Multiparameter Persistence and Generalized Morse Theory

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Overview

Persistent homology gives:

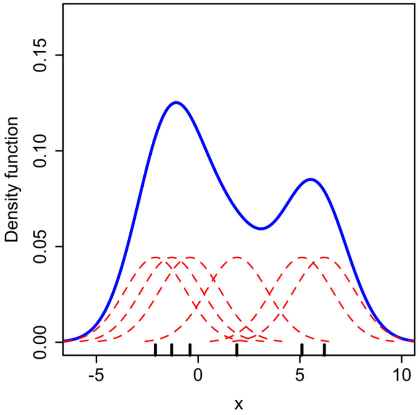
- a global understanding of the homology of sublevel sets of real-valued functions.

Goal

Give a global summary of the homology of sublevel sets of a 1-parameter family of functions.

Kernel Density Estimation

- Given $x_1, \ldots, x_N \in \mathbb{R}^d$.
- Assume x_i sampled from some unknown distribution f.
- We want to estimate f.
- Take sums of Gaussians centered at the x_i.



Persistent homology

Given $f : M \to \mathbb{R}$.

Sublevel sets: $F(a) = \{m \in M \mid f(x) \leq a\} = f^{-1}(-\infty, a]$
$a \leq b : F(a) \hookrightarrow F(b)$

Persistence module: $HF : \mathbf{R} \to \mathbf{Vect}_K$

Nice setting:

- M smooth manifold
- f Morse: smooth, all critical points non-degenerate, with distinct critical values

Results:

1. Morse theory completely determines the persistence module.
2. Generic smooth functions have the properties above.

Our setting for multiparameter persistence

Given $f_t : M \to \mathbb{R},\ t \in [0,1]$.

$$f : I \times M \to I \times \mathbb{R}$$
$$(t, m) \mapsto (t, f_t(m))$$

Our setting for multiparameter persistence

Given $f_t : M \to \mathbb{R},\ t \in [0,1]$.

$$f : I \times M \to I \times \mathbb{R}$$
$$(t, m) \mapsto (t, f_t(m))$$

Get $F(t_1, t_2; a) = f^{-1}([t_1, t_2] \times (-\infty, a])$ and

$HF : \mathbf{Int}_I \times \mathbf{R} \to \mathbf{Vect}_K$

Goal

Understand HF for a generic 1-parameter family f_t.

Generalized Morse Functions

Definition

Morse critical point: $f(x) = -\sum_{i=1}^{j} x_i^2 + \sum_{i=j+1}^{d} x_i^2$

Morse Lemma

For a Morse function, all critical points are Morse.

Definition

Generalized Morse critical point:

$$f(x, z) = -\sum_{i=1}^{j} x_i^2 + \sum_{i=j+1}^{d-1} x_i^2 + z^3$$

Cerf Theory

Let $\mathcal{F}$ be the space of smooth functions $M \to \mathbb{R}$.

Idea

$\mathcal{F}$ has a stratification $\mathcal{F}^0 \sqcup \mathcal{F}^1 \sqcup \cdots$

$\mathcal{F}^0$ Morse functions with distinct critical values

$\mathcal{F}^1 = \mathcal{F}^1_\alpha \sqcup \mathcal{F}^1_\beta$

$\mathcal{F}^1_\alpha$ functions with distinct critical values, one critical point generalized Morse, others Morse

$\mathcal{F}^1_\beta$ Morse functions with two equal critical values, all others distinct

Theorem (Cerf, 1970)

A generic 1*-parameter family of smooth functions is in* $\mathcal{F}^0$ *for all but finitely many parameter values, where it is in* $\mathcal{F}^1$*.*

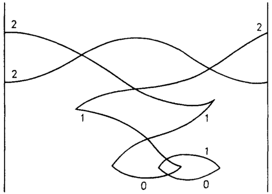
The Cerf diagram

Let $\Sigma(f_t)$ denote the set of critical values of f_t.

Definition

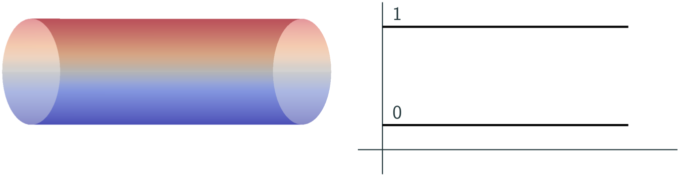
The Cerf diagram of a generic family of Morse functions:

$$\bigcup_{t \in I} \bigcup_{m \in \Sigma(f_t)} (t, f_t(m)) \subset I \times \mathbb{R}$$



R. Kirby (1978) A Calculus for Framed Links in S^3.

Using Cerf Diagrams to Decompose H_*F



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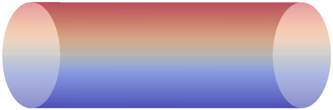
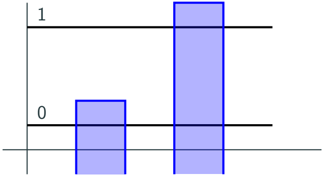
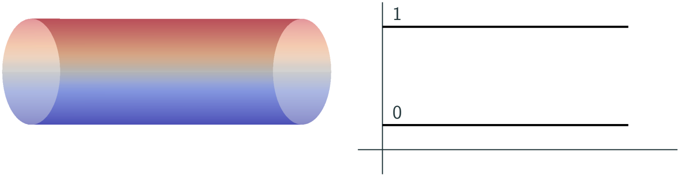




Using Cerf Diagrams to Decompose $H_* F$



$H_* F \cong$

H_0

$\bigoplus$

H_1

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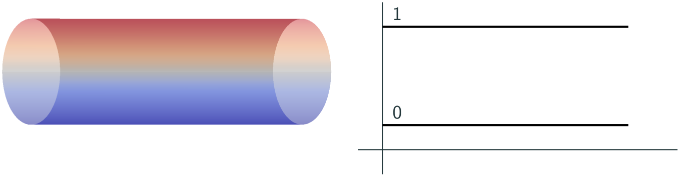

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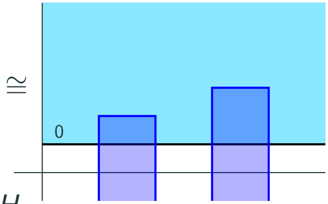

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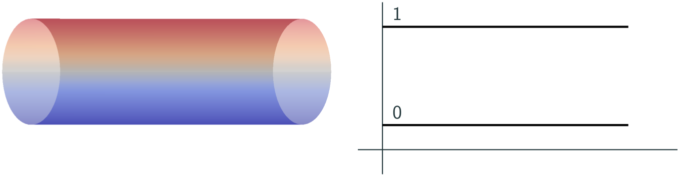

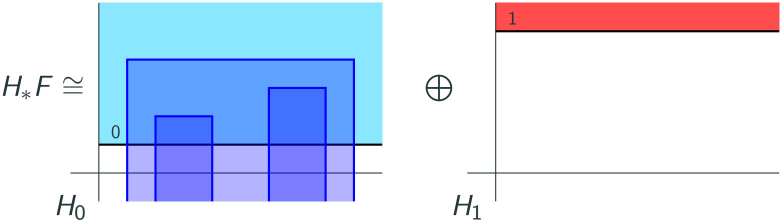

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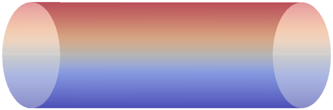

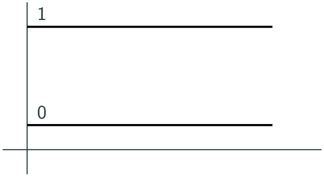

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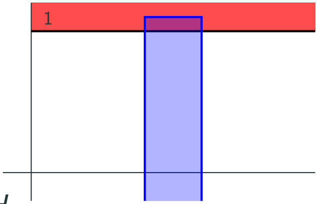


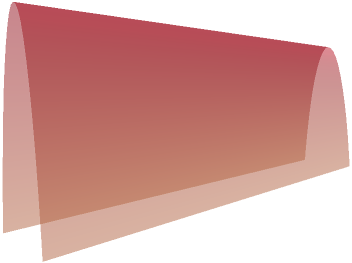
H_1

Fold Singularities

Definition

A point p is a fold singularity of index j, if near p,

$$(t, x) \mapsto \left(t, -\sum_{i=1}^{j} x_i^2 + \sum_{i=j+1}^{d} x_i^2 \right) .$$

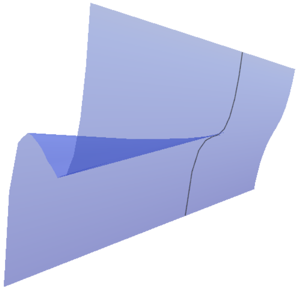


Cusp singularities

Definition

A point p is a cusp singularity of index $j + 1/2$, if near p,

$$(t, x, z) \mapsto \left(t, -\sum_{i=1}^{j} x_i^2 + \sum_{i=j+1}^{d-1} x_i^2 + z^3 + 3tz \right) ,$$



Wrinkles

Definition

For an open subset U, a wrinkle of index $j + 1/2$ is a map equivalent to

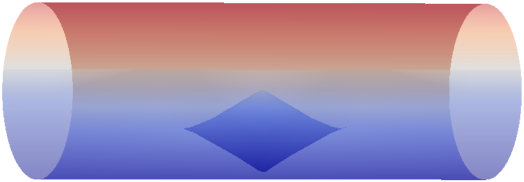
$$w(t, x, z) = \left(t, -\sum_{i=1}^{j} x_i^2 + \sum_{i=j+1}^{d-1} x_i^2 + z^3 + 3\left(t^2 - 1\right) z \right).$$

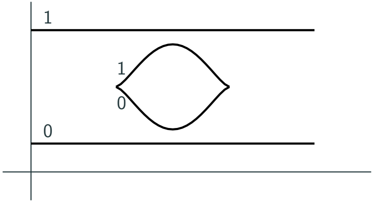
Definition

f is a map with wrinkles: there exist disjoint opens U_i with

- $f|_{U_i}$ is a wrinkle, and

Wrinkled cylinder



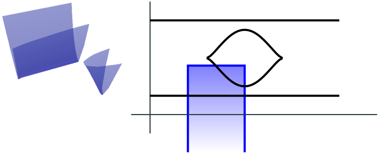


$$f_t : S^1 \to \mathbb{R}$$

$$f : I \times S^1 \to I \times \mathbb{R}$$

$$F(t_1, t_2; a) = f^{-1}\left([t_1, t_2] \times (-\infty, a]\right)$$

Wrinkled cylinder



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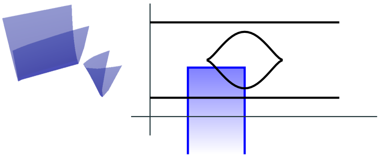
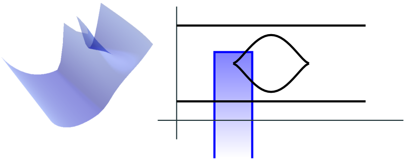
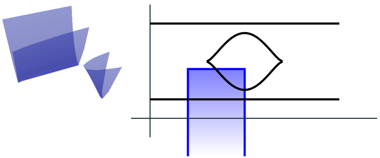
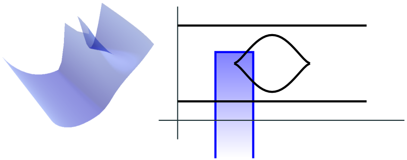
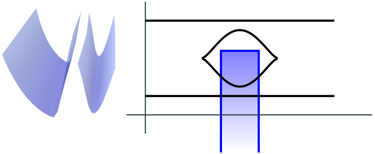




Wrinkled cylinder







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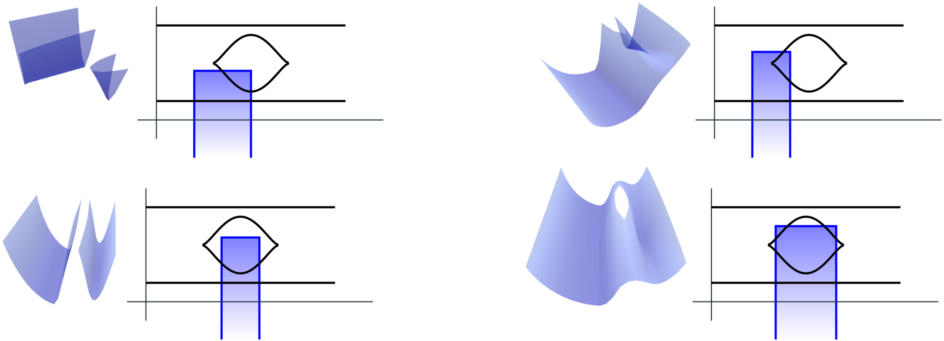

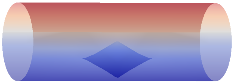

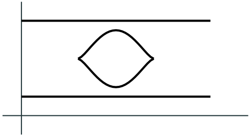

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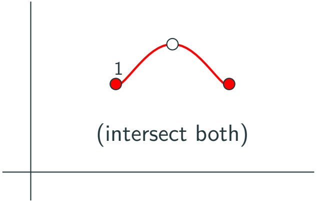

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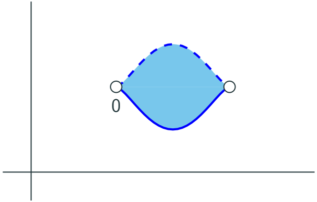

Theorem (B.-Catanzaro)

Let $f : I \times M \to I \times \mathbb{R}$ be a map with wrinkles. Each wrinkle gives rise to two indecomposable summands of $H_*(F)$. For a wrinkle of index $j + 1/2$, one is in H_j and the other in H_{j+1}.

Idea of Proof:

- View each semi-infinite rectangle as a cobordism.
- Use left/right half-handle attachments from Morse theory for manifolds with boundary.

Future directions

- Decompositions for more complicated Cerf diagrams
- Level set persistence, leading to a cosheaf
- Simplifications: Cerf moves
- More parameters: wrinkles and the h-principle
- Algorithms