Representations of compact Lie groups and their orbit spaces

(Joint work with Alexander Lytchak)

Claudio Gorodski

Geometry and Lie theory. Applications to classical and quantum mechanics
Dedicated to Eldar Straume on his 70th birthday.
Norwegian University of Science and Technology, Trondheim
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- Concrete example: $(\mathbf{SL}(n,\mathbb{R}), \mathbf{SO}(n))$ and $\mathbf{SO}(n)$-conjugation of $n \times n$ real symmetric matrices.

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Theorem (Dadok 1985)

Eevery polar representation (Def 1) of a connected compact Lie group is **orbit-equivalent** *to an s-representation.*

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- The converse essentially follows from the fact that a minimizing geodesic between G-orbits can be taken contained in Σ.

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- Existence and uniqueness; hierarchy.
- Luna-Richardson reduction: $V^H/(N_G(H)/H) = V/G$ where H is a principal isotropy group. (Representations of connected groups with non-trivial principal isotropy representations were classified by W.-C. Hsiang and W.-Y. Hsiang in 1970.)

Hsiang, Wu-Yi. Lie transformation groups and differential geometry. Differential geometry and differential equations (Shanghai, 1985), 34-52, Lecture Notes in Math., 1255, Springer, Berlin, 1987.

Let M be a given compact, 1 - connected, homogeneous space, $G = ISO(M)$, $K = ISO(M, X_o)$. Let V be the variety of focal points of X_o which can be decomposed into the union of K-orbits. For each point $X_1 \in V$, it is natural to define the <u>variational cocompleteness</u> of a shortest geodesic segment $\overline{X_o X_1}$, to be the codimension of the space of Killing vector fields in the space of Jacobi vector fields, vanishing at both X_o and X_1. Suggested by the remarkable accomplishment of Bott and Samelson [4,5] in the case of compact symmetric spaces,

Straume, Eldar. On the invariant theory and geometry of compact linear groups of cohomogeneity 3. Differential Geom. Appl. 4 (1994), no. 1, 1-23.

Let $G \subset O(n)$ be a full linear group. A *minimal reduction* of G is a linear group $K \subset O(k)$, k minimal, such that

a) K and G have isomorphic invariant rings (as graded algebras) and

b) S^{n-1}/G and S^{k-1}/K are isometric. We state our main results as follows.

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- For representations, this is equivalent to I inducing an isomorphism between the rings of invariants (Schwarz's theorem).

Riemannian orbifolds

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- If there is an isometry $I : X \to B$, where B is a Riemannian orbifold, then I is a diffeomorphism between the quotient smooth structure of X and the underlying smooth orbifold structure of B. Hence an isometry between orbit spaces is smooth if these are Riemannian orbifolds [Alexandrino-Lytchak].

HAPPY BIRTHDAY, ELDAR!