, \]

with,

\[ \Omega^n_A \simeq \Omega^n. \]
Let, $a_i \in A$, $i = 1, \ldots, r$, and compute in $\Omega^r_A$ the value of, $d^r(a_1a_2\ldots a_r)$. It is clear that this gives the formula,

$$\sum d_{i_1}(a_1)d_{i_2}(a_2)\ldots d_{i_r}(a_r) = 0,$$

the sum being over all permutation $(i_1, i_2, \ldots, i_r)$ of $(0, 1, \ldots, r - 1)$. Here we consider $A$ as a subalgebra of $Ph^n(A)$ via the unique compositions of the $i^s_0 : Ph^s(A) \subset Ph^{s+1}(A)$. In particular, we have,

$$d_0(a_1)d_1(a_2) + d_1(a_1)d_0(a_2) = 0,$$

for all $a_1, a_2 \in A$. This relation and the relation $d_0(a)d_1(b) = d_1(b)d_0(a)$, which follows from commutativity, $d(a)b = bd(a)$, forcing the left and right $A$-action on $\Omega_A$ to be equal, immediately give us,

$$d_0(a)d_1(b) = -d_0(b), d_1(a).$$

It is now clear that the map that sends the element $da_1 \wedge da_2 \wedge \ldots \wedge da_r \in \Omega^r_A$ to $d_0(a_1)d_1(a_2)\ldots d_{r-1}(a_r) \in \Omega^r_A$ is an isomorphism, and the rest should be clear.
Generalization to modules I

Let now, $V$ be an $A$-module, and assume $c(V) = 0$, and pick a connection, $\nabla \in \text{Hom}_k(V, V \otimes_A Ph(A))$ with $c = \iota(\nabla)$. This imply that for $a \in A$ and $v \in V$ we have $\nabla(va) = \nabla(v)a + v \otimes d_0(a)$. Composing $\nabla$ with the cosection, $o : Ph(A) \to A$, corresponding to the 0-derivation of $A$, we therefore obtain an $A$-linear homomorphism $P : V \to V$, a potential. Since $i_0 : A \to Ph(A)$ is a section of $o$, we find a $k$-linear map,

$$\nabla_0 := \nabla - P : V \to V \otimes m_1^1$$

Using the property,

$$d_n \circ i_{j+1}^{n+1} = i_j^n d_{n+1},$$

it is easy to find well defined $k$-linear maps,

$$\nabla_1 : V \to V \otimes \Omega_2^2, \quad \nabla_2 : V \to V \otimes \Omega_3^3, \ldots, \quad \nabla_n : V \to V \otimes \Omega_{n+1}^{n+1} \quad \forall n \geq 0,$$

given by,

$$\nabla_{n+1} := \nabla_n \circ i_1^{n+1}, \quad n \geq 0.$$
Fix the connection $\nabla$. For all, $\nu \in V$, $\omega \in \Omega^n$, the formula,

$$\nabla_n(\nu \otimes \omega) = \nabla_n(\nu)\omega + \nu \otimes d_n(\omega).$$

makes sense, and defines derivations, also called $d$. We obtain a situation just like above,
Generalization to modules

In general, there are no reasons for these $d'$s to define complexes, and we shall make the following definition,

**Definition**

The curvature $R(V, \nabla)$ of the connection $\nabla$ defined on the right $k$ $A$-module $V$, is the $k$-linear map composition of $d_0$ and $d_1$,

$$R(V, \nabla) = d_0 d_1 : V \to V \otimes \Omega^2.$$

The following result is then easily proved,

**Theorem**

Suppose $A$ is commutative, and let $\nabla : \Theta_A \to \text{End}_k(V)$ be the connection corresponding to $\nabla_0$. Suppose moreover that the curvature $R$ of $\nabla$ is 0, then $R(V) = 0$, implying that $d^2 = 0$, and so the diagonals in the diagram above, are all complexes.
Consider now the co-simplicial algebra,

\[ A \rightarrow i_0^0 \, Ph(A) \rightarrow i_1^1 \, Ph^2(A) \rightarrow i_2^2 \, Ph^3(A) \rightarrow i_3^3 \, \ldots \]

where, for each integer \( n \), the symbol \( i_p^n \), for \( p = 0, 1, \ldots, n \) signify the family of \( A \)-morphisms between \( Ph^n(A) \) and \( Ph^{n+1}(A) \) defined above. The inductive (direct) limit,

\[ Ph^\infty(A) = \lim_{\rightarrow} \{ Ph^n(A), i_j^n \}_{n \geq 0} \]

comes with homomorphisms

\[ i_n : Ph^n(A) \rightarrow Ph^\infty(A), \text{ satisfying } i_j^n \circ i_{n+1} = i_n, \quad j = 0, 1, \ldots, n \]

Moreover, the family of derivations, \( \{ d_n \}_{0 \leq n} \) defines a unique Dirac-derivation,

\[ \delta : Ph^\infty(A) \rightarrow Ph^\infty(A), \]

such that \( i_n \circ \delta = d_n \circ i_{n+1} \).
Dynamics in $\text{Rep}(A)$

Since $\text{Ext}^1_A(V, V)$ is the tangent space of the miniversal deformation space of $V$ as an $A$-module, we see that the non-commutative space $Ph(A)$ also parametrizes the set of \textit{generalised momenta}, i.e. the set of pairs of an $A$-module $V$, and a tangent vector of the formal moduli of $V$, at that point. Therefore the above implies that any representation, $\rho : Ph^\infty(A) \to \text{End}_k(V)$, corresponds to a family of $Ph^n(A)$-module-structures on $V$, for $n \geq 1$, i.e. to an $A$-module $\rho_0 : (V_0 := V)$, an element $\eta_0 \in \text{Ext}^1_A(V, V)$, i.e. a tangent of the deformation functor of $V_0 := V$, as $A$-module, an element $\eta_1 \in \text{Ext}^1_{Ph(A)}(V, V)$, i.e. a tangent of the deformation functor of $V_1 := V$ as $Ph(A)$-module, an element $\eta_2 \in \text{Ext}^1_{Ph^2(A)}(V, V)$, i.e. a tangent of the deformation functor of $V_2 := V$ as $Ph^2(A)$-module, etc.
All this is just $\rho_0 : A \to \text{End}_k(V)$, considered as an $A$-module, together with a sequence $\{\eta_n\}, 0 \leq n$, of a tangent, or a momentum, $\eta_0$, an acceleration vector, $\eta_1$, and any number of higher order momenta $\eta_n$.

- Thus, specifying a $Ph^\infty(A)$-representation $V$, implies specifying a formal curve through $\rho_0(v_0)$, the base-point, of the miniversal deformation space of the $A$-module $V$. Formally, this curve is given by the composition of the homomorphism $\epsilon(\tau) := \exp(\tau \delta)$ and $\rho$. 
It is, however, impossible to prepare a physical situation such that a measurement, i.e. an object like $\rho_0$, is given by an infinite sequence $\{\eta_n\}$, of dynamical data. We shall have to be satisfied with a finite number of data, and normally with just the first one, i.e. the momentum $\eta_0$. This is the Problem of Preparation and (of the) Time Evolution of a representation $\rho$, to be treated in the sequel.
Given an $r$-dimensional $k[t] := k[t_1, ..., t_d]$-module, consisting of $r$ points \( \{ P_p = (\alpha_1^0(p), \alpha_2^0(p), ..., \alpha_d^0(p)) \} \mid p = 1, ..., r \). Assume given, \( \alpha_i^n(p) \in k, \ i = 1, ..., d, \ n \geq 0, \ p = 1, ..., r \), and arbitrary coupling constants, \( \sigma_m(p, q) \in k \) with \( \sigma_0 = 0 \). Put
\[
\alpha_i^n(p, q) = \alpha_i^n(p) - \alpha_i^n(q), \ i = 1, ..., d,
\]
and assume, for all \( n \geq 1 \),
\[
\sum_{h} \binom{n}{h} \sigma_{n-h}(p, q)(\alpha_i^h(p, q)\alpha_j^0(p, q) - \alpha_i^0(p, q)\alpha_j^h(p, q))
\]
\[
= \sum_{k, l, m, s} \frac{n!\sigma_{n-k-m}(p, s)\sigma_{k-l}(s, q)}{l!m!(k-l)!(n-k-m)!} (\alpha_j^m(p, s)\alpha_i^l(s, q) - \alpha_i^l(p, s)\alpha_j^m(s, q)),
\]
Representations of $Ph^\infty(k[t])$

Consider the matrix,

$$D_i^n := \begin{pmatrix}
\alpha_i^n(1) & r_i^n(1, 2) & \ldots & r_i^n(1, r) \\
r_i^n(2, 1) & \alpha_i^n(2) & \ldots & r_i^n(2, r) \\
\vdots & \vdots & \ddots & \vdots \\
r_i^n(r, 1) & r_i^n(r, 2) & \ldots & \alpha_i^n(r)
\end{pmatrix}$$

with the relations,

$$\alpha : r_i^0(p, q) = 0, \quad r_i^n(p, q) = \sum_{l=0}^{n} \binom{n}{l} \alpha_i^l(p, q) \sigma_{n-l}(p, q),$$

Then $\rho(d^n t_i) = D_i^n$ define a representation,

$$\rho : Ph^\infty(k[t]) \to M_r(k)$$
Proof

Let us, as above, consider the matrix,

\[ X_i = \rho(\exp(\tau\delta))(t_i) = \sum_{n \geq 0} \frac{\tau^n}{n!} D^n_i \]

Putting

\[ \alpha_i(p) = \sum_{n=0}^{\infty} \frac{\tau^n}{n!} \alpha^n_i(p), \quad \alpha_i(p, q) = \sum_{n=0}^{\infty} \frac{\tau^n}{n!} \alpha^n_i(p, q), \]

and, \( \sigma(p, q) = \sum_{n=0}^{\infty} \frac{\tau^n}{n!} \sigma^n(p, q) \), we find the explicite formulas,

\[
X_i = \begin{pmatrix}
\alpha_i(1) & \sigma(1, 2)\alpha_i(1, 2) & \cdots & \sigma(1, r)\alpha_i(1, r) \\
\sigma(2, 1)\alpha_i(2, 1) & \alpha_i(2) & \cdots & \sigma(2, r)\alpha_i(2, r) \\
\cdots & \cdots & \cdots & \cdots \\
\sigma(r, 1)\alpha_i(r, 1) & \sigma(r, 2)\alpha_i(r, 2) & \cdots & \alpha_i(r)
\end{pmatrix}, \quad i = 1, \ldots, d.
\]

Now, compute, and put, \([X_i, X_j] = 0\) and see that the condition of the theorem emerges.
We may consider the space

$$A(r) = k[\alpha_i^n(p), \sigma_n(p, q)]/a,$$

with coordinates $$\{\alpha_i^n(p), \sigma_n(p, q), i = 1, ..., d, n \geq 0, p, q = 1, ..., r\}$$, and where the ideal $$a$$ is generated by the equations above, as the versal base space for the versal family of the non-commutative deformation theory applied to the family of $$Ph^\infty(k[t])$$ modules defined by the object $$P$$.

In dimensions $$d = 3$$, $$r = 3$$, and order 1, the condition above reads:

$$\forall p, s, q = 1, 2, 3$$

$$\sigma_1(p, q)(\alpha^1(p, q) \times \alpha^0(p, q)) = -\sigma_1(p, s)\sigma_1(s, q)(\alpha^0(p, s) \times \alpha^0(s, q)).$$

This says that for any two of the three points in space, the relative momentum must sit in the plane defined by the three points, the length being determined by the 3 coupling constants. Moreover, the obvious sum of all three relative momenta must be 0.
Let
\[ C := k[t_1, \ldots, t_n] \]

Then,
\[ Ph(C) = k < t_1, \ldots, t_n, dt_1, \ldots, dt_n > / ([t_i, t_j], [dt_i, t_j] + [t_i, dt_j]). \]

A non-degenerate metric, \( g = 1/2 \sum_{i=1}^{d} g_{i,j} dt_idt_j \in Ph(C) \) induces an isomorphism of \( C \)-modules
\[ \Theta_C = \text{Hom}_C(\Omega_C, C) \cong \Omega_C. \]

Consider the bilateral ideal \( (\sigma_g) \) of \( Ph(C) \) generated by
\[ (\sigma_g) = ([dt_i, t_j] - g^{i,j}), \]

and put,
\[ C(\sigma_g) := Ph(C)/(\sigma_g). \]
The Dirac Derivation in $C(\sigma_g)$

II No 2

Let, moreover,

$$T := \sum_j T_j = -1/2 \sum_{i,j,l} \delta_{t_i}(g_{i,j})g^{l,i} dt_j$$  \hspace{1cm} (1)

$$= -1/2(\sum_{k,l} \Gamma_{k,l} dt_l + \sum_{k,p,q} g^{k,q} \Gamma_{p,k,q} g_{p,l} dt_l),$$  \hspace{1cm} (2)

and consider the inner derivation of $C(\sigma_g)$, defined by,

$$\delta := ad(g - T).$$

After a dull computation, we obtain, in $C(\sigma_g)$,

$$\delta(t_i) = dt_i, \quad i = 1, \ldots, d.$$ 

Therefore, by universality, we have a well-defined dynamical structure, i.e. a $\delta$ stable ideal $(\sigma_g) \subset Ph^\infty(C)$, with Dirac derivation, $\delta = ad(g - T)$. It is also easy to see that $(\sigma_g)$ is invariant w.r.t. isometries.
There are in $Ph(C)$ force laws, one is given by,

\[
(*) \quad d^2 t_i = - \sum_{p,q} \Gamma^i_{p,q} dt_p dt_q - 1/2 \sum_{p,q} g_{p,q}(F_{i,p} dt_q + dt_p F_{i,q}) \\
+ 1/2 \sum_{l,p,q} g_{p,q}[dt_p, (\Gamma^i_{l,q} - \Gamma^q_{l,i})]dt_l + [dt_i, T]
\]

\[
= - \sum_{p,q} \Gamma^i_{p,q} dt_p dt_q - \sum_{p,q} g_{p,q} F_{i,p} dt_q + 1/2 \sum_{l,p,q} g_{p,q}[F_{i,q}, dt_p] \\
+ 1/2 \sum_{l,p,q} g_{p,q}[dt_p, (\Gamma^i_{l,q} - \Gamma^q_{l,i})]dt_l + [dt_i, T]
\]

and consistent with the Dirac derivation in $C(\sigma_g)$, as well as with the classical equation for geodesics in $C(\sigma_0)$, where we have $\sigma_0 = \langle \{[dt_i, t_j], [dt_i, dt_j]\} \rangle$, so that $C(\sigma_0) = Ph(C)_{com}$, the commutative classical phase space.
We observe that we have got:

A Field Theory for connections on $C$-representations $V$, i.e. for representations,

$$\rho : C(\sigma_g) \to \text{End}_k(V)$$

with Dirac derivation for representations, $[\delta] = 0$ and Hamiltonian $Q = \rho(g - T)$.

We have also a model for General Relativity, for the scheme $\text{Spec}(\text{Ph}(C)_{\text{com}})$, i.e. for representations,

$$\rho : C(\sigma_0) \to \text{End}_{C(\sigma_0)}(C(\sigma_0)) \to k$$

with Dirac derivation,

$$\delta = \sum_{i,p,q} (-\Gamma^i_{p,q} v_p v_p \delta v_i + v_i \delta t_i)$$

Here $v_i$ are the ”vertical” coordinates in the phase space of $C = k[t_1, .., t_n]$, i.e. the momenta. The Hamiltonian $Q$ is, obviously,
Further consequences of the Force Laws

II No 5

For the Euclidean or the Minkowski metric, the Force Law, or the equation of motion, reduces to,

\[ d^2 t_i = - \sum_{p, q} g_{p,q} F_{i,p} dt_q + \frac{1}{2} \sum_{l, p, q} g_{p,q} [F_{i,q}, dt_p]. \]

Given a representation,

\[ \rho : Ph(C) \rightarrow \text{End}_k(\Theta_C), \]

this implies the following simple evolution equation,

\[ \dot{\rho} (dt_i) = \rho (d^2 t_i) = - \sum_{p} (F_{i,p} \rho (d_p) + \frac{1}{2} \nabla_{\delta_s} (F_{i,p})). \]

which, when we have introduced our favourite Toy Model, will give us the Bloch and the Maxwell equations.
Choose a particular momentum, $\rho^*$, and put
$$\xi_i := \sum_{l=1}^{n} g^{il} \delta_l \in \text{Der}_k(C), \rho^*(dt_i) =: \nabla_{\xi_i}. \text{ Then the space of representations, } \rho \text{ of } C(\sigma) \text{ is given as above, by}$$

$$\rho_\psi(t_i) = t_i, \ \rho_\psi(dt_i) = \nabla_{\xi_i} + \psi_i,$$

where $\psi_i \in \text{End}_C(V).$ (On a free $C$-module $V$, one usually write $\xi_i$ for $\nabla_{\xi_i}.$)

The set of iso-classes is identified with the space of equivalence classes of the corresponding set of potentials, $\psi := (\psi_1, \psi_2, ..., \psi_n)$, naturally isomorphic to,

$$\mathcal{P} = \text{End}_C(V)^n.$$

It does not form an algebraic variety, but it has a nice structure.
The tangent space $T_{\rho}$ of $P$, of $P$ at $\rho_\psi : C(\sigma) \to \text{End}_C(V)$, represented by $\psi \in P$, may also be identified with a quotient of $P$. In fact,

$$\text{Ext}^1_{C(\sigma_g)}(\rho_\psi, \rho_\psi) = \text{Der}_k(C(\sigma_g), \text{End}_k(V))/\text{Triv}.$$ 

Any derivation $\eta \in \text{Der}_k(C(\sigma_g), \text{End}_k(V))$, maps the relations of $C(\sigma_g)$ to zero, so we shall have,

$$[\eta(dt_i), t_j] + \rho(dt_i)\eta(t_j) - \eta(t_j)\rho(dt_i) = \eta(g^{i,j}).$$

If $V$ is a projective $C$-module, such that $\text{Ext}^1_C(V, V) = 0$, there exists a linear map, $\Phi_0 \in \text{End}_k(V)$, such that $\eta(t_j) = t_j\Phi_0 - \Phi_0 t_j$, for all $j$. We may therefore, assume all $\eta(t_i) = 0$, and then the derivation $\eta$ is determined by the family of elements, $\eta(dt_i) \in \text{End}_C(V))$, $i = 1, \ldots, n$. 
For $\rho := \rho_\psi$, corresponding to $\psi \in \mathcal{P}$, we have seen that,

$$T_\rho = \text{Ext}^1_{C(\sigma)}(\rho, \rho) = \mathcal{P} / \text{Triv}$$

where the the elements in $\text{Triv}$, corresponding to trivial derivations also mapping $t_i$ to 0, are exactly those given by the n-tuples,

$$((\xi_1(\Phi) + [\psi_1, \Phi]), ..., (\xi_n(\Phi) + [\psi_n, \Phi])), \text{for some } \Phi \in \text{End}_C(V).$$
The Gauge Group $\text{End}_C(V)$

The expression,

$$\Phi(\psi) := (\xi_1(\Phi) + [\psi_1, \Phi], \ldots, \xi_n(\Phi) + [\psi_n, \Phi]),$$

therefore corresponds to an infinitesimal gauge transformation,

$$\Phi \in \text{Der}_k(\mathcal{P}) =: \mathfrak{h}$$

of the space, $\mathcal{P}$, of representations of $C(\sigma_g)$, acting linearly like

$$\Phi(\rho\psi) = (\rho\psi) + \Phi(\psi)$$

In particular, the tangent space,

$$T_{\rho\psi} = \text{End}_C(V)^n / \mathfrak{h}.$$

The physical relevant space is therefore the quotient,

$$\mathcal{P} = \mathcal{P} / \mathfrak{h}$$

of $\mathcal{P}$ with respect to the action of the Lie algebra $\mathfrak{h}$. 
The Action of the Dirac Derivation on $\mathbf{P}$

II No 10

As in the finite dimensional situation, the Dirac derivation induces a vector field

$$[\delta] \in \Theta_{\mathbf{P}},$$

as long as we, by vector field, understand any map which to an element $\psi$ in $\mathbf{P}$ associate an element in its tangent space, i.e. in $\text{Ext}^1_{C(\sigma_g)}(V_\rho, V_\rho)$, for $\rho = \rho_0 = \psi$. It must, however, vanish at $\rho$, since the Dirac derivation $\delta = \text{ad}(g - T)$, necessarily must be mapped to a trivial derivation in $\text{Der}_k(C(\sigma), \text{End}_k(V))$, therefore to 0 in $\text{Ext}^1_{C(\sigma)}(V_\rho, V_\rho)$. But then it corresponds to an infinitesimal transformation of $V$,

$$\rho_\psi(g - T) = 1/2 \sum g_{i,j} \rho_\psi(dt_i) \rho_\psi(dt_j) + 1/2 \sum_{l}(\bar{\Gamma}_{i,l}^l + \Gamma_{i,l}^l) \rho_\psi(dt_i)$$

This may be interpreted as saying that time acts within each representation, $\rho : C(\sigma_g) \to \text{End}_k(V)$. 

Olav Arnfinn Laudal () Noncommutative Algebraic Geometry, Topology, November 1, 2016 53 / 141
The physicists usually write $\delta\psi := \Phi(\psi)$, not caring to mention $\Phi$, taking for granted that $\delta\psi := \delta\Phi(\psi)$ stands for an infinitesimal movement of $\psi$ in the direction of $\Phi$, and call the transformation above, an \textit{infinitesimal gauge transformation}. The literature on gauge theory, and its relation to non-commutativity of space, and to quantisation of gravity, is huge. I think that the introduction of the \textit{non-commutative phase space}, and in the metric case, the \textit{generic} dynamical system,

$$ (\sigma_g) = ([dt_i, t_j] - g^{i,j}), $$

can, to some degree, elucidate the philosophy behind this effort. See e.g. the papers, [?], and [?], where the authors initially introduce non-commutativity in the ring of \textit{observables} generated by \textit{coordinates}, $\hat{x}^\nu$, by imposing,

$$ [\hat{x}^\nu, \hat{x}^\mu] = \Theta^{\nu,\mu}, $$

where $\Theta^{i,j}$ are constants.
The above treatment of the notion of gauge groups and gauge transformations may also explain why, in physics, one considers potentials as interaction carriers, thus as particles mediating force upon other particles. And maybe one can also see why the notion of Ghost Fields or Particles of Faddeev and Popov, comes in. It seems to me that the introduction of ghost particles is linked to working with a particular section of the quotient map, $\mathcal{P} \to \mathbf{P}$.

The Dirac derivation, which is entirely dependent upon the notion of non-commutative phase space, is not (explicitely) found in present day physics. The parsimony principle is therefore, normally, introduced via the construction of a Lagrangian, and an Action Principle, i.e. a function of the (assumed physically significant) variables, the fields and their derivatives, defined in $\mathcal{P}$, assumed to to be invariant under the gauge transformations, so really defined in $\mathbf{P}$, and supposed to stay stable during time development, see [?].
A non trivial element in the toolbox of the physicists, helping them to guess the Lagrangian, is the Chern-Simons functional, that we now spell out in some generality. Since everything we have done above is functorial, we may work on nonsingular schemes, instead of commutative \( k \)-algebras. Let us, as above, assume given a metric \( g \) on some scheme \( X \), and that we have an affine covering, given in terms of a family of commutative \( k \)-algebras \( C_\alpha \) and a bundle \( V \), defined on \( X \), corresponding to a set of representations, \( \rho_0 : C = C_\alpha \rightarrow \text{End}_k(V_\alpha) \). Let \( \rho : O_X(\sigma_g) \rightarrow \text{End}_k(V) \) be a momentum at \( \rho_0 \), i.e. an extension of \( \rho_0 \) to \( O_X(\sigma_g) \). Then we know that \( \rho \) induces a connection on \( V \), and we have denoted by 
\[
\mathcal{P} := \text{End}_{O_X}(V)^d,
\]
the set of such connections, or if we want to, the set of representations \( \rho : O_X(\sigma_g) \rightarrow \text{End}_k(V) \). Recall that we may pick a 0-object \( \rho^* \) in this set, such that all other representations \( \rho \), is \( \rho^* \) plus a potential \( \psi \in \text{End}_{O_X}(V)^d \).
Consider now for any $O_X$-bundle $\rho$, $V$ the class in $HH^n(O_X, \text{End}_k(V))$ defined by the Hochschild co-chain, $ch^n(\rho)$ the $k$-linear map, defined locally by,

$$ch^n : C_{\alpha}^\otimes^n \rightarrow \text{End}_k(V_{\alpha}),$$

where,

$$ch^n(c_1 \otimes c_2 \ldots \otimes c_n) = \rho(d c_1 d c_2 \ldots d c_n) \in \text{End}_k(V_{\alpha}).$$

We know that $ch^n$ is a Hochschild co-cycle, since,

$$\delta(ch^n)((c_1 \otimes c_2 \ldots \otimes c_{n+1})) = c_1 \rho_1((d c_2 d c_3 \ldots d c_{n+1}))$$

$$+ \sum_{1}^{n} (-1)^{i} \rho_1((d c_1 \ldots d(c_i; c_{i+1}) \ldots d c_{n+1})) + (-1)^{n+1} \rho_1(d c_1 \ldots d c_n)c_{n+1} = 0$$

(3)

(4)
Invariance of the Chern-Simons Classes
II No 15

The Generalized Chern-Simons Class of $\rho_\psi$, is then the class in the obvious
double complex, defined by the covering $\{C_\alpha\}$, and the classes,

$$ch^n(\rho) \in HH^n(C_\alpha, End_k(V_\alpha)),$$

or, usually, by $1/n! \ ch^n$. Let $\Phi \in \mathcal{h} := End_C(V)$, and consider $\Phi$ as a
Hochschild 0-cocycle,

$$\Phi : k \rightarrow C^0 \rightarrow End_k(V), \Phi(\alpha) = \alpha \Phi,$$

then,

$$\delta \Phi(t_i) = \rho_\psi(dt_i) \Phi - \Phi \rho_\psi(dt_i) = [\nabla \xi_i + \psi_i, \Phi] = \xi_i(\Phi) + [\psi_i, \Phi].$$

This proves that

$$ch^1(\rho_\psi) = ch^1(\rho_\psi + \Phi(\psi))$$

and implies that the Ghost Fields have well defined Chern-Simons classes.
The relationship between the two different notions of time, the metric of our moduli space $C$ of our models, and the Dirac derivative of $Ph^\infty(C)$ is tight.

Let $\mathbf{M}$ be the space (of isomorphism classes) of metrics on $C$. For every point $g \in \mathbf{M}$ consider the diagram,

$$
\begin{array}{cccc}
Ph(C) & \rightarrow & Ph^2(C) & \rightarrow & Ph^3(C) & \rightarrow & \cdots & \rightarrow & Ph^\infty(C) \\
\downarrow d & & \downarrow \rho_1 & & \downarrow \rho_2 & & \downarrow \rho_3 & & \downarrow \rho \\
C & \rightarrow & C(\sigma_g) & \rightarrow & End_k(V) & & & & \\
i & & \rho & & & & & & \\
\end{array}
$$

Here $\rho_1$ is the representation of $Ph(C)$ induced by a representation $\rho$ of $C(\sigma_g)$. 

Olav Arnfinn Laudal () Noncommutative Algebraic Geometry, Topology, and Physics November 1, 2016 59 / 141
Consider the family of $k$-algebras,
\[ \mu : \mathcal{C}(\sigma) \to \mathcal{M} \]
indexed by the possible metrics of $\mathcal{C}$, such that $g \in \mathcal{M}$ corresponds to $\mathcal{C}(\sigma_g)$.
Let,
\[ T_{\mathcal{M},g} = \{(h_{i,j})\}, \quad h_{i,j} = h_{j,i} \in \mathcal{C}. \]
be the tangent space to $\mathcal{M}$, at $g$. Define, $h^{i,j}$ by,
\[ h_{i,j} = - \sum_{p,q} g_{i,p} h^{p,q} g_{q,j}, \quad h = \{h_{i,j}\} \in T_{\mathcal{M}}. \]
Consider now the first order deformation $g + \epsilon h$, of the metric $g$, and the corresponding Dirac derivation, $ad(g + \epsilon h - T')$ in $\mathcal{C}(\sigma_{(g+\epsilon h)})$. Then we find, in $\mathcal{C}(\sigma_{(g+\epsilon h)})$,
\[ [d' t_i, t_j] = g^{i,j} - h^{i,j} \epsilon = (g + \epsilon h)^{i,j}. \]
Moreover, the derivation \( \eta : Ph(C) \rightarrow End_k(V) \) defined by,

\[
\eta(t_i) = 0, \quad \eta(dt_i) = \sum_{l,q} h^{i,l} \rho_1(g_{l,q} dt_q) = \sum_{l} h^{i,l} \nabla_{\delta_l}
\]

corresponding to the 2.-order derivative of \( \rho \), i.e. to the morphism,

\[
\eta(\rho) := \rho_2 : Ph^2(C) \rightarrow End_k(V),
\]

for which, \( \rho_2(d^2t_i) = \eta(dt_i) \), induces an element,

\[
\eta(h) \in \text{Ext}^1_{Ph(C)}(V, V)
\]

producing an injective map,

\[
\eta : T_{M,g} \rightarrow \text{Ext}^1_{Ph(C)}(V, V).
\]
Thus, any non-trivial deformation of the metric $g$ induces a non-trivial deformation of the $Ph(C)$-representation $(\rho_1, V)$, and any first order non-trivial deformation of the $Ph(C)$-representation $(\rho_1, V)$ induces a non-trivial deformation of the metric. Given any non-zero tangent $h \in T_{Mg}$, we see that there corresponds a non-zero second-order tangent of the representation $\rho_0$, given by the element $\eta(h) \in Ext^1_{Ph(C)}(V, V)$, determined by the $\rho_1$-derivation $\eta$.

This $\eta(h)$ is the measure of an acceleration of the representation $\rho_0$, and should correspond to a "cataclysmic change" of any massy "particle $\rho_0 : C \rightarrow End(V)$", given by $\ddot{\rho}_0 = h$, the solution of which would be a "wave", $\tilde{\rho}_0$. Combined with the structure of our "Toy Model", see [?], this fits well with the present understanding of gravitational waves. It also fit reasonably well with the present cosmological theory.
Moreover, let us, for every metric $g \in \mathcal{M}$ consider the scalar curvature $R$ as a function on the space $\mathcal{C} := \text{Spec}(\mathcal{C})$, and fix a ”compact” subset $\Omega \subset \mathcal{C}$. Since $R \sim 1/r^d$, where $r$ is the ”radius of curvature” of $\Omega$ at the corresponding point, it is not unreasonable to consider

$$S(g) := \int_{\Omega} R \, dv_g,$$

where $dv_g$ is the volume element in $\mathcal{C}$ defined by the metric $g$, as the ”furniture”, i.e. the ”material” content of the the part of space, $\Omega$, represented by all representations of $\mathcal{C}(\sigma_g)$. But then one could look at $S$, as a functional,

$$S : \mathcal{M} \to \mathbb{R}$$

the stability of which would give us a unique vector field on $\mathcal{M},$

$$\mathcal{G} \in \Theta_{\mathcal{M}}, \quad \mathcal{G}(g) = \{\mathcal{G}_{i,j}\} := \{R_{i,j} - 1/2Rg_{i,j}\} \in T_{\mathcal{M},g}.$$
Then the injective map,
\[ \eta : T_{M,g} \rightarrow \text{Ext}^1_{\Phi(C)}(V, V), \]
would be a *Field Equation*, of Einstein-Hilbert type. An ”increment” \( h \), of the metric \( g \) (in \( \Omega \)) would correspond to the element of \( \eta(h) \in \text{Ext}^1_{\Phi(C)}(V, V) \), of first order, for all representations \( \rho \), corresponding to an increment of energy of the total furniture of our Universe. So, again we find a Schrödinger type equation; equating derivation with respect to time, with \( \mathcal{G} \), and seeing that \( \rho_2 \) corresponds to an Hamiltonian operator on \( V \), we find,
\[ \delta_t(\rho_1) = \rho_2. \]

This is our generalised Einstein field equation, \( \eta(h) \) for all \( \rho \), being identified with the mass, energy, stress-tensor of the general relativity theory.
III Quotients in Geometry

No 1

Let $\mathfrak{g} \subset \text{Der}_k(A)$, be a sub Lie-module, and consider first, in the commutative case, the scheme (algebraic variety), $X = \text{Spec}(A)$, and the Lie algebra $\mathfrak{g}$ as a Lie algebra of vector fields defined on $X$. The set of maximal integral subschemes,

$$X/\mathfrak{g} := \{ M \subset X | \Theta_M = \mathfrak{g}|_M \},$$

is called the quotient of $X$ by $\mathfrak{g}$, and coincides, in good cases with the quotient of $X$ by the group of automorphisms $G$ acting on $X$, with $\text{lie} G = \mathfrak{g}$, when this exists. In the classical, commutative case, one would identify,

$$X/\mathfrak{g} = \text{Spec}(A^\mathfrak{g}),$$

where, $A^\mathfrak{g} := \{ a \in A | \forall \gamma \in \mathfrak{g}, \gamma(a) = 0 \}$. This is, however, only reasonable when $\mathfrak{g}$ is reductive and/or all orbits of $G$ are closed, which they rarely are.
In the non-commutative situation, this last definition of a quotient, has no meaning. The algebra $A$ may have no 1-dimensional representations at all. The solution is to define the relevant points $Simp(A)$, of the geometry $A$, defined by $A$, and then to seek out those points that should correspond to the points of the quotient $A/g$.

Consider a representation $\rho : A \rightarrow End_k(V)$, and let for every $\gamma \in g$ the derivation, $\gamma \circ \rho \in Der_k(A, \text{Hom}_k(V, V))$, map to 0 in $Ext^1_A(V, V)$. This means that the representation $\rho$, is not moved by the action of $g$. As above, it follows from, $\kappa(\gamma \circ \rho) = 0$, that there exist an element, $Q_\gamma \in \text{Hom}_k(V, V)$, the Hamiltonian, such that for all $\gamma \in g$ and all $a \in A$,

$$\rho(\delta(a)) = Q_\gamma \circ \tilde{\rho}(a) - \rho(a) \circ Q_\gamma.$$

which can be written as,

$$Q_\gamma(av) = \gamma(a)v + aQ_\gamma(v), \forall v \in V,$$

i.e. $Q$ is a $g$-connection on $V$. 

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November 1, 2016 66 / 141
Now, we define,

\[ A/g := \text{Simp}(\{\rho | \delta := \gamma \circ \rho = 0 \in \text{Ext}^1_A(V, V)\}) \]

where \text{Simp} of course picks out those representations of this sort, with no such sub-representations.

The aim of, my kind, of non-commutative algebraic geometry is to associate to any family of representations \( V \), of the algebra \( A \), the optimal extension-algebra,

\[ \eta : A \rightarrow O(V) \]

for which the family \( V \) becomes the set of simple \( O(V) \)-representations. Then \( V \) should be called: \textit{A Scheme for} \( O(V) \).
The notion of **Gauge Group** in physics, is intimately related to this non-commutative quotient structure.

We shall, in fact, show that there is an algebra $H$, representing our Cosmos, a Lie algebra, $\mathfrak{g}$ acting on $H(\sigma) := Ph(H)/([h, dh]|h \in H)$, the partially commutativization of $Ph(H)$, such that the main ingredients of the Standard Model, including representations like the Weyl and the Dirac Spinors pop up as points of the quotient,

$$H(\sigma)/\mathfrak{g},$$

suggesting that the standard Quantum Theory should be thought of as part of non-commutative algebraic geometry.
Suppose, we have identified a \( k \)-Lie algebra \( \mathfrak{g}_0 \subset \text{Der}_k(A) \), of infinitesimal automorphisms, i.e. of derivations of \( A \), a \textit{global gauge groupe}, leaving invariant the physical properties of our phenomena \( \mathbf{P} \). We would then be led to consider the \textit{quotient space} \( \mathbf{M}/\mathfrak{g}_0 \), which in our non-commutative geometry, is equivalent to restricting our representations, \( \rho : A \to \text{End}_k(V) \), to those representations \( V \) for which, \( \mathfrak{g}_0 \subset \mathfrak{g}_V \). This would then imply that the corresponding \textit{Hamiltonians}, \( Q_\gamma \) define a \( \mathfrak{g}_0 \)-connection on \( V \),

\[
Q : \mathfrak{g}_0 \longrightarrow \text{End}_k(V),
\]

such that, for all \( c \in A \), and for all \( \gamma \in \mathfrak{g}_0 \), \( \rho(\gamma(c)) = [Q_\gamma, \rho(c)] \).
Global Gauge Groups and Forces

This is usually written,

\[ \rho(\gamma(c)) = [\gamma, \rho(c)]. \]

The curvature,

\[ R(\gamma_1, \gamma_2) := [Q_{\gamma_1}, Q_{\gamma_2}] - Q_{[\gamma_1, \gamma_2]} \in \text{End}_A(V), \]

corresponds to a *global force* acting on the representation \( \rho \). These forces, *mediated* by the gauge-particles, \( \lambda \in g_0 \), will be the first to be studied in some details. Put,

\[ \text{Rep}(A, g_0) := \{ \rho \in \text{Rep}(A) | \kappa(\gamma \rho) = 0, \forall \gamma \in g_0 \} = \{ \rho \in \text{Rep}(A) | g_0 \subset g_\rho \}, \]

where \( \text{Rep}(A) \) is the category of all representations of \( C \), and notice that, in the commutative situation, if we consider the case where the gauge group \( g_0 = \text{Der}_k(A) \) then \( \text{Rep}(A, g_0) \) is the category of \( A \)-Connections, for which the space of isomorphism classes is discrete. Notice that this is the situation in the classical quantum theory, where the *Hilbert Space* is always considered as the unique state space of interest.
An object $V \in \text{Rep}(A, g_0)$ is called simple if there are no non-trivial sub-objects of $V$ in $\text{Rep}(A, g_0)$. The generalized quotient $\text{Simp}(A)/g_0$, is by definition, the set, $\text{Simp}(A : g_0)$, of iso-classes of simple objects in $\text{Rep}(A, g_0)$.

If the curvature also vanish, there is a canonical homomorphism,

$$\phi : U(g_0) \rightarrow \text{End}_k(V).$$

where $U(g_0)$ is the universal algebra of the Lie algebra $g_0$. 
Gauge Representations

In the general case let,
\[ A'(g_0) \subset \text{End}_k(A), \]
be the sub-algebra generated by \( A \) and \( g_0 \). Then we put,
\[ A(g_0) = A'(g_0)/(\gamma a - a\gamma - \gamma(a)) \]
and we have an identification between the set of \( g_0 \)-connections on \( V \), and the set of \( k \)-algebra homomorphisms,
\[ \rho_g : A(g_0) \to \text{End}_k(V), \]
since any such would respect the relation above, such that,
\[ \rho_{g_0}(\gamma a) = \rho_{g_0}(\gamma)\rho_g(a) = \rho_{g_0}(a)\rho_{g_0}(\gamma) + \rho_{g_0}(\gamma(a)). \]
In the above situation, we have the following isomorphisms,

$$\text{Simp}(A)/g_0 := \text{Simp}(A : g_0) \cong \text{Simp}(A(g_0)).$$

Notice that the commutant in $A(g_0)$, of $g_0$ is the subring,

$$A^{g_0} := \{ a \in A | \forall \gamma \in g_0, \gamma(a) = 0 \} \subset A.$$

Notice also that the commutativisation, $A(g_0)^{com}$, of $A(g_0)$ is the quotient of $A(g_0)$ by an ideal containing $\{ \gamma(a) | a \in A, \gamma \in g_0 \}$. Therefore there is a natural map,

$$A^{g_0} \rightarrow A(g_0)^{com}.$$

However this map may not be injective, so we cannot, in general, identify the rank 1 points of $\text{Simp}(A, g_0)$, with $\text{Simp}_1(A^{g_0})$. 
If $A = C$ is assumed commutative, the classical invariant theory identifies the two schemes, $\text{Spec}(C)/\mathfrak{g}_0$ and $\text{Spec}(C^{\mathfrak{g}_0})$, which in the above light, is not entirely kosher. However, if $\mathfrak{a} \subset C$ is an ideal, stable under the action of $\mathfrak{g}_0$, then since any derivation $\gamma$ of $C$ acts on the multiplicative operators $a \in C$ as $\gamma(a) = \gamma a - a \gamma$, it is clear that the quotient $C/\mathfrak{a}$ is an $(C, \mathfrak{g}_0)$-representation. Moreover,

$$C/\mathfrak{a} \in \text{Simp}_1(C)/\mathfrak{g}_0$$

if and only if the subset $\text{Simp}_1(C/\mathfrak{a}) \subset \text{Simp}_1(C)$ is the closure of a maximal integral subvariety for $\mathfrak{g}_0$. The space of such integral subvarieties is what we, previously have termed the non-commutative quotient, $\text{Spec}(C)/\mathfrak{g}_0$. 
Suppose now there is a $C$-Lie algebra $g_1$, acting $C$-linearly on those $C$-modules $V$, which we would consider of physical interest. $g_1$ would then be called a local gauge group. One may then want to know whether the given action of $g_0$ moves $g_1$ in its formal moduli as $C$-Lie algebra. If so, the action of $g_1$ would not be invariant under the the gauge transformations induced by $g_0$, and we should not consider $(\rho, g_1)$ as physically kosher. If, on the other hand, the action of $g_0$ does not move $g_1$ in its formal moduli, it should follow that there is a relation between the $g_0$-action (i.e. the connection) on $V$, and the action of $g_1$. Now, in the case $g_0 \subset \text{Der}_k(C)$, it follows from the Kodaira-Spencer map,

\[ ks : \text{Der}_k(C) \to A^1(C, g_1 : g_1), \]

see Lemma(2.3), of [6], that the following holds,
Assume $\mathfrak{g}_1$ as a $C$-module is such that $\mathfrak{g}_0 \subset \mathfrak{g}_1$, and let $c_{i,j}^k \in C$ be the structural constants of $\mathfrak{g}_1$ with respect to some $C$-basis $\{x_i\}$, and let $\delta : \mathfrak{F} \to \mathfrak{g}_1$ be a surjective morphism of a free $C$-Lie algebra $\mathfrak{F}$ onto $\mathfrak{g}_1$, mapping the generators $\xi_i$ of $\mathfrak{F}$ onto $x_i$. Let 

$$\mathfrak{F}_{i,j} = [x_i, x_j] - \sum_k c_{i,j}^k \xi_k \in \ker \delta,$$

and let $\gamma \in \text{Der}_k(C)$. Then, $ks(\gamma)$ is the element of $A^1(C, \mathfrak{g}_1 : \mathfrak{g}_1)$ determined by the element of $\text{Hom}_{\mathfrak{F}}(\ker(\delta), \mathfrak{g}_1)$, given by the map,

$$\mathfrak{F}_{i,j} \to -\sum_k \gamma(c_{i,j}^k)x_k.$$

For $ks(\gamma)$ to be 0, there must exist a $C$- derivation $D_\gamma : \mathfrak{F} \to \mathfrak{g}_1$, (a potential), such that

$$D_\gamma(\mathfrak{F}_{i,j}) = D_\gamma([x_i, x_j] - \sum_k c_{i,j}^k \xi_k) = -\sum_k \gamma(c_{i,j}^k)x_k$$

for $ks(\gamma)$ to be 0, there must exist a $C$- derivation $D_\gamma : \mathfrak{F} \to \mathfrak{g}_1$, (a potential), such that
Let now $\nabla : g_0 \to Der_k(g_1)$ be a connection on the $C$-module $g_1$. Then, $ks(\gamma) = 0$ iff there exist a $g_0$-Lie-connection of the form $\mathcal{D} = \nabla - D$ on $g_1$, i.e. a $k$-linear map,

$$\mathcal{D} : g_0 \to Der_k(g_1),$$

such that, for $\gamma \in g_0$, $c \in C$, $\kappa \in g_1$,

$$\mathcal{D}_\gamma(c \cdot \kappa) = c\mathcal{D}_\gamma(\kappa) + \gamma(c) \cdot \kappa.$$
If the curvature,

\[ R(\gamma_1, \gamma_2) := [\mathcal{D}\gamma_1, \mathcal{D}\gamma_2] - \mathcal{D}[\gamma_1, \gamma_2] \]

representing the 2. order action of \( g_0 \) on \( g_1 \), (the force, as a physicist might have said), vanish, the map,

\[ \mathcal{D} : g_0 \rightarrow \text{Der}_k(g_1), \]

would be a Lie-algebra morphism. And then this notion is related to the notion of a Lie-Cartan pair. In fact, the structure given by \( \mathcal{D} \) defines a Lie algebra structure on the sum

\[ g := g_0 \oplus g_1, \]

with Lie-products of the sum defined as the product in each Lie algebra, and the cross-products defined for, \( \gamma \in g_0, \xi \in g_1 \), as,

\[ [\gamma, \xi] = \mathcal{D}\gamma(\xi), \]
A structure like this, a Lie-Cartan pair, is now often called a Lie algebroid, and the operation of the $C$-Lie algebra $g_1$ is, when it is supposed to be physically interesting, called a local gauge group.

The category of representations $\rho : C \to \text{End}_k(V)$ with this property vis a vis the Lie algebra $g$, should be called $g$-insensitive, and the space of simple $g$-insensitives shall be denoted.

$$\text{Simp}(A)/g.$$  

This is going to be our central object of study.
Physicists have a way of classifying, or naming states, i.e. the elements of the representation vector space $V$, according to certain numbers associated to them, like spin, charge, hyperspin, etc. We find this in the situation above, as follows.

Consider the Cartan algebra $\mathfrak{h} \subset \mathfrak{g}_1$. It will operate on the above representation space $V$ as diagonal matrices, and the eigenvectors may be labeled by the corresponding eigenvalues.

Notice also that if $V_i \in \text{Rep}(C, \mathfrak{g})$, $i = 1, 2$, then it follows from (1.15) that an extension of the $C(\mathfrak{g}_0)$-module $V_1$ with $V_2$ will also sit in $\text{Rep}(C, \mathfrak{g})$.

Put, as above, $A := C(\sigma)$ where $\sigma$ is a dynamical structure, and assume given an action of the Lie algebra $\mathfrak{g}_0$ on $C$. Notice that since $\text{Ph}(\cdot)$ is a functor in the category of algebras and algebra morphisms, the action of $\mathfrak{g}_0$ extends to $\text{Ph}(C)$, but not necessarily to a dynamical system of the type $C(\sigma)$.

This will turn out to be important for our version of the Standard Model.
We fix a field $k$. All algebras occurring, will be associative $k$-algebras. A deformation of an algebra $A$ parametrized by the (commutative) $k$-algebra, $B$, is a flat $k$-algebra homomorphism, $B \to A_B$, such that there is a ”point”, i.e. a homomorphism, $\rho : B \to k$, such that $k \otimes_B A_B \cong A$.

For any associative $k$-algebra $A$, there is a formal moduli, i.e. a complete local $k$-algebra, $H(A)$, and a deformation, a versal family $\mu : H(A) \to \tilde{A}$, of associative algebras, such that for any other deformation $B \to A_B$, with $B$ a finite dimensional local $k$-algebra, there is a diagram,

\[
\begin{array}{ccc}
\mu : H(A) & \to & \tilde{A} \\
\downarrow & & \downarrow \\
B & \to & A_B
\end{array}
\]

with $A_B \cong B \otimes_{H(A)} \tilde{A}$. 
The tangent space $T^*$ of the formal moduli of an associative algebra

The tangent space of $\mathbf{H}(A)$, i.e. the dual $k$ vectorspace of $\mathfrak{m}/\mathfrak{m}^2$, where $\mathfrak{m}$ is the maximal ideal of $\mathbf{H}(A)$, is calculated as:

$$T^* = A^1(k, A; A) = \text{Hom}_F(\ker(\rho), A)/\text{Der}$$

where, $\rho : F \to A$, is a surjective homomorphism of a free $k$-algebra $F$ onto the given algebra $A$, $\text{Hom}_F$ is the set of $F$-bilinear maps, and $\text{Der} \subset \text{Hom}_F$, denotes the restrictions to $\ker(\rho)$, of the derivations $\text{Der}_k(F, A)$. 
Deformations of $U$ as associative algebras

• $A = k[x_1, x_2, x_3]$ is the commutative coordinate ring of the affine 3-space.

• $U = A/(x)^2$ is, geometrically, a thick point in affine 3-space, but

• $U$ is also a quotient of the free associative $k$-algebra, $F = k \langle x_1, x_2, x_3 \rangle$,

• Let $\rho : F \rightarrow U$ be the quotient map, then the kernel, $\ker(\rho) = (x_i x_j), \ i, j = 1, 2, 3$

A deformation of $U$ parametrized by the (commutative) $k$-algebra, $B$, is a flat $k$-algebra homomorphism, $B \rightarrow B \langle x_1, x_2, x_3 \rangle / (x_i x_j + \sum b_{i,j}^l x_l + b_{i,j}^0)$

It is easy to compute the tangent space of the formal moduli of $U$, $dim_k A^1(k, U; U) = dim_k Hom_F(\ker(\rho), U)/Der = 27$. 
The restricted formal moduli of $U$

VI No 4

Pick any two 3-vectors,

$$\bar{o} := (o_1, o_2, o_3), \bar{p} := (p_1, p_2, p_3).$$

The bi-linear homomorphisms,

$$\kappa : \ker(\rho) = (x_i x_j) \to U, \quad \kappa(x_i x_j) := o_i x_j + x_i p_j$$

represents linearly independent elements in $A^1(k, U; U)$, and span a subspace of the tangent space of the formal moduli of $U$ generating a quotient of the versal family,

$$\mu : H(U) \longrightarrow \tilde{U}$$

$$\downarrow$$

$$H(U) \longrightarrow U$$
The restricted versal family

VI No 5

• Let \( o := (o_1, o_2, o_3) \), and \( p := (p_1, p_2, p_3) \) be sets of independent coordinates, and let,

\[ H(U) =: H = k[o_1, o_2, o_3, p_1, p_2, p_3] \]

• Put, \( H := Spec(H) = A^3 \times A^3 \), and we have got a deformation of \( U \),

\[ \tilde{\rho}: H \to U := H< x_1, x_2, x_3 > /(x_i x_j - o_i x_j - x_i p_j + o_i p_j) \]

parametrized by the 6-dimensional scheme \( H \).

• Let \( o := (o_1, o_2, o_3) \), and \( p := (p_1, p_2, p_3) \) be two vectors, then

\[ U(o, p) := k< x_1, x_2, x_3 > /(x_i x_j - o_i x_j - x_i p_j + o_i p_j) \]

• Put, \( B = k[t], b^l_{i,j} = \epsilon_{i,j,l} t, \ b^0_{i,j} = \delta_{i,j} \) then the deformation of \( U \) along \( t \) for \( t \neq 0 \) is constant, equal to the Quaternions.
The Toy Model

My favourite "Toy Model", of General Relativity, and Quantum Theory is the philosophically reasonable (?) Physical Model, of an Observer and an Observed in 3-dimensional space, mathematically modelled by the Hilbert scheme $\mathcal{H}$ of length 2 sub-schemes in $\mathbb{A}^3$. Consider the diagonal, $\Delta \subset \mathbb{A}^3 \times \mathbb{A}^3 = H$, and let $\tilde{H}$ be the blow up of $H$ in $\Delta$. We find a diagram,

$$
\begin{array}{c}
\tilde{H} \longrightarrow H \supset \Delta \longleftarrow \tilde{\Delta} \subset \tilde{H} \\
\downarrow Z_2 \\
H
\end{array}
$$

where $H = \text{Hilb}^2(\mathbb{A}^3) = \tilde{H}/Z_2$. Call he generator, $P$ of $Z_2$, the Parity operator.

Coordinates for $H$: $(\lambda, \omega, \rho)$, with $\lambda \in \Delta$, $\omega$ a spherical coordinate of the blow-up of $\Delta$ in $H$, and, $\rho$ the length from $\Delta$ along the line defined by $\omega$. 
Coordinates of $\tilde{H}$

VII No 2

This is a convenient parametrisation of $\tilde{H}$. Consider, as above, for each $t \in \tilde{H}$ the length $\rho$, in $E^3$, the Euclidean space, of the vector $(o, p)$. Given a point $\lambda \in \Delta$, and a point $\xi \in E(\lambda) = \pi^{-1}(\lambda)$, of the fiber of,

$$\pi : \tilde{H} \to H,$$

at the point $\lambda$, for $o = p$. Since $E(\lambda)$ is isomorphic to $S^2$, parametrized by $\omega = (\phi, \theta)$, any element of $\tilde{H}$ is now uniquely determined in terms of the triple $t = (\lambda, \omega, \rho)$, such that $c(t) := c(o, p) = \lambda$, and such that $\xi$ is defined by the line $op$, and the action of $\theta$ keeps $\xi$ fixed. Here $\rho \geq 0$, and notice that, at the exceptional fiber, i.e. for $\rho = 0$, the momentum corresponding to $d\rho$ is not defined.
Metrics or Time

Any metric, and therefore also the notion of Time of the Model $\tilde{H}$, can be given the form,

$$ g = h_\rho(\lambda, \phi, \rho) d\rho^2 + h_\phi(\lambda, \phi, \rho) d\phi^2 + h_\lambda(\lambda, \phi, \rho) d\lambda^2, $$

where $d\phi^2$ is the natural metric in $S^2 = E(\lambda)$. Pick the simplified space, in which $\omega$ is reduced to the angle $\phi$, and the coordinates $\lambda$ is reduced to one parameter $\lambda = |\lambda|$.

$$ g = \left(\frac{\rho - h(\lambda)}{\rho}\right)^2 d\rho^2 + (\rho - h(\lambda))^2 d\phi^2 + \kappa(\lambda) d\lambda^2, $$

This correspond to considering the sub-universe of $M(BB)$, parametrized by $(\lambda, \phi, \rho)$. 
Computing the geodesics (the Force Law), taking into account that Time is the metric, we find,

\[
\frac{d^2 \lambda}{dt^2} = -\left(\frac{\rho - h(\lambda)}{\rho}\right)\left(\frac{1}{\kappa(\lambda)}\right)\left(\frac{dh}{d\lambda}\right)\left(\frac{d\rho}{dt}\right)^2
\]

\[-\left(\frac{\rho - h(\lambda)}{\rho}\right)\left(\frac{1}{\kappa(\lambda)}\right)\left(\frac{dh}{d\lambda}\right)\left(\frac{d\phi}{dt}\right)^2
\]

\[+ \left(\frac{d\log(\kappa)}{d\lambda}\right)\left(\frac{d\lambda}{dt}\right)^2\]
Moreover we find,

\[
\frac{d^2 \rho}{dt^2} = -\left( \frac{h(\lambda)}{\rho(\rho - h(\lambda))} \right) \left( \frac{d\rho}{dt} \right)^2 \\
+ \left( \frac{2}{\rho - h(\lambda)} \right) \left( \frac{dh}{d\lambda} \right) \left( \frac{d\rho}{dt} \right) \left( \frac{d\lambda}{dt} \right) + \left( \frac{\rho^2}{\rho - h(\lambda)} \right) \left( \frac{d\phi}{dt} \right)^2,
\]

\[
\frac{d^2 \phi}{dt^2} = -\frac{2}{\rho - h(\lambda)} \frac{d\rho}{dt} \frac{d\phi}{dt} + \frac{2}{\rho - h(\lambda)} \left( \frac{dh}{d\lambda} \right) \left( \frac{d\phi}{dt} \right) \left( \frac{d\lambda}{dt} \right)
\]

where \( t \), is the time parameter of the model. From these formulas we see that the Gravitation is expanding inside the Horizon and contracting outside. Conservation of mass implies, \( h(\lambda) = h_0/\lambda \). From this follows that the Horizon at the BB, i.e. for \( \lambda = 0 \), is all of space. Interpreting \( \lambda \) as Cosmological time we find a striking cosmological model, complete with Inflation and Hubble formulas, \( v = r/t \) and \( v/\sqrt{1 - v^2} = r/\lambda \).
Now let \( h(\lambda) = h \) be constant, then the geodesics have the equations,

\[
\frac{d^2 \rho}{dt^2} = -\left(\frac{h}{\rho(\rho - h)}\right)\left(\frac{d\rho}{dt}\right)^2 + \left(\frac{\rho^2}{(\rho - h)}\right)\left(\frac{d\phi}{dt}\right)^2, \\
\frac{d^2 \phi}{dt^2} = -\frac{2}{(\rho - h)}\frac{d\rho}{dt}\frac{d\phi}{dt}, \quad \text{Kepler's 2.Law.} \\
\frac{d^2 \lambda}{dt^2} = 0,
\]

The definition of time gives us,

\[
\rho^{-2}\left(\frac{d\rho}{dt}\right)^2 = (\rho - h)^{-2}K^2 - \left(\frac{d\phi}{dt}\right)^2.
\]

where, \( K^2 = (1 - \left(\frac{d\lambda}{dt}\right)^2) \), is the kinetic energy.
Put this into the first equation above, and obtain,

$$\frac{d^2 \rho}{dt^2} = -hK^2 \left( \frac{\rho}{\rho - h} \right) \frac{1}{(\rho - h)^2} + \left( \frac{\rho + h}{\rho - h} \right) \rho \left( \frac{d\phi}{dt} \right)^2.$$  

Assume now $r := \rho - h \approx \rho$, we find,

$$\frac{d^2 r}{dt^2} = -\frac{hK^2}{r^2} + r \left( \frac{d\phi}{dt} \right)^2, \text{ Kepler's 1.Law.}$$

The constant $h$, the radius of the exceptional fibre, is thus also related to mass. Recall that the Schwarzschild radius, the Einstein equivalent to $h$, is assumed to be,

$$r_s = \frac{2GM}{c^2},$$

where, $G = \text{Newton's gravitational constant}$, $M = \text{mass}$, $c = \text{speed of light}$, which here, of course, is put equal to 1.
For every \((o, p) \in \mathbb{A}^3 \times \mathbb{A}^3 = \mathcal{H}\), there is an associative 4-dimensional \(k\)-algebra \(U(o, p)\), the fibre of \(U\) at \((o, p)\). Moreover, if \(o = p\), then \(U(o, p) \simeq U\). We obtain the extended diagram,

\[
\begin{array}{cccccc}
\tilde{H} & \rightarrow & H & \leftarrow & \Delta & \leftarrow \tilde{\Delta} \subset \tilde{H} \\
\downarrow & & \uparrow & & \uparrow & \\
\mathcal{H} & \leftarrow & U & \leftarrow & U & \\
\end{array}
\]

This follows from the following elementary relations: For \((o, p) \in \mathcal{H}\), for every \(c \in \mathbb{A}^3\) and for any non-zero \(\kappa \in k\), we have,

- \(U(\kappa o, \kappa p) \simeq U(o, p)\)
- \(U(o, p) \simeq U(o - c, p - c)\)
- \(U(-p, -o) \simeq U(o, p)\)
A Universal Gauge Group

Consider the bundle of Lie algebras, defined on $H$ by,

- $\mathfrak{g} := \text{Der}_H(U)$
- $\mathfrak{g}(o, p) = \text{Der}_{k(o, p)}(U(o, p)), (o, p) \in H$.

Any element $\delta \in \mathfrak{g}(o, p)$ must have the form,

$$
\delta(x_i) = \delta_i^0 + \delta_i^1 x_1 + \delta_i^2 x_2 + \delta_i^3 x_3.
$$

Consider the 4-vectors,

$$
\delta_i = (\delta_i^0, \delta_i^1, \delta_i^2, \delta_i^3), \quad \bar{o} = (1, o_1, o_2, o_3), \quad \bar{p} = (1, p_1, p_2, p_3)
$$

**Theorem**

- $\delta \in \mathfrak{g}(o, p)$ if and only if $\delta_i \cdot \bar{o} = \delta_i \cdot \bar{p} = 0$,
- If $o \neq p$, then, $\mathfrak{g}(o, p) \simeq \begin{pmatrix} 0 & * & * \\ 0 & * & * \\ 0 & * & * \end{pmatrix}$
- $\text{rad}(\mathfrak{g}) = \{u, r_1, r_2\}, \quad \mathfrak{g}/\text{rad} \simeq \mathfrak{sl}(2) = \{h, e, f\} \subset \mathfrak{g}$
- $h \in \mathfrak{h} \subset \mathfrak{g}$; the generator of the Cartan algebra.
Canonical Basis for $\Theta_H$

VIII No 3

Notice also that, in this case, the unique 0 (resp light)-velocity tangent line at the point $t_0 = (o, p)$, $o = (0, 0, 0)$, $p = (1, 0, 0)$, killed by $g$, is represented by,

- $d_3 := ((1, 0, 0), (1, 0, 0))$, the unique 0-velocity, resp.
- $c_3 := ((1, 0, 0), (-1, 0, 0))$, the unique light-velocity.

Let, $d_1 := ((0, 1 , 0), (0, 1, 0))$, $d_2 := ((0, 0, 1), (0, 0, 1))$ and let,
$c_1 := ((0, 1 , 0), (0, -1, 0))$, $c_2 := ((0, 0, 1), (0, 0, -1))$.

- Then $\{c_1, c_2, c_3, d_1, d_2, d_3\}$ is a basis for the tangent space $\Theta_{t_0}$, $\{c_1, c_2, c_3\}$ for $\tilde{c}_{t_0}$, and $\{d_1, d_2, d_3\}$ for $\tilde{\Delta}_{t_0}$.

Put, $B_o := \langle c_1, d_1 \rangle$ and $B_p := \langle c_2, d_2 \rangle$. 
Denote by $\tilde{\Delta} \subset \tilde{\Theta}_{\tilde{H}}$ the 3-dimensional distribution, generated by the translations $\{(o, p) \rightarrow (o + c, p + c), \ c \in A^3\}$, and complexify all bundles. Introduce a Riemannian metric $g$ on the space $H$, and see that we have the following structures uniquely defined,

- $\tilde{\Delta}_C$, with the action of $su(3)$
- $\Theta_{\tilde{H}} \simeq B_o \oplus B_p \oplus A_{o,p} \simeq \tilde{\Delta} \oplus \tilde{c}$; resp. 0- and light-velocities.
- There exists a canonical action of $g_C$ on $\Theta_{\tilde{H},C}$, respecting the decomposition, such that,
  - $g(o, p)$ acts on the tangent space $T_H, (o, p) = T_{A^3,o} \times T_{A^3,p}$ killing the vector $p - o$, in both factors.

The canonical action of these principal Lie bundles on the complexified tangent bundle $\Theta_H$ ”determins all Fields”.
Action of the Gauge group

VIII No 5

The generators, $h, e, f \in \mathfrak{sl}(2) \subset \mathfrak{g}$ act, in the above basis, like,

\[
h = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]

\[
e = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]
Action of the Gauge group

VIII No 6

\[ f = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \]

The generators, \( u, r_1, r_2 \in \text{rad}(g) \) act, in the above basis, like,

\[ u = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \]
Action of the Gauge group

$\mathbf{VIII \ No \ 7}$

\[ r_1 = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
\end{pmatrix} . \]

\[ r_2 = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
\end{pmatrix} . \]
We observe here that the rotations (spin) around the 3 different axes in $\tilde{c}$, respectively in $\tilde{\Delta}$, are given by, $(e - f), (e - r_1), (f - r_2)$, about the axe defined by $(op) = c_3, c_2, c_1$, respectively $(op) = d_3, d_2, d_1$.

Recall that $\tilde{\Delta} \subset \Theta_{\tilde{H}}$, is the sub-bundle defined, at the point $t$ as the space of tangents of the form $(\xi, \xi)$. Given a metric on $\tilde{H}$, we may look at the action of $su(3)$ on $\tilde{\Delta} \otimes \mathbb{C}$. Knowing that $\Theta = \tilde{c} \oplus \tilde{\Delta}$, $su(3)$ acts in the obvious way on the lower right corner, like

$$
\gamma = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & * & * & * & * \\
0 & 0 & 0 & * & * & * & * \\
0 & 0 & 0 & * & * & * & * \\
0 & 0 & 0 & * & * & * & * \\
\end{pmatrix}.
$$
Action of the Gauge group

VIII No 9

Since the 0-velocity direction defined at \((o, p)\), is \((o - p, o - p)\), which here is \(d_3\), we may in an essential unique way decompose the Cartan subalgebra \(\mathfrak{h} \subset \mathfrak{su}(3)\), into the Cartan subalgebra \(\mathfrak{h}_1\), for the \(\mathfrak{su}(2)\)-component leaving \(\delta_3\) invariant, and the part \(\mathfrak{h}_2 \subset \mathfrak{h}\) perpendicular, in the Killing metric, to \(\mathfrak{h}_1\).

\[
\mathfrak{h}_1 = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1/2 & 0 & 0 \\
0 & 0 & 0 & 0 & -1/2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}, \quad \mathfrak{h}_2 = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1/3 & 0 & 0 \\
0 & 0 & 0 & 0 & 1/3 & 0 \\
0 & 0 & 0 & 0 & 0 & -2/3 \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix},
\]

Classically one denotes the other 6 base elements of \(\mathfrak{su}(3) \otimes \mathbb{C}\), as, \(e^i_\pm, i = 1, 2, 3\).
The restriction to $\tilde{\Delta}$ of these operators, in the basis $\{d_1, d_2, d_3\}$ are given by,

$$e_+^1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, e_+^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, e_+^3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$  

and their duals,

$$e_-^1 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, e_-^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, e_-^3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$
We need the commutators, \([h_1, h_2] = 0\), and,

\[ [h_1, e^1_\pm] = \pm e^i_\pm, [h_1, e^2_\pm] = \mp 1/2e^2_\pm, [h_1, e^3_\pm] = \pm 1/2e^3_\pm \]

together with the following ones,

\[ [h_2, e^1_\pm] = 0, [h_2, e^2_\pm] = \pm e^2_\pm, [h_2, e^3_\pm] = \pm e^3_\pm \]

Notice, for later use, that the only non-zero products of the \(e^i_\pm\) are,

\[ e^1_+ \cdot e^2_+ = e^3_+, e^2_- \cdot e^1_- = e^3_- . \]
Together these formulas show that the quotients of the \( g^* \)-representation \( \Theta_{\tilde{H}} \) are the following,

- \( \tilde{c} \) and therefore also the *photon* \( \{c_1, c_2\} \) and a singleton, \( \{c_3\} \), both simple.
- \( \tilde{\Delta} \) and therefore the *electron* \( \{d_1, d_2\} \) and a singleton, \( \{d_3\} \), both simple.
- Weyl spinors, \( B_o, B_p \), and Dirac spinors, \( B_o \oplus B_p \)

The non-trivial simple quotients of the \( su(3) \)-representation \( \Theta_{\tilde{H}} \) are reduced to,

- The quarks, \( \tilde{\Delta} \)
We observe that the action of $\phi \in u(1)$ is, $\exp(i\phi \cdot (e - f))$, on the (transverse) light-wave is given by,

$$A(\phi) = \begin{pmatrix}
cos\phi & -\sin\phi & 0 & 0 & 0 & 0 \\
\sin\phi & \cos\phi & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}.$$  

It is clear that this, together with the formulas above give good reasons to believe that there is a relation between this model, and the Standard Model (and so also to the 8-fold way of Gell-Mann). Moreover, here all ingredients are universally given by the information contained in the singularity U, the Big Bang, in my tapping. The choice of metric depends, however, on the nature of what I have called the Furniture of the model.
It is now easy to see that the Pauli matrices are found as follows,

\[
\sigma^1 = e + f = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},
\]

\[
\sigma^2 = -ie + if = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix},
\]

\[
\sigma^3 = h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
\]

Moreover, the parity operator \( P \), the generator \( \gamma \) of the symmetry group \( \mathbb{Z}_2 \), operating on \( \tilde{H} \), acts on \( \tilde{c} \), as multiplication by \((-1)\), see WS. Therefore, it maps the basis \( \{(c_1 + d_1), (c_2 + d_2)\} \) into \( \{(-c_1 + d_1), (-c_2 + d_2)\} \). Consequently, we have an isomorphism,

\[
P : B_o \rightarrow B_p,
\]
Chirality, in the physicists language, is explained as follows. The morphism $P$, extended to $B_o \oplus B_p$, in the basis chosen above, is given by the matrix,

$$P = \begin{pmatrix} 0 & id \\ id & 0 \end{pmatrix},$$

which turns left-handedness to right-handedness, with respect to the direction $(o, p)$, resp. $(p, o)$. 

Chirality
The Dirac Matrices

IX No 5

This shows that it is meaningful to consider the representations given by
the Dirac matrices,

\[ \gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \gamma^k = \begin{pmatrix} 0 & \sigma^k \\ -\sigma^k & 0 \end{pmatrix}, \quad k = 1, 2, 3. \]

as well as the new operators,

\[ \gamma^{k+3} = \begin{pmatrix} \sigma^k & 0 \\ 0 & -\sigma^k \end{pmatrix}, \quad k = 1, 2, 3, \]

acting on, \( B_o \oplus B_p \), such that,

\[ \forall p \neq q, \quad \gamma^p \gamma^q = -\gamma^q \gamma^p, \quad \gamma^p \gamma^p = 1, \quad p, q = 1, 2, 3, 4, 5, 6. \]
Fix a metric $g$, for $\tilde{H}$. Consider a representation, 

$$\rho_0 : \tilde{H} \rightarrow \text{End}_k(\mathcal{B}), \rho \in \tilde{H}/\bar{g}$$

and an extension $\rho_\psi : \tilde{H}(\sigma_g) \rightarrow \text{End}_k(\mathcal{B})$, as a momentum of $\rho_0$, in the tangent direction $\psi \in T_{\rho_0} = \text{End}_{\tilde{H}}(\mathcal{B})^n$, then

$$\rho_\psi(dt_i) = \xi_i + \psi_i, \ \psi_i \in \text{End}_{\tilde{H}}(\mathcal{B})$$

corresponds to a $\rho_0$-derivation,

$$\eta \in \text{Der}_k(\tilde{H}, \text{End}_k(\mathcal{B})), \text{ mapping to: } 0 \in \text{Ext}_{\tilde{H}}^1(\mathcal{B}, \mathcal{B})$$

Then $\text{ad}(\rho_\psi(g - T)) = \text{ad}(Q_h + [\psi] + Q_v)$ where

$$[\psi] := \sum_i \psi_i \delta_{t_i}, \ Q_h = 1/2 \sum_{i,j} g_{i,j} \xi_i \xi_j, \ Q_v = 1/2 \sum g_{i,j} \psi_i \psi_j$$
The Generic Time-Action

Time is then given by,
\[ \delta = \text{ad}(\rho_\psi(g - T)) = \text{ad}(Q_h + [\psi] + Q_v) \]
meaning that,
\[ \rho_\psi(d^{n+1}t_i) = [Q_h + [\psi] + Q_v, \rho_\psi(d^n t_i)]. \]
This takes care of the complete time development of an operator in \( V \), as well as a state vector \( \phi \in \mathcal{B} \).
The second order time-development is also given in terms of the Force Law valid in \( \tilde{(H)(\sigma_g)} \),
\[ d^2 t_i = -\sum_{p,q} \Gamma^i_{p,q} dt_p dt_q - \sum_{p,q} g_{p,q} F_{i,p} dt_q + 1/2 \sum_{l,p,q} g_{p,q}[F_{i,q}, dt_p] \\
+ 1/2 \sum_{l,p,q} g_{p,q}[dt_p, (\Gamma^i_{i,q} - \Gamma^{q,i}_{i,q})] dt_l + [dt_i, T]. \text{ Maxwell} \]
The Generic Time-Action

Given the gauge group \( \mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \), there is a particular tangent direction, \( \mathfrak{v} = \{ \mathfrak{v}_i \} \), with \( \mathfrak{v}_i = \sum_j \gamma_j g^{j,i} \), and \( \gamma_i \in \mathfrak{g}_1 \), ”normal” to \( C \) in \( V \), naturally related to a derivation, \( \mathcal{D} : \mathfrak{g}_0 \to \text{Der}_k(\mathfrak{g}_1) \). Put,

\[
[\mathfrak{v}] := \sum_i \gamma_i \nabla \xi_i, \quad Q_h = 1/2 \sum g_{i,j} \xi_i \xi_j, \quad Q_v = 1/2 \sum g_{i,j} \nu_i \nu_j
\]

then with this choice of momentum, the time operator becomes,

\[
\delta = ad(Q_h + [\mathfrak{v}] + Q_v) \in \text{Der}(\text{End}_k(\tilde{V}))
\]

with the corresponding first order time-action in the state-space being defined by,

\[
\delta = [\mathfrak{v}] + Q \in \text{End}_k(V).
\]

and the curvature equal to,

\[
F_{i,j} = ad([\mathfrak{v}_i, \mathfrak{v}_j]) \quad \text{Bloch}
\]
XI Time and Furniture

No 1

Let $\mathbf{M}$ be the space of (isomorphism classes of) metrics on $\tilde{H}$. For every point $g \in \mathbf{M}$ consider the diagram,

$$\begin{array}{c}
Ph(\tilde{H}) \xrightarrow{d} Ph^2(\tilde{H}) \xrightarrow{\rho_1} Ph^3(\tilde{H}) \ldots \xrightarrow{\rho_3} Ph^\infty(\tilde{H}) \circ \delta \\
\tilde{H} \xrightarrow{i} \tilde{H}(\sigma_g) \xrightarrow{\rho} \text{End}_k(\mathcal{B})
\end{array}$$

Where $\rho_1$ is the representation of $Ph(\tilde{H})$ induced by the representation $\rho$ of $\tilde{H}(\sigma_g)$. Any extension $\rho_2$ of $\rho_1$ correspond to a tangent of $\rho_1$, therefore to an acceleration of $\rho_0 := i \circ \rho$. Let, $T_{\mathbf{M},g}$ be the tangent space to $\mathbf{M}$, at $g$, i.e. $T_{\mathbf{M},g} = \{(h_{i,j}), \ h_{i,j} = h_{j,i} \in C\}$. This implies that,

$$T_{\mathbf{M},g} \subset Ext^1_{Ph(\tilde{H})}(\mathcal{B}, \mathcal{B})$$

More precisely, any first order deformation of the $Ph(\tilde{H})$-representation $(\rho_1, \mathcal{B})$, induces a deformation of the metric $g$. 
Let us pick a basis, \( \{ \gamma_i \} \), \( i = 1, \ldots, 6. \), in the finite dimensional Lie algebra \( g \), and consider the operator,

\[
[\delta] := \sum_{p,q} g^{p,q} \gamma_p \delta t_q : \text{End}_k(\Theta) \to \text{End}_k(\Theta).
\]

We may consider a family of representations,

\[
\tilde{\rho} : P\text{h}(\tilde{H}) \to \text{End}_{\tilde{H}}(\Theta)
\]

defined by,

\[
\tilde{\rho}(t_i) = t_i, \quad \tilde{\rho}(dt_i) = \sum_j g^{i,j} \gamma_j.
\]
Dynamics

The dynamics comes out as,

\[ \tilde{\rho}(d(a)) = [\delta](\tilde{\rho}(a)) + [Q, \tilde{\rho}(a)]. \]

where the Hamiltonian, \( Q := \tilde{\rho}(g - T) \). Moreover, using the definition (1) above, we find after a short computation,

\[ \tilde{\rho}(d^2 t_i) = [\delta](\tilde{\rho}(dt_i)) + [Q, \tilde{\rho}(dt_i)]. \]

In fact this follows from the well known formula,

\[ \delta_{ta}g^{i,r} = -(\sum_l g^{i,l}\Gamma_{q,l}^r + g^{i,l}\Gamma_{q,l}^r), \]

using the physically equivalent to the Force Law of \( \tilde{H}(\sigma_g) \), given by the formula (1) mentioned above,

\[ d^2 t_i = -1/2 \sum_{p,q}(\Gamma_{p,q}^i + \bar{\Gamma}_{q,p}^i)dt_pdt_q \quad (5) \]
\[ + 1/2 \sum_{p,q}g_{p,q}(R_{p,i}dt_q + dt_pR_{q,i}) + [dt_i, T] \quad (6) \]
The energy-mass-equation looks very much like an equations of Dirac type, and in fact, assuming the metric is trivial, and having picked constant gauge fields \( \{ \gamma_i \} \) properly, we find that \( Q = \sum_i \gamma_i^2 \) is the identity, and then Theorem (1.9) actually produces the Weyl equation, for Weyl-spinors,

\[
[\delta](\xi) := \sum_i \sigma^i \delta t_i(\xi) = E \xi, \; \xi \in B_o,
\]

and the Dirac equation, for Dirac-spinors,

\[
[\delta](\xi) := \sum_i \gamma^i \delta t_i(\xi) = E \xi, \; \xi \in B_o, \oplus B_p.
\]

But, beware, the parity operator,

\[
P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},
\]

does not preserve the Weyl spinors.
Quantum Klein-Gordon

Put,

\[
\kappa := \sum_{p,q=1,2,3} g^{p,q} \gamma^p \delta_{t_q}, \quad \mu := \sum_{p,q=4,5,6} g^{p,q} \gamma^p \delta_{t_q},
\]

and notice that since, by definition \(\tilde{c}\) and \(\tilde{\Delta}\) are normal, we have

\[
\sum_p g^{p,q} g^{p,q'} = 0, \text{ for } q \in \tilde{c}, q' \in \tilde{\Delta}.
\]

Then we find,

\[
\{\kappa, \mu\} = 0, \quad (\kappa + \mu)^2 = \sum_{i=1}^{6} \xi_i^2, \quad \xi_i := \sum_{q=1}^{6} g^{i,q} \delta_{t_q}.
\]

proving that, \([\kappa^2, \mu^2] = 0\), and \((\kappa + \mu)^2 = \kappa^2 + \mu^2\). So if \(\omega\) is a nonsingular eigenvector for both \(\kappa\) and \(\mu\), then,

\[
\kappa(\omega) = \sum_{p,q=1,2,3} g^{p,q} \gamma^p \delta_{t_q}(\omega) = K\omega, \quad \mu(\omega) = \sum_{p,q=4,5,6} g^{p,q} \gamma^p \delta_{t_q}(\omega) = m\omega.
\]
Consequently,\
\[
\sum_{i=1}^{6} \xi_i^2(\omega) = E^2 \omega = (K^2 + m^2)\omega.
\]

This is, in our model, the combined Einstein, Klein-Gordon and Dirac equation. Notice that,\
\[
\sum_{p} g_{i,j} \xi_i \xi_j = \sum_{p,q} g^{p,q} \delta_{t_p} \delta_{t_q},
\]
is the Laplace-Beltrami operator.
Summarizing, we observe that our Toy Model has furnished,

- a Field Theory (including a Y-M model) for connections on $\tilde{H}$-bundles $\mathcal{B}$, i.e. for representations,

  $$\rho : \text{Ph}(\tilde{H})/(\sigma_g) =: \tilde{H}(\sigma_g) \to \text{End}_k(\mathcal{B})$$

  with Dirac derivation for representations, $[\delta] = 0$ and Hamiltonian $Q = \rho(g - T)$.

- a Quantum Field Theory for gauge fields, i.e. for representations,

  $$\rho : \text{Ph}(\tilde{H})/([dt_i, t_j]) =: \text{Ph}(\tilde{H})_{\text{part}} \to \text{End}_{\tilde{H}}(\Theta_{\tilde{H}})$$

  with Dirac derivation $[\delta] = \sum_{p,q} g^{p,q} \gamma_p \delta_{t_q}$ and Hamiltonian $Q := \tilde{\rho}(g - T)$.
Both of these models are compatible with,

- a General Relativistic model, for the scheme $\text{Spec}(\mathcal{P}h(\tilde{H})_{com})$, i.e. for representations,

$$\rho : \tilde{H}(\sigma_0) := \mathcal{P}h(\tilde{H})_{com} \rightarrow \text{End}_{\tilde{H}(\sigma_0)}(\tilde{H}(\sigma_0)) \simeq \tilde{H}(\sigma_0)$$

with Dirac derivation, $[\delta] = \sum (-\Gamma^i)\delta_\xi^i + \xi_i\delta_t$, and Hamiltonian $Q = 0$, 
Exterior Forces

The exterior force fields in physics are in our Field Theory Model representations like $\rho : \tilde{H}(\sigma_g) \to \text{End}_k(\mathcal{B})$. The dynamics of $\rho$ is given by the time derivative $\dot{\rho}$ of $\rho$, being defined by,

$$
\dot{\rho}(dt_i) = \rho(d^2 t_i) = - \sum_{p,q} \Gamma^i_{p,q} \nabla_p \nabla_q - 1/2 \sum_{p,q} g_{p,q}(F_{i,p} \nabla_q + \nabla_p F_{i,q}) + 1/2 \sum_{l,p,q} g_{p,q}[\nabla_p, (\Gamma^i_{l,q} - \Gamma^q_{l,i})] \nabla_l + [\nabla_i, T].
$$

For trivial metric (or for the Minkowski metric), the equation reduces to,

$$
\dot{\rho}(dt_i) = \rho(d^2 t_i) = - \sum_p (F_{i,p} \nabla_p + 1/2 \delta_{tp}(F_{i,p})),
$$

and the Hamiltonian $Q = \text{ad}(g - T) = \Delta$, reduces to the Laplace differential operator.
In this talk I shall sketch a mathematical model for a Big Bang scenario, based on relatively simple non commutative algebraic geometry. Let $U$ be the 4-dimensional real affine algebra of a point with a 3-dimensional tangent space, in $\mathbb{A}_\mathbb{R}^3$. The versal family of deformation of $U$, as associative $\mathbb{R}$-algebras, contains, in a natural way, the 6-dimensional moduli space, $H := \text{Hilb}^2(\mathbb{R}^3)$, which is my Toy Model for the Big Bang-scenario in cosmology, introduced in (WS).

The study of the corresponding family of derivations of the 4-dimensional algebras, leads to a natural way of introducing an action of the gauge Lie algebras of the Standard Model, in $\Theta_H$.

Introducing the notion of quotient spaces in non-commutative algebraic geometry, we obtain a geometry that seems to fit well with the set-up of the Standard Model.

These subjects are all treated within the set-up of (WS).
Standard Model

Fixing the Riemannian metric $g$ on the space $H$, we have the following structures uniquely defined,

- **Fermions**: $\tilde{\Delta}_C$, with the action of $\mathfrak{su}(3)$
- **Bosons**: $g_C$, with the regular representation of $g$

**Ingredients of SM**

- $\Theta_H \cong B_o \oplus B_p \oplus A_{o,p} \cong \tilde{\Delta} \oplus \tilde{c}$; resp. 0- and light-velocities.
- $u(1) \cong B_o \cong B_p$.
- $\mathfrak{sl}(2) \subset \mathfrak{g} = \text{Der}_H(U)$
- $\mathfrak{su}(3) := \mathfrak{su}_g(\tilde{\Delta}_C)$

The canonical action of these principal Lie bundles on the complexified tangent bundle $\Theta_H$ gives us all Fields.
Choose a point $BB \in \Delta$, considered as the point $(0, 0) \in \mathbb{A}^3 \times \mathbb{A}^3$, and treat $H$ as a vector space. Recall that $\Delta$, and therefore the point $BB$, are not contained in $H$. Let, moreover $\mathcal{M}(BB)$ be the 4-dimensional sub-scheme of $H$, consisting of the set \{$(o, p)$\} where $o$ and $p$ are linear dependent. Consider the diagram,

\[
\begin{array}{ccc}
\mathfrak{g} & \rightarrow^{\kappa} & \mathcal{M}(BB) \\
\downarrow & & \downarrow \\
H & \rightarrow & \Theta_H \leftarrow \tilde{\Delta}
\end{array}
\]

**Theorem**

- There exists an $H$-linear map $\kappa$, defined by,
  \[
  \kappa(\delta) = \sum_{i=1}^{3} \delta_i^0 \left( \frac{\partial}{\partial o_i} + \frac{\partial}{\partial p_i} \right)
  \]
- If $o$ and $p$, are linear independent, i.e. if $(o, p)$ is not in $\mathcal{M}(BB)$, then
  \[
  \mathfrak{sl}(2) = \mathfrak{g}/\ker(\kappa) \cong \tilde{\Delta}, \text{ as } H\text{-modules.}
  \]
Define: \( Q_i : \mathfrak{sl}(2)_C \oplus \tilde{\Delta} \rightarrow \mathfrak{sl}(2)_C \oplus \tilde{\Delta}, \) \( Q_1 = (\kappa, 0), \) \( Q_2 = (0, \kappa^{-1}) \) Then we have a \( \text{N}=1, \text{ SUSY} \) with \( \{Q_1, Q_2\} = 1, \) \( [Q_i, P_j] = 0 \)

The graded Lie algebra \( L \subset End_H(\mathfrak{sl}(2)_C \oplus \tilde{\Delta}), \) generated by \( Q_i \) and \( \mathfrak{su}(3) \) and \( g, \) give us a SUSY-like duality between the Bosons and the Fermions of the model.

And one recognises the ingredients of the Standard Model, in the following canonical representations of, \( \mathfrak{sl}(2) \subset g = \text{Der}_H(U), \) and \( \mathfrak{su}(3) := \mathfrak{su}_g(\tilde{\Delta}_C), \) respectively,

- \( B_o \overset{P}{\simeq} B_p \subset \Theta_{\tilde{H}}; \) Weyl spinors.
- \( B_o \oplus B_p \subset \Theta_{\tilde{H}}; \) Dirac spinors.
- \( \tilde{\Delta} \subset \Theta_{\tilde{H}}; \) Gell-Mann, 8-fold way, colour.
- \( d_3 \in \tilde{\Delta} \cap A_{o,p}; \) up-quark
- \( d_1, d_2 \in \tilde{\Delta}, < d_1, d_2 > \perp d_3; \) left-right down-quark.
Consider the list of markers, or labels, $h_1$, $h_2$, $l_3^{\pm} = 3/4h_2 \pm 1/2h_1$, and $Y_W = 1/2h_2 \pm h_1$.

<table>
<thead>
<tr>
<th>Markers</th>
<th>$1/2 \cdot h$</th>
<th>$h_1$</th>
<th>$h_2$</th>
<th>$l_3$</th>
<th>$Y_W$</th>
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</thead>
<tbody>
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<td>$0$</td>
<td>$-2/3$</td>
</tr>
<tr>
<td>$d_2$</td>
<td>$-1/2$</td>
<td>$-1/2$</td>
<td>$-1/3$</td>
<td>$-1/2$</td>
<td>$1/3$</td>
</tr>
<tr>
<td>$d_3$</td>
<td>$0$</td>
<td>$0$</td>
<td>$2/3$</td>
<td>$1/2$</td>
<td>$1/3$</td>
</tr>
</tbody>
</table>

$p_1 = d_1 d_3 d_3$  

|          | $1/2$ | $1/2$ | $1$  | $1$  | $0$   |

$p_2 = d_2 d_3 d_3$  

<p>|          | $-1/2$| $-1/2$| $1$  | $1$  | $0$   |</p>
<table>
<thead>
<tr>
<th>Markers</th>
<th>$1/2 \cdot h$</th>
<th>$\nu_1$</th>
<th>$\nu_2$</th>
<th>$l_3$</th>
<th>$Y_W$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n_1 = d_1 d_1 d_3$</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>$1/2$</td>
<td>$-1$</td>
</tr>
<tr>
<td>$n_2 = d_2 d_2 d_3$</td>
<td>$-1$</td>
<td>$-1$</td>
<td>0</td>
<td>$-1/2$</td>
<td>$1$</td>
</tr>
<tr>
<td>$n_{1,2} = d_1 d_2 d_3$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$e_L = d_1 d_1 d_1$</td>
<td>$3/2$</td>
<td>$3/2$</td>
<td>$-1$</td>
<td>0</td>
<td>$-2$</td>
</tr>
<tr>
<td>$e_R = d_1 d_1 d_2$</td>
<td>$1/2$</td>
<td>$1/2$</td>
<td>$-1$</td>
<td>$-1/2$</td>
<td>$-1$</td>
</tr>
</tbody>
</table>
To go from left to right handedness comes out by just exchanging $d_1$ and $d_2$. Here one may see that, $e_L + \nu_L = e_R$, and one easily find reasons for the decays,

$$n \rightarrow p + e + \nu_e, \quad p^+ \rightarrow e^+ + \pi^0 \rightarrow e^+ + 2\gamma, \quad d_1 \rightarrow d_3 + w^-, \quad p^+ = e^- \rightarrow n + \nu_e.$$ 

Notice also that we may, in an obvious way, identify $\nu_L$ with the element $e + f$ in one coordinate, swapping $d_1, d_2$. 

$$W_1 = d_1 d_2 d_2$$
$$\nu_L = d_1^{-1} d_2$$
Further consequences of the Force Laws, Maxwell

It is not difficult to see that the evolution equation,

$$
\dot{\rho}(dt_i) = \rho(d^2t_i) = -\sum_p (F_{i,p}\rho(d_p) + 1/2\nabla\delta_s(F_{i,p})).
$$

restricted to the subspace $M(B)$ contains Maxwells equation. Interpreting,

$$
C = M(B), \quad v_p = \dot{t}_p := \nabla_p = \nabla\delta_p(A_p)(t) \in \text{End}_{M(B)}(\Theta), \quad \dot{v}_p := E \cdot \rho(d^2t_i)(t)
$$

we find that the observed electromagnetic fact, i.e. the Lorentz force law,

$$
\dot{v}_i = -\sum_p F_{i,p}v_p
$$

implies that the tangent to $\dot{\rho}$, given by the potential

$$
\{(1/2 \sum_p \nabla\delta_p(F_{i,p})), i = 1, \ldots, 4\},
$$

must be zero, (or as physicists might say, gauge). This means, that there exist $\Phi \in \text{End}_{M(B)}(\Theta_{M(B)})$, such that,

$$
\{1/2 \sum_p \nabla\delta_p(F_{i,p}), i = 1, \ldots, 4\} = \{\nabla\delta_i(\Phi) + [1/2 \sum_p \nabla\delta_p(F_{i,p})), \Phi], i = 1, \ldots, 4\}
$$
Further consequences of the Force Laws, Maxwell

Assuming that the electromagnetic field potentials, $A_i$ are scalars, the curvatures, $F_{i,p} = [\nabla_{\delta_p}, A_i] - [\nabla_{\delta_i}, A_p]$ are also scalars, so all commutators $[1/2 \sum_p [\nabla_{\delta_p}, F_{i,p}], \Phi]$ vanish, implying,

$$1/2 \sum_p \nabla_{\delta_p} (F_{i,p}) = \nabla_{\delta_i} (\Phi), \ i = 1, \ldots 4.$$

Now, let us choose coordinates such that $t_4 =: t_0 = \lambda$ (the proper time for physicists) is the only 0-velocity coordinate of $M(B)$, and use the classical notations, Simple computation then shows, where $\nabla_s$ is the (light-) space part of $\nabla$. From this, and from the vanishing of the tangent to $\dot{\rho}$, i.e. the equation above, the Maxwell equations follows, with $\Phi = \nabla \cdot A$, electric current $\mathcal{J} = \nabla_s (\Phi)$, and electric charge $\rho = \nabla_{t_0} (\Phi)$. Together, $(\rho, \mathcal{J})$ is a 0-tangent to the representation $\rho$, the first component in $\tilde{c}$, i.e. in space (or light) direction, and the second in $\tilde{\Delta}$, or in 0-velocity direction.
Conclusions and Apology

The physical interpretation of the mathematical models presented here, and in the literature referred to in the next slide, should not, at the moment, be considered as a serious proposition for a new physics. With respect to the Big Bang, for example, I think any existing model, must be approached with utmost care. However, I think the idea of defining time as a metric on the moduli space of the iso-classes of the mathematical models used to specify/define the physical phenomenon of interest, may be of some interest. I therefore proposed to name the study of the geometry of reasonable moduli spaces of this kind: **Geometry of Time-Spaces.**
Assuming given a dynamical structure and a Dirac derivation $\delta$, we have associated to any representation $\rho : A(\sigma) \to \text{Hom}_k(V)$ its unique extension, or force acting on $V$ via,

$$\psi(a) : V \to V, \psi = \delta \circ \rho.$$ 

For gauge fields, given as above by a representation of $\tilde{H}$ of the form $\mathcal{B}$, where $\mathcal{B}$ is a representation of the principal Lie-bundle $g \oplus \mathfrak{su}(3)$ defined on $\tilde{H}$, we have studied self-interactions of representations,

$$\rho : \text{Ph}(\tilde{H})(\sigma_g) \to \text{End}_{\tilde{H}}(\mathcal{B}),$$

and the force fields created by the derivations $\tilde{H} \to \text{End}_k(\mathcal{B})$ induced by $\rho$. However, there are, in this case, other forces to be considered. The $\tilde{H}$ module $\mathcal{B}$ may be a free, and $\mathfrak{sl}(2)$ and $\mathfrak{su}(3)$ being semi-simple, do not have nontrivial extensions, but the $\text{rad}(g)$ part of $g = \mathfrak{sl}(2) \oplus \text{rad}(g)$, has non-trivial extension modules, and as we shall see, this give us a model for the Weak Interaction.
Consider the action of $g = \mathfrak{sl}(2) \oplus \text{rad}(g)$, on $\tilde{c}$ and $\tilde{\Delta}$ then there is an isomorphism,

$$\text{Ext}^1_{H(g)}(\Theta, \Theta) \simeq \text{Ext}^1_g(\Theta, \Theta) \simeq \text{Ext}^1_{\text{rad}(g)}(\Theta, \Theta)$$

and,

$$\text{Ext}^1_{\text{rad}(g)}(\Theta, \Theta) = \text{Ext}^1_{\text{rad}(g)}(\tilde{c}, \tilde{c}) \oplus \text{Ext}^1_{\text{rad}(g)}(\tilde{\Delta}, \tilde{\Delta}) \oplus \text{Ext}^1_{\text{rad}(g)}(\tilde{c}, \tilde{\Delta}) \oplus \text{Ext}^1_{\text{rad}(g)}(\tilde{\Delta}, \tilde{c})$$

Moreover,

$$\text{Ext}^1_{\text{rad}(g)}(\tilde{c}, \tilde{c}) \simeq \text{Ext}^1_{\text{rad}(g)}(\tilde{\Delta}, \tilde{\Delta})$$

and, any $\psi \in \text{Ext}^1_{\text{rad}(g)}(\tilde{\Delta}, \tilde{\Delta})$ is given by the derivation,

$$\psi : \text{rad}(g) \to \text{End}_{\tilde{H}}(\tilde{\Delta}, \tilde{\Delta})$$

in the bases, $\{d_1, d_2, d_3\}$ of $\tilde{\Delta}$, the quarks, and $\{u, r_1, r_2\}$ of $\text{rad}(g)$, modulo the trivial derivations, given by an element $a := (a_{i,j}) \in \text{End}_{\tilde{H}}(\tilde{\Delta}, \tilde{\Delta})$. 
We find,

\[\psi(u) = \begin{pmatrix} \alpha_{1,1} & 0 & \alpha_{1,3} \\ 0 & \alpha_{2,2} & \alpha_{2,3} \\ \alpha_{3,1} & \alpha_{3,2} & \alpha_{3,3} \end{pmatrix} \mod \begin{pmatrix} 0 & 0 & -a_{1,3} \\ 0 & 0 & -a_{2,3} \\ a_{3,1} & a_{3,2} & 0 \end{pmatrix}\]

where, \( \alpha_{1,1} = \alpha_{2,2} = \alpha_{3,3} \), and

\[\psi(r_1) = \begin{pmatrix} -\alpha_{1,3} & 0 & 0 \\ -\alpha_{2,3} & 0 & 0 \\ \alpha_{3,1} & \alpha_{3,2} & \alpha_{3,3} \end{pmatrix} \mod \begin{pmatrix} a_{1,3} & 0 & 0 \\ a_{2,3} & 0 & 0 \\ a_{3,3} - a_{1,1} & -a_{1,2} & -a_{1,3} \end{pmatrix}\]

\[\psi(r_2) = \begin{pmatrix} 0 & -\alpha_{1,3} & 0 \\ 0 & -\alpha_{2,3} & 0 \\ \alpha_{3,1} & \alpha_{3,2} & \alpha_{2,3} \end{pmatrix} \mod \begin{pmatrix} 0 & a_{1,3} & 0 \\ 0 & a_{2,3} & 0 \\ -a_{2,1} & a_{3,3} - a_{2,2} & -a_{2,3} \end{pmatrix}\]

for some matrix \( (a_{i,j}) \).
Recall the structure of $\text{rad}(g)$. We have $[u, r_i] = \pm r_i$, $[r_1, r_2] = 0$, and the actions on $\Theta$, in the basis $\{c_1, c_2, c_3, d_1, d_2, d_3\}$, given by,

\[
u = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}.
\]

\[
r_1 = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0
\end{pmatrix} \quad r_2 = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0
\end{pmatrix}.
\]
Now, $\Ext^1_{\text{rad}(\mathfrak{g})}(\tilde{\Delta}, \tilde{\Delta}) = \text{Der}(\text{rad}(\mathfrak{g}), \text{End}_k(\tilde{\Delta}, \tilde{\Delta}))/\text{Triv}$. The rest is just computation. In case (1), choose $a = \alpha$. Then we find,

$$\psi(u) = \begin{pmatrix} \alpha_{1,1} & 0 & 0 \\ 0 & \alpha_{2,2} & 0 \\ 2\alpha_{3,1} & 2\alpha_{3,2} & \alpha_{3,3} \end{pmatrix}$$

where, $\alpha_{1,1} = \alpha_{2,2} = \alpha_{3,3}$, and

$$\psi(r_1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \alpha_{3,1} & \alpha_{3,2} & 0 \end{pmatrix}$$

$$\psi(r_2) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \alpha_{3,1} & \alpha_{3,2} & 0 \end{pmatrix}$$

since $\alpha_{1,2} = \alpha_{2,1} = 0$. 
If, in case (2), we choose $a = -\alpha$, we find,

$$\psi(u) = \begin{pmatrix} \alpha_{1,1} & 0 & 2\alpha_{1,3} \\ 0 & \alpha_{2,2} & 2\alpha_{2,3} \\ 0 & 0 & \alpha_{3,3} \end{pmatrix}$$

where, $\alpha_{1,1} = \alpha_{2,2} = \alpha_{3,3}$, and

$$\psi(r_1) = \begin{pmatrix} -2\alpha_{1,3} & 0 & 0 \\ -2\alpha_{3,2} & 0 & 0 \\ \alpha_{3,1} & \alpha_{3,2} & 2\alpha_{1,3} \end{pmatrix}$$

$$\psi(r_2) = \begin{pmatrix} 0 & -2\alpha_{1,3} & 0 \\ 0 & -2\alpha_{2,3} & 0 \\ \alpha_{3,1} & \alpha_{3,2} & 2\alpha_{2,3} \end{pmatrix}$$

since $\alpha_{1,2} = \alpha_{2,1} = 0$. 
Case (1) can be interpreted as follows: $Z_0 := \psi(u) - \alpha id$, $\alpha = \alpha_1, i$ mediates a force, mapping $\{d_1, d_2\}$ to $d_3$, $W_- := \psi(r_1)$ and $W_+ := \psi(r_2)$ do the same, but do not permute $(d_1, d_2)$. In case (2), $Z_0$ maps $d_3$ to $\{d_1, d_2\}$, and $W_-, W_+$ sends $\{d_1, d_2\}$ to $d_3$ and permute $(d_1, d_2)$, i.e. they change the orientation, see (6.3), and the relationship to left-and right-handedness, that is, to chirality.

This seems to be very close to the weak interaction on quarks, mediated by the weak interaction bosons, $Z, W_+, W_-$, identified above with the Bosons $u, r_1, r_2 \in g$, respectively.
A Canonical Filtration

Put

\[ Ph^{(n)}(A) := \text{im } i_n \subseteq Ph^\infty(A) \]

The $k$-algebra $Ph^\infty(A)$ has a descending filtration of two-sided ideals, 
\( \{ F_n \}_{0 \leq n} \) given inductively by

\[ F_1 = Ph^\infty(A) \cdot \text{im}(\delta) \cdot Ph^\infty(A) \]

and

\[ \delta F_n \subseteq F_{n+1}, \quad F_{n_1} F_{n_2} \cdots F_{n_r} \subseteq F_n, \quad n_1 + \ldots + n_r = n \]

such that the derivation $\delta$ induces derivations $\delta_n : F_n \longrightarrow F_{n+1}$. Using the canonical homomorphism $i_n : Ph^n(A) \longrightarrow Ph^\infty(A)$ we pull the filtration 
\( \{ F_p \}_{0 \leq p} \) back to $Ph^n(A)$, obtaining a filtration of each $Ph^n(A)$ with,

\[ F_1^n = Ph^n(A) \cdot \text{im}(\delta) \cdot Ph^n(A) \]
And inductively,

\[ \delta F^n_p \subseteq F^{n+1}_{p+1}, \quad F^n_{p_1} F^n_{p_2} \cdots F^n_{p_r} \subseteq F^n_p, \quad p_1 + \ldots + p_r = p. \]
The Algebras of Higher Differentials

Let \( D(A) := \lim_{\leftarrow \ n \geq 1} Ph^\infty(A)/F_n \), the completion of \( Ph^\infty(A) \) in the topology given by the filtration \( \{ F_n \}_{0 \leq n} \). The \( k \)-algebra \( Ph^\infty(A) \) will be referred to as the \( k \)-algebra of higher differentials, and \( D(A) \) will be called the \( k \)-algebra of formalized higher differentials. Put

\[
D_n := D_n(A) := Ph^\infty(A)/F_{n+1}
\]

Clearly \( \delta \) defines a derivation on \( D(A) \), and an isomorphism of \( k \)-algebras

\[
\epsilon := \exp(\delta) : D(A) \to D(A)
\]

and in particular, an algebra homomorphism,

\[
\tilde{\eta} := \exp(\delta) : A \to D(A)
\]

inducing the algebra homomorphisms

\[
\tilde{\eta}_n : A \to D_n(A)
\]
References


