The classification of three-dimensional homogeneous spaces with non-solvable transformation groups

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Outline

History

Structure of 1-dimensional invariant foliations

Algebraic techniques

Classification of 3-dimensional homogeneous spaces

Historical remarks

- One-dimensional homogeneous spaces:

$$\begin{aligned}
\mathbb{R}^1 :& \langle \partial_x \rangle; \\
\mathbb{A}^1 :& \langle \partial_x, x\partial_x \rangle; \\
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- Two-dimensional homogeneous spaces: 18 cases (S. Lie)
- Three-dimensional homogeneous spaces: classification of all spaces that do not admit 1-dimensional invariant folations (S. Lie, Theorie der Transformationgruppen, Band 3, 1893).

oder durch die allgemeine lineare oder durch die specielle lineare Gruppe in $n-1$ Veränderlichen transformirt. Die betreffenden Normalformen sind Verallgemeinerungen der Gruppen [1] ... [12] auf

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*) Lie hat die Bestimmung aller imprimitiven Gruppen des Raumes x, y, z, allerdings durch ausserordentlich weitläufige Rechnungen, schon in den Jahren 1878 und 1879 durchgeführt und dabei den mitgetheilten Satz über die transitiven Gruppen gefunden. Durch die hier entwickelten Methoden, die Lie erst während dieser Rechnungen fand, ist die Bestimmung aller imprimitiven Gruppen des Raumes x, y, z ungemein erleichtert, denn durch diese Methoden ist die ganze Aufgabe in eine grosse Anzahl von einzelnen Aufgaben zerlegt, deren jede für sich erledigt werden kann, ohne dass man vorher die übrigen erledigt zu haben braucht. Der Leser ist durch die Entwickelungen des Textes in den Stand gesetzt, jede Kategorie von imprimitiven Gruppen des Raumes x, y, z, deren Kenntniss ihm wünschenswerth ist, ohne Schwierigkeit zu bestimmen.

English translation

Lie has already carried out the determination of all the primitive groups of the space x, y, z in the years 1878 and 1879, though by extraordinarily extensive calculations, and the corresponding statement on the transitive groups has been found.

By the methods developed here, which Lie first found in these calculations, the description of all the imprimitive groups of the space x, y, z is greatly simplified. Using these methods **the whole task is split into a great number of individual tasks, each of which can be done independently without the need to perform the rest in advance***.*

By means of the developments in this text the reader is able to easily determine any category of imprimitive groups of the space x, y, z, whose knowledge is desirable to him.

Classification of all primitive homogeneous spaces

A Lie algebra of vector fields $\mathfrak{g}$ is called *primitive*, if it does not admit any invariant foliations. Locally this is equivalent to the fact that the stationary subalgebra $\mathfrak{g}_0$ is maximal.

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- Classification of maximal subalgebras in complex simple Lie algebras: E. Dynkin.
- Real case: F. Karpelevich, B. Komrakov.

How to "glue" primitive actions?

- Let $M = G/G_0$ be a homogeneous space with an invariant foliation. Any invariant foliation on M has the form $\{(gH)G_0\}$, where H is some subgroup in G containing G_0:

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- Question: how to reconstruct M, if we have only "effectivized" spaces G/H and H/G_0?

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- Examples of invariant foliations on $\mathbb{R}^2 \to \mathbb{R}$:

$$\langle \partial_x, 2x\partial_x + ny\partial_y, x^2\partial_x + nxy\partial_y, \quad \partial_y, x\partial_y, \dots, x^n\partial_y \rangle,$$
$$\langle \partial_x, 2x\partial_x - 2y\partial_y, x^2\partial_x - (1+2xy)\partial_y \rangle,$$

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Local structure of one-dimensional invariant folations

Assume that $N = \tilde{G}/\tilde{H}$ is known (but not G/H!).

Theorem

Locally we have one and only one of the following cases:

- *G/H is effective. Then to construct G/G_0 from G/H we just need to describe all subgroups of codimension 1 in G_0.*
- *M is the direct product of N and $\mathbb{RP}^1$, $G = \tilde{G} \times PSL(2, \mathbb{R})$.*
- *M is an invariant a vector bundle over N, and G is an extension of $\tilde{G}$:*

$$\{e\} \to V \to G \to \tilde{G} \to \{e\}$$

  by some finite-dimensional $\tilde{G}$-invariant space of sections.
- *M is an invariant a vector bundle over N, and G is a trivial extension of $\tilde{G} \times \mathbb{R}^*$:*

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- Let V be any *finite-dimensional* submodule of the $\tilde{G}$-module $\mathcal{F}(\pi)$ and let $G = \tilde{G} \ltimes V$ be the semidirect product of $\tilde{G}$ and V. The group G acts naturally on E:

$$(g, \sigma).p = g.p + \sigma(g.\pi(p)).$$

  This action is transitive on E, if V contains any non-zero section. The stationary subgroup G_0 of this action is equal to $\tilde{G}_0 \ltimes V_0$, where V_0 is the set of all sections in V vanishing at $0 = e\tilde{G}_0$.

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- The semidirect product $\tilde{G} \ltimes V$ can be replaced by any extension of $\tilde{G}$ by means of the abelian subgroup V.

Proofs: algebraic techniques

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- Based on the theory of Frattini subalgebras (E. Stitzinger, 1970), we prove that if $\mathfrak{g}_0$ is maximal and almost effective, then it is either effective ($=$ the pair $(\mathfrak{g}, \mathfrak{g}_0)$ corresponds to a locally primitive action), or it has a very special structure:

$$\mathfrak{g} = \mathfrak{a} \ltimes (V \otimes k^n), \quad \mathfrak{g}_0 = \mathfrak{a} \ltimes (V \otimes k^{n-1}),$$

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- Using this technique we prove the above description of all invariant 1-dimensional foliations on homogeneous spaces.

Describing the extensions: cohomology

- Let $(\mathfrak{g}, \mathfrak{g}_0)$ be an effective pair of Lie algebras. A pair (V, V_0) is called an effective $(g, \mathfrak{g}_0)$-module, if V is any $\mathfrak{g}$-module, $\mathfrak{g}_0 V_0 \subset V_0$ and V_0 does not contain any non-zero submodules of the $\mathfrak{g}$-module V.

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- Extensions of $(\mathfrak{g}, \mathfrak{g}_0)$ by means of (V, V_0) are defined as exact sequences:

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- By analogy to the standard extensions, they are described by the 2nd cohomology $H^2((\mathfrak{g}, \mathfrak{g}_0), (V, V_0))$ of the subcomplex:

$$C^k((\mathfrak{g}, \mathfrak{g}_0), (V, V_0)) \subset C^k(\mathfrak{g}, V),$$
$$\alpha \in C^k((\mathfrak{g}, \mathfrak{g}_0), (V, V_0)) \quad \text{iff} \quad \alpha(\mathfrak{g}_0, \ldots, \mathfrak{g}_0) \subset V_0.$$

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(a) $M^3 = M^2 \times_{\tilde{H}} M^1$, $G = \tilde{G}$;
(b) $M^3 \to M^2$ is an invariant line bundle, G is an extension of $\tilde{G}$ or $\tilde{G} \times \mathbb{R}^*$;
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  (c) $M^3 = M^2 \times \mathbb{R}\mathrm{P}^1$, $G = \tilde{G} \times SL(2, \mathbb{R})$;
- **Completely classified for all non-solvable** G. We need only to consider cases (a) and (b), when $\tilde{G}$ is non-solvable (10 cases over $\mathbb{C}$ and 3 additional cases over $\mathbb{R}$), or the trivial case (c) for all possible $\tilde{G}$.

Example I

- One of the $SL(2)$-actions on the plane is infinitesimally given by the Lie algebra $\tilde{\mathfrak{g}}$:

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- There is a one-parameter family of invariant line bundles L_α. The action of $\tilde{\mathfrak{g}}$ on the sections of L_α is generated by linear differential operators:

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- We get the following Lie algebras of vector fields:

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- $\epsilon = 0$: trivial extension;
  $\epsilon = 1$: non-trivial extension.

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$$V_{n,m} = \langle \partial_x^i (x^{n+m}(1+xy)^m), 0 \leq i \leq n+2m \rangle, \quad m \geq 0.$$
- Non-trivial extensions exist only if $n = 0$. In this case we get the following families of Lie algebras:
$$\langle \partial_x, x\partial_x - y\partial_y + \partial_z, x^2\partial_x - (1+2xy)\partial_y + 2x\partial_z, f(x,y)\partial_z \\ f \in V_{0,m_1} + \cdots + V_{0,m_k} \rangle.$$

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- Finite-dimensional invariant subspaces of polynomials is x, y are in one-to-one correspondence with ideals of finite codimension $\mathbb{R}[\partial_x, \partial_y]$.
- Such ideals can not be explicitly parametrized by a finite number of parameters!

Happy birthday, Eldar!