Comparison of categories of traces in directed topology

Abel Symposium, Geiranger 2018

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Directed Algebraic Topology
Data: D-spaces Result: Dipaths and traces

Definition
A d-space (M. Grandis) is a topological space X together with a subspace $\vec{P}(X)$ (the dipaths) of path space X^I that

- contains all constant paths
- is closed under concatenation
- is closed under non-decreasing reparametrizations $\vec{I} \to \vec{I}$

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Glossary

- Trace = Path up to non-decreasing onto reparametrization
- $\vec{P}(X)_x^y$: Dipaths from x to y (CO-topology)
- $\vec{T}(X)_x^y$: Traces from x to y (quotient topology)
- OBS: The “end point map” $q : \vec{T}(X) \to X \times X$ is not a fibration, in general!

Trace spaces
Pre-cubical sets

- Pre-cubical sets and their geometric realization: glued from cubes along subcubes (like for pre-simplicial sets).
- D-structure: Paths that are non-decreasing along every coordinate. Glued together at common boundaries.

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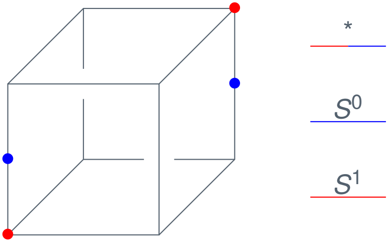
- D-structure: Paths that are non-decreasing along every coordinate. Glued together at common boundaries.
- For pre-cubical sets, there exist polyhedral spaces homotopy equivalent to $\vec{T}(X)_x^y$ – finite if X does not admit non-trivial directed loops (MR, K. Ziemiański).

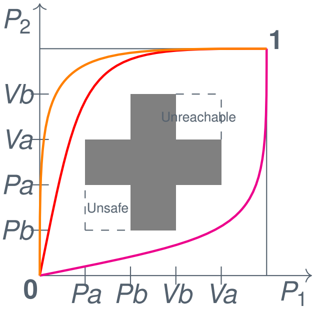
Example: $X = \partial\square^n$

$\vec{T}(\partial\square^n)_{\bullet}^{\bullet}$ is empty or contractible or homotopy equivalent to $S^k, 0 \leq k \leq n-2$.



Primary motivation: Concurrency

A simple PV program: “Swiss flag”



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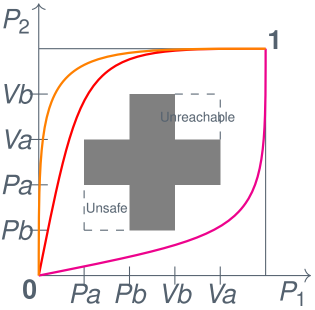

Executions are directed paths – since time flow is irreversible – avoiding a forbidden region (shaded).
Dipaths that are dihomotopic (through a 1-parameter deformation consisting of dipaths) correspond to equivalent executions.

2 categories associated with a d-space X

1. Trace category

Trace category $\mathcal{T}(X)$

Objects Points $x \in X$

Morphisms Traces $\vec{T}(X)_x^y$ for $x \preceq y$, i.e., $\vec{T}(X)_x^y \neq \emptyset$

Identities trivial dipath $\sigma_x \in \vec{T}(X)_x^x$

Composition Concatenation $*$: $\vec{T}(X)_x^y \times \vec{T}(X)_y^z \to \vec{T}(X)_x^z$

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Analogy: Poset category from filtered space $X = \bigcup_{r \in \mathbf{R}} X_r$

Objects	$r \in \mathbf{R}$
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- Apply algebraic topology functors – to **HoTop**, **Ab** etc.
- Which morphisms go to isos?
- Morphisms between certain pairs of objects may be empty.

2 categories associated with a d-space X

2. Its envelopping category (extension category)

Extension category $\mathcal{E}\vec{T}(X)$

- **Objects** Pairs $(x, y), x \preceq y$
- **Morphisms** (extension) maps $(\tau_{x'}^{x}, \tau_{y}^{y'}) \in \vec{T}(X)_{x}^{y} \times \vec{T}(X)_{x'}^{y'}$
- **Identities** (σ_x, σ_y)
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Functor $\mathcal{E}\vec{T}(X) \to$ HoTop

- $(x, y) \mapsto \vec{T}(X)_{x}^{y}$
- $(\tau_{x'}^{x}, \tau_{y}^{y'}) \mapsto [\sigma \in \vec{T}(X)_{x}^{y} \mapsto \tau_{x'}^{x} * \sigma * \tau_{y}^{y'} \in \vec{T}(X)_{x'}^{y'}]$

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Goal: Find “small” equivalent categories
Two strategies

First explained for filtered space:

1. Replace reals by maximal (ordered disjoint) intervals J such that $X_r \hookrightarrow X_s$ is a (weak) homotopy equivalence (resp. induces an iso of algebraic invariant) for all $r, s \in J,\ r \leq s$.
   Morphism: $I \leq J$ – maps to homotopy class of $X_r \hookrightarrow X_s$ – up to homotopy equivalence.
2. Pick representatives in each of these intervals and consider the full subcategory on only these objects.

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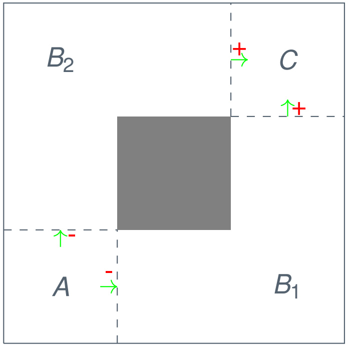
The "reals" category and the resulting enriched poset categories are equivalent:

1. The resulting quotient map is fully faithful (in the enriched sense) and surjective on objects.
2. The subcategory is fully faithful and essentially surjective on objects.

Example: 2D-Mutual exclusion

Trace category



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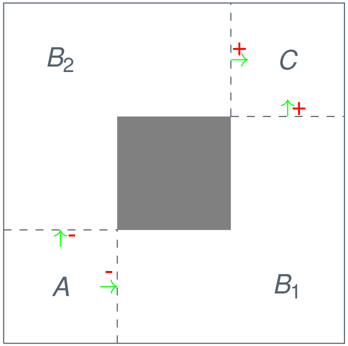
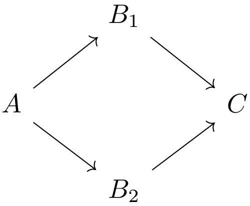


Compressed trace category



- taking into account both composition in the past and in the future.
- Can compress to two objects if composition is considered only in the future (resp. only in the past) .

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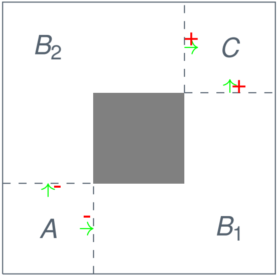
Extension category



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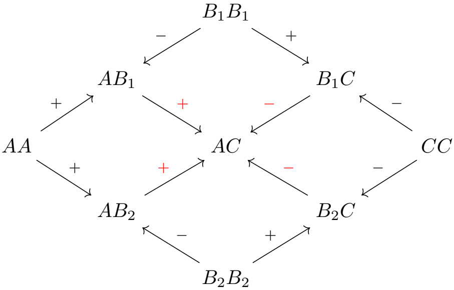
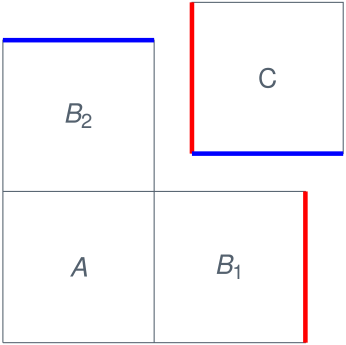


homology functor = "natural homology"
(Dubut-Goubault[2] 2017)

Necessary?

Example: A cubical complex
Glue along blue and red edges

Modification of an example from J. Dubut's thesis (2017):



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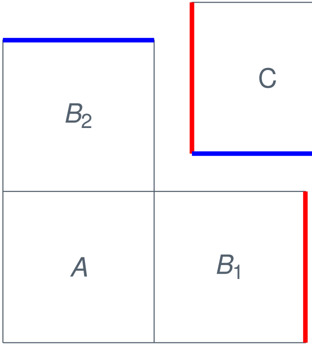
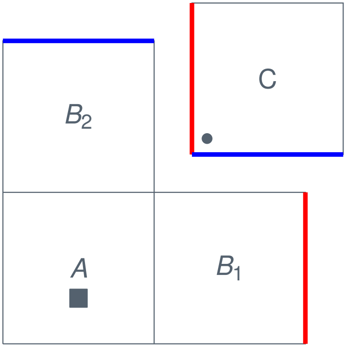


Directed paths?

Increasing in each cube. Fit at glueings.

Example: A cubical complex

No dipath from ■ to •



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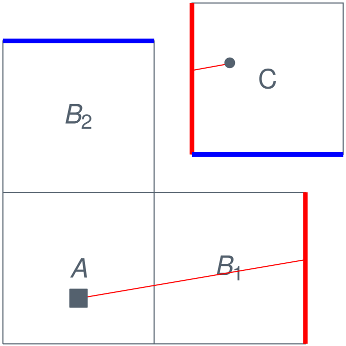
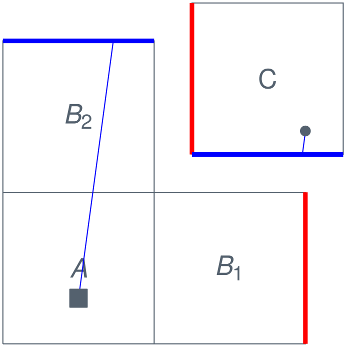
One dipath from ■ to • (through B_1)



A cubical complex

One dipath from ■ to • (through B_2)



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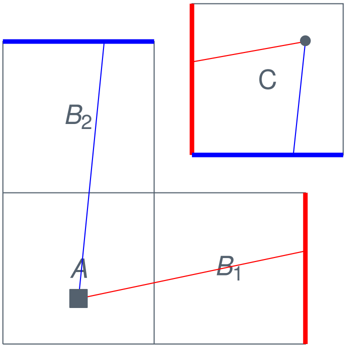

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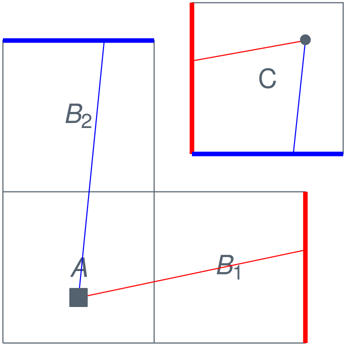

Lesson

The relative position of ■ and of • (within their cubes) is decisive for the homotopy type of $\vec{T}(X)_{■}^{•}$!

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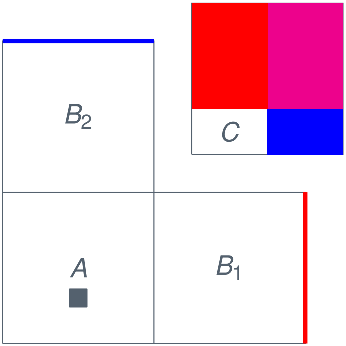
No isomorphisms in the trace category



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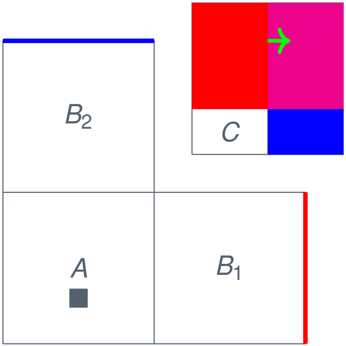

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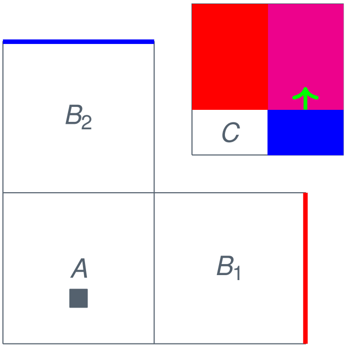

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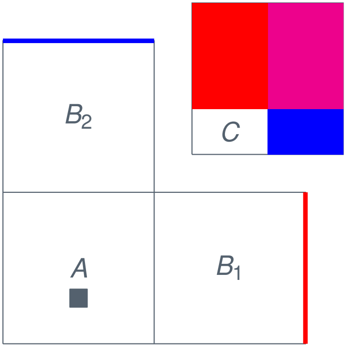

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OBS: No non-trivial dipath σ within the cell C preserves every path space up to homotopy equivalence:
For every such σ there exists $a \in A$ such that
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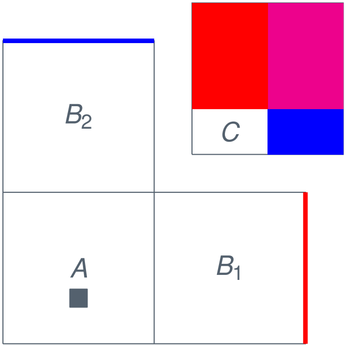

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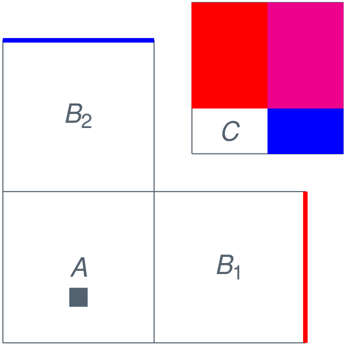
Trace category cannot be compressed!

Example: A cubical complex

Better luck with the extension category



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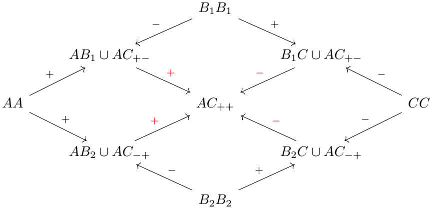

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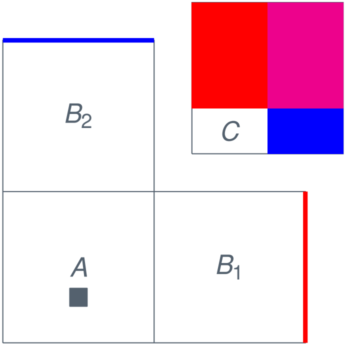

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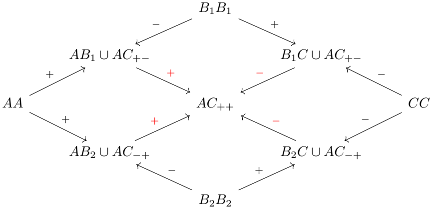

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Isomorphic to the extension category from previous example

Path space preserving homotopy flows

Definition (Psp homotopy flow)

- A homotopy flow on a d-space X is a directed homotopy $H : X \times \vec{I} \to X$ with $H_0 = id_X$.
- A homotopy flow is path space preserving (psp) if, for all $x \leq y$ and all $t \in I$, the maps $H(-, t)$ induce homotopy equivalences

$$\circ H_t(x, y) : \vec{T}(X)_x^y \simeq \vec{T}(X)_{H(x,t)}^{H(y,t)},$$
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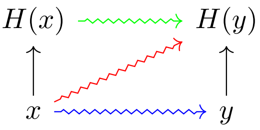
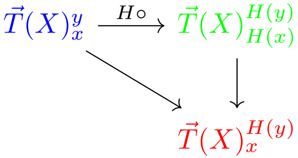
$H\circ$ fits into a diagram of morphisms

$\vec{T}(X)_x^y \to \vec{T}(X)_x^{H(y)}$

$\vec{T}(X)_{H(x)}^{H(y)} \to \vec{T}(X)_x^{H(y)}$





Localization and quotient category

Extension category

- For every psp homotopy flow H and objects (x, y) introduce formal morphisms $H : (x, y) \to (Hx, Hy)$ and inverses H^{-1} – with obvious compositions (“zig-zags”) and cancellation rules:

$$(H(x, -), \sigma_y) \circ H = (\sigma_x, H(y, -)).$$

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- Compression: From the resulting category, form a category (of components) with path components (wrt. isos) as objects. For every psp homotopy flow, (x, y) and $(H(x), H(y))$ give rise to the same object – most often also $(x, H(y))$.
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- Alternative construction (losing information, but universal): "Stable" components (K. Ziemiański, arXiv 1805.05061, recent)

Discretization via subcategory

for a pre-cubical set X

Pre-cubical set: like pre-simplicial set, composed from cubes.

- Barycentric subdivision $bd(X)$.
  Pick barycenters (one for each cell).
- Category with pairs of barycenters as objects.
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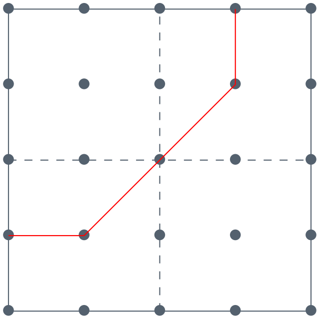

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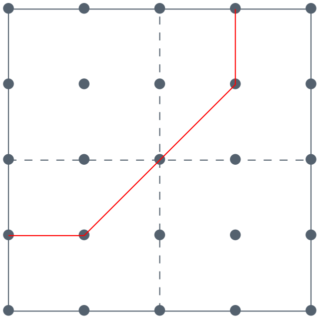
- Category with pairs of barycenters as objects.
- Morphisms: Extensions given by piecewise linear dipaths through barycenters (and composition).
- Result: A category that is certainly bisimilar to the extension category – also equivalent? (details to be checked).
- Extends a result by J. Dubut for Euclidean cubical complexes (subdivision not needed).



Dir. homotopy equivalence $f : X \to Y$?

Yielding equivalent extension categories

Requirements

- A reverse dimap $g : Y \to X$ and directed (zig-zag) psp!! dihomotopies H between id_X and $g \circ f$, resp. K between id_Y and $f \circ g$.

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Properties

- Homotopy types of path spaces are preserved: $\vec{T}(X)_x^y \simeq \vec{T}(Y)_{fx}^{fy}$.
- in a coherent way: Equivalent extension categories (+,-) (or their discrete counterparts).
- 2-out-of-3-property?

Thanks!
The end

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- the organizers
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