A NONLINEAR EIGENVALUE PROBLEM

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1. INTRODUCTION

My lectures at the *Minicorsi di Analisi Matematica* at Padova in June 2000 are written up in these notes[1]. They are an updated and extended version of my lectures [37] at Jyväskylä in October 1994. In particular, an account of the exciting recent development of the asymptotic case is included, which is called the ∞-eigenvalue problem. I wish to thank the University of Padova for financial support. I am especially grateful to Massimo Lanza de Cristoforis for his kind assistance. I thank Harald Hanche-Olsen for his kind help with final adjustments of the typesetting.

These lectures are about a nonlinear eigenvalue problem that has a serious claim to be the right generalization of the linear case. By now I have lectured on four continents about this theme and my reason for sticking to this seemingly very peculiar problem is twofold. First, one can study the interesting questions without any previous knowledge of spectral theory. Second, to the best of my knowledge there are many open problems easy to state. The higher eigenvalues are "mysterious".

The leading example of a linear eigenvalue problem is to find all non-trivial solutions of the equation $\Delta u + \lambda u = 0$ with boundary values zero in a given bounded domain in $\mathbf{R}^n$. This is the Dirichlet boundary value problem. (In the Neumann boundary value problem the normal derivative is zero at the boundary.) Needless to say, this has been generalized in numerous ways: to Riemann surfaces and manifolds, to equations $\Delta u + \lambda u + Vu = 0$ with a potential V, to more general differential operators than the Laplacian, and so on.

[1]A short comment on the uniqueness proof in [11] has been added later.

However, when talking about nonlinear eigenvalue problems, there is seldom any eigenvalue at all involved. For example, one just considers the existence of *positive* solutions. The extremely popular and very interesting Emden-Fowler equation

$$\Delta u + |u|^{\alpha-1}\, u = 0$$

is of this type. If $\alpha \neq 1$, the parameter λ plays no role in the equation $\Delta u + \lambda\, |u|^{\alpha-1}\, u = 0$, since it can be scaled out: multiply u by a suitable constant to see this. In equations of the type

$$\Delta u + \lambda u + |u|^{\alpha-1}\, u = 0$$

the parameter λ is stabilizing, when the exponent α is critical. Though interesting as they are, I will not consider these problems here. I refer to Professor Donato Passaseo's lectures about *Nonlinear Elliptic Equations Involving Critical Sobolev Exponents.*

My objective is the nonlinear eigenvalue problem

$$\operatorname{div}(|\nabla u|^{p-2}\, \nabla u) + \lambda\, |u|^{p-2}\, u = 0 \tag{1.1}$$

with $u = 0$ on the boundary of a bounded domain Ω in the n-dimensional Euclidean space. Here $1 < p < \infty$ and for $p = 2$ we are back to the linear case $\Delta u + \lambda u = 0$. Note that

$$\lambda = \frac{\int_\Omega |\nabla u|^p\, dx}{\int_\Omega |u|^p\, dx}, \tag{1.2}$$

if u is a solution, not identically zero. (Here $dx = dx_1\, dx_2 \cdots dx_n$ is the Lebesgue measure.) Thus it appears that $\lambda > 0$. Minimizing this so called nonlinear *Rayleigh quotient* among all admissible functions we arrive at Eqn (1.1) as the corresponding Euler-Lagrange equation. The first one to study it in any serious way seems to have been F. de Thélin in 1984, cf. [51]. The so-called p-harmonic operator $\operatorname{div}\big(|\nabla u|^{p-2}\, \nabla u\big)$ appears in many contexts in physics: non-Newtonian fluids, reaction-diffusion problems, non-linear elasticity, and glaceology, just to mention a few applications.

The range of p in the p-harmonic operator

$$\begin{aligned} \Delta_p u &= \operatorname{div}(|\nabla u|^{p-2} \nabla u) \\ &= |\nabla u|^{p-4} \left\{ |\nabla u|^2 \Delta u + (p-2) \sum \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} \frac{\partial^2 u}{\partial x_i \partial x_j} \right\} \end{aligned}$$

is usually $1 \leq p \leq \infty$. The case $p = 1$ is the mean curvature operator (with a minus sign)

$$H = -\Delta_1 u = -\operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right)$$

and the fascinating asymptotic case $p = \infty$ is related to the operator

$$\Delta_\infty u = \sum_{i,j=1}^{n} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} \frac{\partial^2 u}{\partial x_i \partial x_j}.$$

In Section 6 the theory of viscosity solutions is used to treat the latter case. An amazing "differential equation" replaces (1.1). Arcane phenomena occur.

Many results are readily extended to equations of the more general form

$$\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} \left(\left| \sum_{k,m=1}^{n} a_{km}(x) \frac{\partial u}{\partial x_k} \frac{\partial u}{\partial x_m} \right|^{\frac{p-2}{2}} a_{ij}(x) \frac{\partial u}{\partial x_j} \right) + \lambda \rho(x) \left|u\right|^{p-2} u = 0$$

where the matrix (a_{km}) satisfies the *ellipticity condition*

$$\sum_{i,j=1}^{n} a_{ij}(x) \xi_i \xi_j \geq \left|\xi\right|^2$$

for all $\xi = (\xi_1, \xi_2, \ldots, \xi_n)$, and, by assumption, $\rho(x) \geq \varepsilon > 0$. The weaker restriction $\rho(x) \geq 0$ leads to considerable technical difficulties, not to mention the case when the density $\rho(x)$ is allowed to change signs. See [54]. It is likely that the theory works, when (a_{km}) is a Muckenhoupt weight. The essential feature here is that *solutions may be multiplied by constants.* Indeed, among all the equations

$$\operatorname{div} \left(\left|\nabla u\right|^{p-2} \nabla u \right) + \lambda \left|u\right|^{s-2} u = 0$$

only the homogeneous case $s = p$ has the proper structure of a "typical eigenvalue problem", to quote an expression in [5].

In passing, I mention that the density $\rho(x)$ in the equation

$$\operatorname{div}(|\nabla u|^{p-2} \nabla u) + \lambda \rho(x) |u|^{p-2} u = 0$$

is very decisive. Indeed, if we take $\rho(x)^{-p}$ to be the *distance function* $\delta(x) = \operatorname{dist}(x, \partial\Omega)$ in a *convex* domain Ω, then there is no eigenfunction at all: 0 is the only solution. Moreover, the sharp lower bound in the inequality ("Hardy's inequality").

$$\left(1 - \frac{1}{p}\right)^p < \frac{\int_\Omega |\nabla \varphi|^p dx}{\int_\Omega |\frac{\varphi}{\delta}|^p dx}, \quad \varphi \in C_0^\infty(\Omega),$$

is not attained for any admissible function, if Ω is convex. It is curious that this sharp bound depends only on p. This phenomenon was observed by S. Agmon. See [40].

The reader who wants to learn this topic does well in reading the first volume of the celebrated book by Courant & Hilbert and, perhaps, the book by Polya & Szegö, before passing on to so called modern expositions like [9] and [50]. The lecture [32] by E. Lieb is very illuminating. See [12] about spectral theory on manifolds. About elliptic partial differential equations we refer to the books [24], [30], and [23]. See also [19]. The book *Metodi diretti nel calculo variazioni* by E. Giusti is an excellent source of information.

NOTE ADDED IN PROOF. The reference E. Lieb [31] has come to my attention. It contains an interesting result about the minimum of the nonlinear Rayleigh quotient. Thus it appears that E. Lieb was the first one to study the nonlinear eigenvalue problem in several variables.

2. PRELIMINARY RESULTS

Throughout these lectures Ω will denote a *bounded* domain in $\mathbf{R}^n$. For most of the theorems no regularity assumptions are needed about the boundary $\partial\Omega$. The equation will be interpreted in the weak sense.

DEFINITION 1 *We say that* $u \in W_0^{1,p}(\Omega)$, $u \not\equiv 0$, *is an eigenfunction, if*

$$\int_\Omega |\nabla u|^{p-2}\, \nabla u \cdot \nabla \eta \, dx = \lambda \int_\Omega |u|^{p-2}\, u\eta \, dx \tag{2.1}$$

whenever $\eta \in C_0^\infty(\Omega)$. *The corresponding real number* λ *is called an eigenvalue.*

The Sobolev space $W_0^{1,p}(\Omega)$ is the completion of $C_0^\infty(\Omega)$ with respect to the norm

$$\|\varphi\| = \left\{ \int_\Omega \left(|\varphi|^p + |\nabla\varphi|^p\right) dx \right\}^{\frac{1}{p}} .$$

As usual, $C_0^\infty(\Omega)$ is the class of smooth functions with compact support in Ω. By standard elliptic regularity theory *an eigenfunction is continuous*, i.e., it can be made continuous after a modification in a set of measure zero. See for example [23], [24], [30]. Indeed, even the *gradient* ∇u *is locally Hölder continuous*, the Hölder exponent depending only on n and p. See [17] or [53] for this deep regularity result, the first proof of which is credited to N. Uraltseva.

In regular domains the boundary value zero is attained in the classical sense. For example, any domain satisfying an exterior cone condition is surely regular enough. As a matter of fact, the regular boundary points can be characterized by a version of the celebrated Wiener criterion, formulated by Mazj'a [39] in a nonlinear setting. See [22] and [29]. It is known that those boundary points ξ where the requirement

$$\lim_{x \to \xi} u(x) = 0$$

fails is a set of p-capacity zero. That is to say that the irregular boundary points form a very small set. If $p > n$, then every boundary point is regular!

It is not difficult to see that *every eigenvalue λ is positive.* Indeed, by approximation, u itself will do as test-function in (2.1). Therefore

$$\lambda = \frac{\int\limits_\Omega |\nabla u|^p \, dx}{\int\limits_\Omega |u|^p \, dx}.$$

It is useful to have an explicit lower bound. The familiar Sobolev inequality $\|u\|_{np/(n-p)} \leq C \|\nabla u\|_p$, where $C = C(n,p)$ and $1 < p < n$, implies

$$\lambda \geq \frac{1}{C^p \, |\Omega|^{p/n}}. \tag{2.2}$$

This lower bound for the eigenvalues is valid also for $p \geq n$. It is instructive to prove it directly. Suppose that $\varphi \in C_0^\infty(\Omega)$ where Ω is the parallelepiped $0 < x_1 < a_1$, $0 < x_2 < a_2, \dots, 0 < x_n < a_n$. Then

$$\varphi(x_1, x_2, \dots, x_n) = \int\limits_0^{x_1} \frac{d\varphi(t, x_2, \dots, x_n)}{dt} dt,$$

$$|\varphi(x_1, x_2, \dots, x_n)|^p \leq x_1^{p-1} \int\limits_0^{a_1} \left| \frac{d\varphi(t, x_2, \dots, x_n)}{dt} \right|^p dt,$$

$$\int\limits_0^{a_1} |\varphi(x_1, x_2, \dots, x_n)|^p \, dx_1 \leq \frac{a_1^p}{p} \int\limits_0^{a_1} |D_1 \varphi(t, x_2, \dots, x_n)|^p \, dt$$

and an integration with respect to the remaining variables $x_2, \dots, x_n$ gives the estimate

$$\int\limits_0^{a_1} \int\limits_0^{a_2} \cdots \int\limits_0^{a_n} |\varphi(x_1, x_2, \dots, x_n)|^p \, dx_1 dx_2 \cdots dx_n$$
$$\leq \frac{a_1^p}{p} \int\limits_0^{a_1} \int\limits_0^{a_2} \cdots \int\limits_0^{a_n} |D_1 \varphi(x_1, x_2, \dots, x_n)|^p \, dx_1 dx_2 \cdots dx_n.$$

Therefore

$$\frac{\int\limits_{\Omega} |\nabla\varphi|^p \, dx}{\int\limits_{\Omega} |\varphi|^p \, dx} \geq \frac{p}{a_1^p}. \tag{2.3}$$

We only used the fact that $\varphi(0, x_2, \ldots, x_n) \equiv 0$. (Since also

$$\varphi(a_1, x_2, \ldots, x_n) = 0$$

we can readily improve the lower bound a little, replacing a_1 by $a_1/2$.) Note that we may write $a_2, a_3 \ldots,$ or a_n instead of a_1. The shortest side yields the best estimate.

Essentially the same reasoning can be used to prove the estimate

$$\lambda \geq \frac{\text{Const.}}{R^p}$$

in a *regular domain*, R denoting the radius of the largest inscribed ball in the smallest "box" containing Ω. This means that the eigenvalues are large even in very long, yet narrow domains. See [41] in the linear case.

The Harnack inequality: *If u is a non-negative eigenfunction, then*

$$\max_{B_r} u \leq C \min_{B_r} u$$

whenever $B_{2r} \subset \Omega$. Here B_r and B_{2r} are concentric balls of radii r and $2r$. The constant C depends only on n and p. This result is due to Trudinger [55], who proved it in 1967 using the celebrated Moser iteration. The inequality implies that, if $u \geq 0$ in Ω, then $u > 0$. As we will see in Section 4, a positive eigenfunction must correspond to the smallest eigenvalue

$$\lambda_1 = \inf_{\varphi} \frac{\int\limits_{\Omega} |\nabla\varphi|^p \, dx}{\int\limits_{\Omega} |\varphi|^p \, dx} \tag{2.4}$$

where the infimum is taken over all $\varphi \in C_0^\infty(\Omega)$, $\varphi \not\equiv 0$. By standard compactness arguments it is easily seen that the infimum is attained for a function u in $W_0^{1,p}(\Omega)$. But, if u is minimizing, so is $|u|$. By the Harnack inequality $|u| > 0$. By continuity, either $u > 0$ in Ω or $u < 0$ in Ω. Hence *a first eigenfunction does not change signs.*

To prove existence, the following slightly simplified version of the Rellich-Kondrachov theorem is useful.

Lemma 2 (Rellich–Kondrachov) *Let $p > 1$. Suppose that $u_1, u_2, \ldots$ are functions in $W_0^{1,p}(\Omega)$ and that $\|\nabla u_k\|_{p,\Omega} \leq L < \infty$, when $k =$*

$1, 2, 3, \ldots$. *Then there is a function* $u \in W_0^{1,p}(\Omega)$ *such that* $u_{k_j} \to u$ *strongly in* $L^p(\Omega)$ *and* $\nabla u_{k_j} \rightharpoonup \nabla u$ *weakly in* $L^p(\Omega)$ *for some subsequence.*

Proof: This is a combination of the weak compactness of L^p and the Rellich-Kondrachov imbedding theorem. See [49, §11, pp. 82-85] or [57, Theorem 2.5.1, p. 62]. As a matter of fact, the convergence is better than we have stated. □

We end this section by proving two results. First, the spectrum is a closed set. This fact would properly belong to Section 5. Second, we bound the eigenfunctions. This fact is needed in Section 4.

THEOREM 3 *The spectrum is a closed set.*

Proof: Suppose that a sequence $\lambda_1, \lambda_2, \ldots$ of eigenvalues converges to $\lambda \neq \infty$ and let $u_1, u_2, \ldots$ denote the eigenfunctions, normalized by the condition $\|u_k\|_{p,\Omega} = 1$. We have

$$\int_\Omega |\nabla u_k|^{p-2} \nabla u_k \cdot \nabla \eta \, dx = \lambda_k \int_\Omega |u_k|^{p-2} u_k \eta \, dx \tag{2.5}$$

for each $\eta \in C_0^\infty(\Omega)$. We claim that λ is an eigenvalue. By the normalization

$$\lambda_k = \int_\Omega |\nabla u_k|^p \, dx.$$

By the Rellich-Kondrachov Theorem there is a subsequence and a function $u \in W_0^{1,p}(\Omega)$ such that $u_{k_j} \to u$ strongly in $L^p(\Omega)$ and $\nabla u_{k_j} \rightharpoonup \nabla u$ weakly in $L^p(\Omega)$. We have to prove that this u is the eigenfunction corresponding to λ. By the equation itself we have

$$\int_\Omega \left[\left|\nabla u_{k_j}\right|^{p-2} \nabla u_{k_j} - |\nabla u|^{p-2} \nabla u \right] \cdot \nabla (u_k - u) \, dx$$

$$= \lambda_{k_j} \int_\Omega \left|u_{k_j}\right|^{p-2} u_{k_j} \left(u_{k_j} - u\right) dx - \int_\Omega |\nabla u|^{p-2} \nabla u \cdot \left(\nabla u_{k_j} - \nabla u\right) dx.$$

The first integral on the right-hand side approaches zero, because of the convergence $\|u_{k_j} - u\|_{p,\Omega} \to 0$, and so does the second integral by the weak convergence of the gradients. Therefore we have obtained that

$$\lim_{j \to \infty} \int_\Omega \left[\left|\nabla u_{k_j}\right|^{p-2} \nabla u_{k_j} - |\nabla u|^{p-2} \nabla u \right] \cdot \left[\nabla u_{k_j} - \nabla u\right] dx = 0.$$

The elementary inequality

$$2^{1-p} |w_2 - w_1|^p \leq \left[|w_2|^{p-2} w_2 - |w_1|^{p-2} w_1 \right] \cdot (w_2 - w_1), \qquad p \geq 2,$$

for vectors in $\mathbf{R}^n$ shows that the limit above implies the strong convergence

$$\lim_{j\to\infty}\int_\Omega \left|\nabla u_{k_j}-\nabla u\right|^p dx = 0.$$

There is a similar inequality for $p<2$. Thus we can pass to the limit under the integral sign in (2.5) to obtain

$$\int_\Omega |\nabla u|^{p-2}\,\nabla u\cdot\nabla\eta dx = \lambda\int_\Omega |u|^{p-2}\,u\eta dx.$$

This shows that λ is an eigenvalue, since the normalization prevents u from being identically zero. □

It is evident that an eigenfunction is bounded in a regular domain. But there are continuous functions in the Sololev space $W_0^{1,p}(\Omega)$ that are unbounded. Therefore we had better write down a proof of

$$\sup_{x\in\Omega}|u(x)|<\infty.$$

LEMMA 4 *The bound*

$$\|u\|_{\infty,\Omega}\le 4^n\lambda^{\frac{n}{p}}\|u\|_{1,\Omega} \tag{2.6}$$

holds for the eigenfunction u in any bounded domain Ω in $\mathbf{R}^n$.

Proof: The interesting method in [30, Lemma 5.1, p. 71] yields this estimate. (The constant 4^n is not optimal.) To this end, we may assume that u is positive at some point. The function

$$\eta(x)=\max\{u(x)-k,0\}$$

will do as test-function in (2.1) and so we obtain

$$\int_{A_k}|\nabla u|^p\,dx=\lambda\int_{A_k}|u|^{p-2}\,u(u-k)dx \tag{2.7}$$

where

$$A_k=\{x\in\Omega|\,u(x)>k\}.$$

Clearly $k\,|A_k|\le\|u\|_{1,\Omega}$ and $|A_k|\to 0$ as $k\to\infty$.

By the elementary inequality $a^{p-1}\le 2^{p-1}(a-k)^{p-1}+2^{p-1}k^{p-1}$ we have

$$\int_{A_k}u^{p-1}(u-k)dx\le 2^{p-1}\int_{A_k}(u-k)^p dx+2^{p-1}k^{p-1}\int_{A_k}(u-k)dx. \tag{2.8}$$

The Sobolev inequality yields

$$\int_{A_k}(u-k)^p dx\le\left(2^{-1}\,|A_k|\right)^{\frac{p}{n}}\int_{A_k}|\nabla u|^p\,dx, \tag{2.9}$$

when applied to each component of the open set A_k. (The constant $\frac{1}{2}$ is not essential.)

Combining (2.7), (2.8), and (2.9) we arrive at

$$\left[1 - 2^{p-2}\lambda\,|A_k|^{\frac{p}{n}}\right]\int\limits_{A_k}(u-k)^p dx \leq 2^{p-2}k^{p-1}\lambda\,|A_k|^{\frac{p}{n}}\int\limits_{A_k}(u-k)dx.$$

In the first factor $2^{p-2}\lambda\,|A_k|^{\frac{p}{n}} \leq \frac{1}{2}$, when $k \geq k_1 = 2^{n(p-1)/p}\lambda^{n/p}\|u\|_1$. Using the Hölder inequality and dividing out we finally obtain the estimate

$$\int\limits_{A_k}(u-k)dx \leq 2\lambda^{\frac{1}{p-1}}k\,|A_k|^{1+\frac{p}{(p-1)n}} \tag{2.10}$$

for $k \geq k_1$. This is the inequality needed in [30, Lemma 5.1, p.71] to bound $\operatorname{ess\,sup} u$.

Indeed, writing

$$f(k) = \int\limits_{A_k}(u-k)dx = \int\limits_k^\infty |A_t|\,dt,$$

we have $f'(k) = -\,|A_k|$ and hence (2.10) can be restated as

$$f(k) \leq 2\lambda^{\frac{1}{p-1}}k\left[-f'(k)\right]^{1+\frac{p}{(p-1)n}},$$

when $k \geq k_1$. If f is positive in the interval $[k_1, k]$, then an integration of the differential inequality leads to

$$k^{\frac{\varepsilon}{1+\varepsilon}} - k_1^{\frac{\varepsilon}{1+\varepsilon}} \leq \left[2\lambda^{\frac{1}{p-1}}\right]^{\frac{1}{1+\varepsilon}}\left[f(k_1)^{\frac{\varepsilon}{1+\varepsilon}} - f(k)^{\frac{\varepsilon}{1+\varepsilon}}\right]$$

where $\varepsilon = p/(p-1)n$. Since $f(k_1) \leq f(0) = \|u\|_1$ and $f(k) \geq 0$ on the right-hand side, this clearly bounds k and hence $f(k)$ is zero sooner or later. The quantitative bound is seen to be

$$k \leq 2^{1+\frac{2n(p-1)}{p}}\lambda^{\frac{n}{p}}\|u\|_1. \tag{2.11}$$

This means that $f(k) = 0$, if (2.11) is not fulfilled, i.e. $\operatorname{ess\,sup} u$ is never greater than the right-hand side.

To bound $\operatorname{ess\,inf} u$, consider the function $-u$. $\square$

Let me mention a difficult question. Can an eigenfunction be zero at all the points of an open subset of Ω? This is *the problem of unique continuation.* Except for the first eigenfunction this seems to be an open problem. Zero has a special status. No eigenfunction can have a constant value different from zero in an open subdomain. This is evident from the equation.

3. THE ONE-DIMENSIONAL CASE

In the case of one independent variable all the eigenvalues are explicitly known. This was first studied by Ôtani in connexion with the determination of the best constant in some Sobolev type inequalities. The equation is

$$\left(\left|u'\right|^{p-2}u'\right)' + \lambda\left|u\right|^{p-2}u = 0$$

where $u = u(x)$, $a \leq x \leq b$, and $u(a) = 0$, $u(b) = 0$. The equation is readily integrated and, via the first integral

$$|u'|^p + \frac{\lambda|u|^p}{p-1} = \text{Constant},$$

one arrives at the expression

$$\lambda(p) = (p-1)\left\{\frac{2}{b-a}\int_0^1 \frac{dt}{(1-t^p)^{1/p}}\right\}^p$$

for the first eigenvalue, cf. [42]. This is the minimum of the Rayleigh quotient

$$\frac{\int_a^b |u'(x)|^p\,dx}{\int_a^b |u(x)|^p\,dx}$$

taken among all $u \in C^1[a,b]$ with $u(a) = u(b) = 0$. The expression for $\lambda(p)$ is easily evaluated and the result is

$$\sqrt[p]{\lambda(p)} = \frac{2\pi\sqrt[p]{p-1}}{(b-a)p\sin\frac{\pi}{p}}.$$

The rather striking result

$$\sqrt[p]{\lambda(p)} = \sqrt[q]{\lambda(q)}, \quad \frac{1}{p} + \frac{1}{q} = 1$$

can be read off for conjugated exponents p and q. In terms of Rayleigh quotients

$$\min\frac{\|u'\|_p}{\|u\|_p} = \min\frac{\|v'\|_q}{\|v\|_q}, \quad \frac{1}{p} + \frac{1}{q} = 1.$$

See [34].

The spectrum can be completely determined. The eigenvalues are precisely

$$\lambda(p), 2^p\lambda(p), 3^p\lambda(p), \ldots, k^p\lambda(p), \ldots$$

The eigenfunctions are obtained from the first one. Let u_1 denote the first eigenfunction in [0,1]. Extend it as an odd function to $[-1,0]$ and, then, periodically to the whole real axis, i.e., $u_1(x) = -u_1(-x)$, $u_1(x+2) = u_1(x)$. The higher eigenfunctions are

$$u_k(x) = u_1(kx), \qquad k = 1, 2, 3, \ldots$$

In the linear case we have the eigenvalue $k^2\pi^2$ corresponding to the normalized eigenfunction

$$\sqrt{2}\sin(k\pi x), \qquad k = 1, 2, 3, \ldots$$

The spectrum is discrete in the one-dimensional case. The eigenvalues are simple and the k^{th} eigenfunction has $k-1$ nodes (zeros inside the interval) and k nodal intervals of equal length.

An example in [10] shows that the Fredholm alternative does not hold for the equation

$$(|u'|^{p-2}u')' + \lambda|u|^{p-2}u = f(x)$$

in the nonlinear case $p \neq 2$. A solution can exist even if

$$\langle u_1, f\rangle = \int_0^1 u_1(x)f(x)dx \neq 0.$$

Some other orthogonality condition seems to be called for.

4. THE FIRST EIGENFUNCTION

The first eigenfunction (the Ground State) has many special properties. It is the only positive eigenfunction. *The restriction of a higher eigenfunction to a nodal domain is a first eigenfunction* (with respect to this smaller domain).

The first eigenvalue or the principal frequency is

$$\lambda_1 = \inf_{\varphi} \frac{\int\limits_\Omega |\nabla\varphi|^p \, dx}{\int\limits_\Omega |\varphi|^p \, dx} \tag{4.1}$$

where $\varphi \in C_0^\infty(\Omega)$, $\varphi \not\equiv 0$. By (2.2) $\lambda_1 > 0$. Using a (normalized) minimizing sequence $\varphi_1, \varphi_2, \ldots$ we obtain a function $u_1 \in W_0^{1,p}(\Omega)$ such that

$$\lambda_1 = \frac{\int\limits_\Omega |\nabla u_1|^p \, dx}{\int\limits_\Omega |u_1|^p \, dx}.$$

The compactness argument needed in the existence proof is provided by the Rellich-Kondrachov Theorem. A well-known device due to Lagrange shows that u_1 is a weak solution to the equation

$$\operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) + \lambda_1 |u|^{p-2} u = 0.$$

If u_1 is minimizing, so is $|u_1|$ and therefore also $|u_1|$ satisfies the equation. Since $|u_1| \geq 0$, we must have $|u_1| > 0$ by Harnack's inequality. By continuity either $u_1 > 0$ in Ω or $u_1 < 0$ in Ω. We have established the following result.

LEMMA 5 *There exists a positive eigenfunction corresponding to the principal frequency λ_1. This eigenfunction minimizes the Rayleigh quotient among all functions in the Sobolev space $W_0^{1,p}(\Omega)$. Moreover, a minimizer is a first eigenfunction and does not change signs.*

Some basic facts can easily be read off from the Rayleigh quotient. First, if $\Omega_1 \subset \Omega_2$, then $\lambda_1(\Omega_1) \geq \lambda_1(\Omega_2)$, since there are more competing functions in Ω_2. Second, the quantity $p\lambda(p)^{1/p}$ increases with p. (The notation $\lambda_1 = \lambda(p)$ indicates the dependence of p.)

Being a solution to Eqn (2.1), the eigenfunction shares many properties with solutions to more general quasilinear eigenvalue problems. But here we would like to emphasize the following specific features:

I **"Isoperimetric" property**. Among all domains with the same volume (area) the ball (the disc) has the smallest principal frequency.[2]

II **Concavity**. For any bounded convex domain $\log u$ is concave, u denoting a positive eigenfunction [48, Theorem 1].

III **Uniqueness**. The first eigenfunctions are essentially unique in any bounded domain: given p, they are merely constant multiples of each other. Moreover, they have no zeros in the domain and they are the only eigenfunctions not changing sign.

IV **Stability**. For any bounded sufficiently regular domain the principal frequency is continuous as a function of p. [35, Theorem 6.1]. [3] In very irregular domains there is some anomaly.

V **Superharmonicity.** In a convex domain the first eigenfunction is superharmonic, for $p \geq 2$. (We mean that "$\Delta u \leq 0$".)

VI **Asymptotic formula**. As $p \to \infty$ we have

$$\lim_{p \to +\infty} \sqrt[p]{\lambda(p)} = \frac{1}{\max\limits_{x \in \Omega} \operatorname{dist}(x, \partial\Omega)}$$

In other words, the reciprocal number of the radius of the largest inscribed ball in the domain gives the principal frequency for the case $p = \infty$!

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[2]For the second eigenvalue there is a characterization in the linear case, cf [44].

[3]No similar result is known for the second eigenfunction.

The uniqueness (III) for *arbitrary* bounded domains was first proved in [33]. A new proof was found in [3]. Recently, an elegant variational proof was found by Belloni and Kawohl, cf. [11]. The radial case has been studied by F. de Thélin [52] and a good reference for C^2-domains is [48, Theorem A.1]. Other references for regular domains are [7], [6] and [2]. As we said, the restriction of a higher eigenfunction to a nodal domain is a first eigenfunction there. Though the original domain is as regular as we please, it is not clear that this is inherited by the nodal domains. Therefore it is important to prove the uniqueness in arbitrary domains. The proof will be discussed below. The logarithmic concavity [4] mentioned in (II) is due to S. Sakaguchi [48], when $p \neq 2$, and the linear case is credited to H. Brascamp & E. Lieb. The proof by Sakaguchi is based on a convexity principle of N. Korevaar. The superharmonicity (V) is a consequence of (II) and the formula

$$(p-2)|\nabla u|^3 \Delta_1 u + \frac{\Delta_p u}{|\nabla u|^{p-4}} = (p-1)|\nabla u|^2 \Delta u,$$

which connects the Laplacian Δu with the p-Laplacian $\Delta_p u = \nabla \cdot (|\nabla u|^{p-2}\nabla u)$ and the *mean curvature operator* $-\Delta_1 u = -\operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right)$. The formula has to be interpreted in the *viscosity sense*. Property (I) follows by spherical symmetrization (Schwarz symmetrization), cf. [28, p.90]. The ball is (essentially) the only optimal shape, cf [8]. For $p = 2$ this is the celebrated conjecture of Lord Rayleigh, proved by E. Krahn [5] and G. Faber. The asymptotic formula (VI) is postponed to Section 6.

Let us begin by discussing the uniqueness (III). The first eigenvalue is simple. That is, all the first eigenfunctions in a fixed domain are merely constant multiples of each other.

THEOREM 6 *The first eigenvalue is simple in any bounded domain.*

Proof: Suppose that u and v are two first eigenfunctions. So are $|u|$ and $|v|$. Thus the situation is reduced to the case $u > 0$ and $v > 0$. As Anane has observed in [2], the result would follow by certain balanced calculations, if the function $\eta = u - v^p u^{1-p}$ were, a priori, admissible as test-function in

$$\int_\Omega |\nabla u|^{p-2}\, \nabla u \cdot \nabla \eta dx = \lambda_1 \int_\Omega |u|^{p-2}\, u\eta dx$$

[4]The reader might find it strange that the property does not depend on p. The corresponding result for the equation $\operatorname{div}\left(|\nabla u|^{p-2}\,\nabla u\right) = -1$ is that $u^{1-1/p}$ is a concave function.

[5]See "Edgar Krahn 1894-1961", a centenary volume edited by Š. Lumiste & J. Peetre, IOS Press, Amsterdam 1994, pp. 81-106.

and $v - u^p v^{1-p}$ in the similar equation for v. We use the modified test-functions

$$\eta = \frac{(u+\varepsilon)^p - (v+\varepsilon)^p}{(u+\varepsilon)^{p-1}} \text{ and } \frac{(v+\varepsilon)^p - (u+\varepsilon)^p}{(v+\varepsilon)^{p-1}},$$

ε being a positive constant. Then

$$\nabla\eta = \left\{1 + (p-1)\left(\frac{v+\varepsilon}{u+\varepsilon}\right)^p\right\}\nabla u - p\left(\frac{v+\varepsilon}{u+\varepsilon}\right)^{p-1}\nabla v$$

and, by symmetry, the gradient of the test-function in the corresponding equation for v has a similar expression, yet with u and v interchanged. Using the fact that u and v are bounded (Section 2), we easily see that $\eta \in W_0^{1,p}(\Omega)$.

Instead of reproducing the whole proof in [33] we write down the calculations only for $p = 2$, that is, a non-linear proof of the linear case is presented. Inserting the chosen test-functions into their respective equations and adding these, we obtain the simple expression

$$(4.2) \quad \int_\Omega \left(u_\varepsilon^2 + v_\varepsilon^2\right)\left|\nabla \log u_\varepsilon - \nabla \log v_\varepsilon\right|^2 dx = \lambda_1 \int_\Omega \left[\frac{u}{u_\varepsilon} - \frac{v}{v_\varepsilon}\right]\left(u_\varepsilon^2 - v_\varepsilon^2\right) dx,$$

where we have written $u_\varepsilon = u(x) + \varepsilon$ and $v_\varepsilon = v(x) + \varepsilon$. As ε approaches zero, it is plain that the right hand side tends to zero. By Fatou's lemma

$$\int_\Omega \left(u^2 + v^2\right)\left|\nabla \log u - \nabla \log v\right|^2 dx = 0.$$

The integrand must be zero. Hence $u\nabla v = v\nabla u$ a.e. Thus $u = Cv$ or $v = Cu$ for some constant. This proves the case $p = 2$.

If $p \geq 2$, then the inequality [6]

$$|w_2|^p \geq |w_1|^p + p\,|w_1|^{p-2}\, w_1 \cdot (w_2 - w_1) + \frac{|w_2 - w_1|^p}{2^{p-1} - 1}$$

should be used. Take $w_2 = \nabla \log v_\varepsilon$ and $w_1 = \nabla \log u_\varepsilon$. There is a counterpart valid, when $1 < p < 2$. For the details we refer to [33]. □

As a byproduct of the proof we can conclude the following

Theorem 7 *A positive eigenfunction is always a first eigenfunction.*

[6] The inequality seems to be due to L. Evans, see [18, p. 250]. The best constant is not the abovementioned $1/(2^{p-1} - 1)$. It has been determined in [ ].

Proof: Suppose that $v > 0$ is an eigenfunction with the eigenvalue λ. Let $u > 0$ denote the first eigenfunction. In the case $p = 2$ the previous calculation yields that

$$\int_\Omega \left(u_\varepsilon^2 + v_\varepsilon^2\right) |\nabla \log u_\varepsilon - \nabla \log v_\varepsilon|^2 \, dx = \int_\Omega \left[\lambda_1 \frac{u}{u_\varepsilon} - \lambda \frac{v}{v_\varepsilon}\right] \left(u_\varepsilon^2 - v_\varepsilon^2\right) dx.$$

This exhibits that the right-hand member is non-negative. Hence, letting ε tend to zero, we have

$$(\lambda_1 - \lambda) \int_\Omega \left(u^2 - v^2\right) dx \geq 0.$$

If $\lambda \neq \lambda_1$, then $\lambda > \lambda_1$ and $\int_\Omega \left(u^2 - v^2\right) dx \leq 0$. This is an impossible situation, since u can be replaced by $2u, 3u, \ldots$. We have proved that $\lambda = \lambda_1$. The case $p \neq 2$ is rather similar. □

A simple proof of the simplicity of λ_1 has recently been given by Belloni and Kawohl, cf [11]. It is based on the admissible function

$$w = \left(\frac{u^p + v^p}{2}\right)^{1/p}$$

in the Rayleigh quotient. A short calculation yields

$$|\nabla w|^p = \frac{u^p + u^p}{2} \left| \frac{u^p \nabla \log u + v^p \nabla \log v}{u^p + v^p} \right|^p.$$

Because the positive quantities $u^p/(u^p + v^p)$ and $v^p/(u^p + v^p)$ add up to 1, we can use Jensen's inequality for convex functions to obtain the estimate

$$\left| \frac{u^p \nabla \log u + v^p \nabla \log v}{u^p + v^p} \right|^p \leq \frac{u^p |\nabla \log u|^p + v^p |\nabla \log v|^p}{u^p + v^p}.$$

Thus we have

$$|\nabla w|^p \leq \frac{1}{2} |\nabla u|^p + \frac{1}{2} |\nabla v|^p.$$

The inequality is strict at points where $\nabla \log u \neq \nabla \log v$. Now we can conclude that

$$\lambda_1 \leq \frac{\int_\Omega |\nabla w|^p dx}{\int_\Omega w^p dx} \leq \frac{\frac{1}{2}\int_\Omega |\nabla u|^p dx + \frac{1}{2}\int_\Omega |\nabla v|^p dx}{\frac{1}{2}\int_\Omega u^p dx + \frac{1}{2}\int_\Omega v^p dx} = \lambda_1.$$

If $\nabla \log u \neq \nabla \log v$ in a set of positive measure, then we would have a strict inequality above, which is a contradiction. This proves that u and v are constant multiples of each other. — This elegant proof is not, as is were, capable of establishing that a positive eigenfunction is a first one.

About the concavity of $\log u$ we refer directly to [48]. It is worth noting that the first eigenfunction u itself is never concave, the one dimensional case beeing an exception. In a ball in $\mathbf{R}^n$ even u^α is concave for some α, $1/n < \alpha < 1$. In [36] I have conjectured that, among all convex domains, the ball has the best concavity exponent. Even the linear case seems to be unsettled.

The **stability of the principal frequency** $\lambda_1 = \lambda(p)$, **when** p **varies** is rather intriguing. This topic is discussed in [35]. By the Hölder inequality

$$p\lambda(p)^{\frac{1}{p}} < s\lambda(s)^{\frac{1}{s}}, \text{ when } 1 < p < s < \infty,$$

so that the one-sided limits in

$$\lim_{s\to p-} \lambda(s) \leq \lambda(p) = \lim_{s\to p+} \lambda(s). \tag{4.3}$$

exist. The last equality is almost evident. Normalizing the eigenfunctions so that $\|u_s\|_{s,\Omega} = 1$ we actually have

$$\lim_{s\to p+} \int_\Omega |\nabla u_s - \nabla u_p|^p \, dx = 0$$

as s approaches p from above. When s approaches p from below, even the adjusted version

$$\lim_{s\to p-} \int_\Omega |\nabla u_s - \nabla u_p|^s \, dx = 0 \tag{4.4}$$

is plainly false in irregular domains, when $p \leq n$. However, (4.4) implies that

$$\lim_{s\to p-} \lambda(s) = \lambda(p). \tag{4.5}$$

We think that (4.5) implies (4.4).

Given any p, $1 < p \leq n$, there is a bounded domain Ω in $\mathbf{R}^n$ such that

$$\lim_{s\to p+} \lambda(s) < \lambda(p),$$

and, *a fortiori*, (4.4) cannot hold for the normalized eigenfunctions. The explanation is a rather interesting phenomenon. A sequence of eigenfunctions u_s will converge to a function $u \in W^{1,p}(\Omega)$. One has $u \in W_0^{1,s}(\Omega)$ for every $s < p$. This u is a weak solution to the equation $\operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) + \lambda |u|^{p-2} = 0$ in Ω, except that it fails to be in the right space $W_0^{1,p}(\Omega)$. To cause such a delicate effect, one needs a closed set Ξ_p such that $\operatorname{cap}_s(\Xi_p) = 0$, when $s < p$, yet $\operatorname{cap}_p(\Xi_p) > 0$. It is known how to construct such sets as generalized Cantor sets. The final domain Ω will be of the form $B \setminus \Xi_p$, where B is a sufficiently large ball containing Ξ_p in its interior. For a complete discussion of the "p-stability" we refer to our fairly technical paper in "Potential Analysis". See also [26].

The question about **variations of the domain**, instead of of the exponent p, is relatively simple. Let $\Omega_1 \subset \Omega_2 \subset \Omega_3 \subset \cdots$ be an exhaustion of Ω,

$$\Omega = \bigcup_{j=1}^{\infty} \Omega_j.$$

Then

$$\lim_{j\to\infty} \lambda_1^{(p)}(\Omega_j) = \lambda_1^{(p)}(\Omega), \tag{4.6}$$

where the notation is evident. (By a remark in the book by Courant-Hilbert, this is not true for the corresponding Neumann problem, when $p = 2$. One has to define the admissible variations of the domain in a careful way, when the normal derivative at the boundary is involved.)

To prove (4.6), we note that

$$\lambda_1^{(p)}(\Omega_1) \geq \lambda_1^{(p)}(\Omega_2) \geq \cdots \geq \lambda_1^{(p)}(\Omega)$$

Given $\varepsilon > 0$, there is a function $\varphi \in C_0^\infty(\Omega)$ such that

$$\lambda_1^{(p)}(\Omega) > \frac{\displaystyle\int_\Omega |\nabla\varphi|^p\,dx}{\displaystyle\int_\Omega |\varphi|^p\,dx} - \varepsilon,$$

since $\lambda_1^{(p)}(\Omega)$ is the infimum. Being a compact set, the support of φ is covered by a finite number of the sets $\Omega_1, \Omega_2, \ldots$. Hence supp$\varphi \subset \Omega_j$ for j large enough. Thus

$$\lambda_1^{(p)}(\Omega_j) \leq \frac{\displaystyle\int_{\Omega_j} |\nabla\varphi|^p\,dx}{\displaystyle\int_{\Omega_j} |\varphi|^p\,dx} = \frac{\displaystyle\int_\Omega |\nabla\varphi|^p\,dx}{\displaystyle\int_\Omega |\varphi|^p\,dx}$$

so that

$$\lambda_1^{(p)}(\Omega) > \lambda_1^{(p)}(\Omega_j) - \varepsilon$$

for all large j. It is plain that $\lambda_1^{(p)}(\Omega) \geq \lim \lambda_1^{(p)}(\Omega_j)$. This proves the desired result.

Indeed, extending the eigenfunctions $u_j \in W_0^{1,p}(\Omega_j)$ as zero in $\Omega \setminus \Omega_j$ so that $u_j \in W_0^{1,p}(\Omega)$, the strong convergence $\|\nabla u - \nabla u_j\|_{p,\Omega} \longrightarrow 0$ holds for the normalized eigenfunctions (that is, $\|u_j\|_{p,\Omega} = 1$). Here u is the first eigenfunction in Ω. The proof is not difficult.

5. HIGHER EIGENVALUES

The operator $-\Delta$ has a discrete spectrum $\lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots$ and $\lambda_k \longrightarrow \infty$ as $k \longrightarrow \infty$. Each eigenvalue is repeated according to its multiplicity. Weyl's theorem about the asymptotic behaviour of the eigenvalues states that

$$\lim_{k\to\infty} \frac{\lambda_k^{n/2}}{k} = \frac{\text{Const.}}{|\Omega|}.$$

The corresponding eigenfunctions $u_1, u_2, u_3, \ldots$ can be chosen to satisfy

$$\langle u_k, u_j \rangle = \int_\Omega u_k u_j dx = \delta_{ij}.$$

This orthogonality is the key to the linear case $\Delta u + \lambda u = 0$. We recommend the classical book by Courant & Hilbert.

It is more difficult to prove that also the equation

$$\operatorname{div}\left(|\nabla u|^{p-2}\, \nabla u\right) + \lambda\, |u|^{p-2}\, u = 0$$

has infinitely many eigenvalues. There are several methods that work. However, the main open problem is quite the opposite. Are there more eigenvalues than the chosen method produces? If so, how can one exhaust the spectrum. Can all the eigenvalues be numerated? To the best of my knowledge the nonlinear spectrum has not been proved to be discrete, not even when the domain Ω is a ball or a cube.

In order to describe how higher eigenvalues are produced we have to introduce an auxiliary concept, the genus of Krasnoselskij. The proof will be skipped. About the method we refer to [9], [46], and [50].

If A is a symmetric[7] and closed subset of a Banach space, then its *genus* $\gamma(A)$ is defined as the smallest integer k for which there exists a continuous odd mapping $\varphi\colon A \longrightarrow \mathbf{R}^k \setminus \{0\}$. Thus $\varphi(v) = -\varphi(-v)$, when $v \in A$. If no such integer exists, then we define $\gamma(A) = \infty$. Especially, $\gamma(A) = \infty$, if A contains the origin, since $\varphi(0) = 0$ for odd mappings. See [46] and [St, Chapter II] about this concept.

Let $\sum_k$ denote the collection of all symmetric subsets A of $W_0^{1,p}(\Omega)$ such that $\gamma(A) \geq k$ and the set $\{v \in A \mid \|v\|_{p,\Omega} = 1\}$ is compact. *The*

[7] Symmetric means that $-v \in A$, if $v \in A$.

numbers

$$\lambda_k = \inf_{A\in\sum_k} \max_{v\in A} \frac{\displaystyle\int_\Omega |\nabla v|^p\,dx}{\displaystyle\int_\Omega |v|^p\,dx} \tag{5.1}$$

are eigenvalues and there are infinitely many of them, cf. [21] and [6]. The fact that this minimax procedure yields eigenvalues is often explained through the Palais-Smale condition.

These "minimax eigenvalues" $\lambda_1 < \lambda_2 \leq \lambda_3 \cdots$ satisfy an estimate of the type encountered in Weyl's theorem. According to [21] there are two positive constants depending only on n and p such that

$$\frac{c_1}{|\Omega|} \leq \frac{\lambda_k^{\frac{n}{p}}}{k} \leq \frac{c_2}{|\Omega|}$$

as $k \to \infty$. See also [6]. Unfortunately, *it is not known whether the described procedure exhausts the spectrum.* Are there other eigenvalues than those listed in (5.1)? Therefore the asymptotic result is of limited interest, so far.

As the notation in (5.1) indicates, λ_1 is the first eigenvalue. As we will see, λ_1 is isolated. It is possible to show that λ_2 is the second one. An unpublished manuscript [4] of A. Anane and M. Tsouli contains a minimax characterization of the second eigenvalue in terms of the functional

$$I(v) = \left(\int_\Omega |\nabla v|^p\,dx\right)^2 - \int_\Omega |v|^p\,dx.$$

Their proof is easily adapted to the Rayleigh quotient: for $k = 2$ (5.1) yields *the second eigenvalue* λ_2, that is $\lambda_2 = \min\limits_{\lambda<\lambda_1} \lambda$. No such identification is yet known for eigenvalues higher than the second. The second eigenvalue is not known to be isolated, when its multiplicity is ignored.

The *nodal domains* are defined as the connected components of the sets $\{u > 0\}$ and $\{u < 0\}$. See [14] and [1].

THEOREM 8 *Any eigenfunction has only a finite number of nodal domains.*

Proof: Let u be an eigenfunction corresponding to λ. If N_j denotes a component of one of the sets $\{x \in \Omega | u(x) > 0\}$ and $\{x \in \Omega | u(x) < 0\}$, then $u \in W_0^{1,p}(N_j)$. By the Sobolev inequality

$$|N_j| \geq C(n,p)\lambda^{-\frac{n}{p}}.$$

Summing up, we have

$$|\Omega| \geq \sum_j |N_j| \geq C(n,p)\lambda^{-\frac{n}{p}} \sum_j 1$$

so that the number of nodal domains is bounded by a constant times $\lambda^{n/p} |\Omega|$. □

Theorem 9 *The first eigenvalue is isolated.*[8]

Proof: Suppose that there is a sequence of eigenvalues λ'_k tending to λ_1 (these are not supposed to be minimax eigenvalues). If u_k denotes the corresponding normalized eigenfunction, then

$$\int_\Omega |\nabla u_k|^p \, dx = \lambda'_k, \quad \int_\Omega |u_k|^p \, dx = 1.$$

By compactness arguments there are a subsequence and a function $u \in W_0^{1,p}(\Omega)$ such that $\nabla u_{k_j} \rightharpoonup \nabla u$ weakly and $u_{k_j} \to u$ strongly in $L^p(\Omega)$. By weak lower semicontinuity

$$\frac{\int_\Omega |\nabla u|^p \, dx}{\int_\Omega |u|^p \, dx} \leq \lim_{j\to\infty} \lambda'_{k_j} = \lambda_1$$

so that u is the first eigenfunction. Since u does not change signs, we may take $u > 0$.

If $\lambda'_k \neq \lambda_1$, then u_k must change signs in Ω. Both sets $\Omega_k^+ = \{u_k > 0\}$ and $\Omega_k^- = \{u_k < 0\}$ are non-empty and their measures cannot tend to zero, since

$$\left|\Omega_k^+\right| \geq C(n,p)(\lambda'_k)^{-\frac{n}{p}}, \quad \left|\Omega_k^-\right| \geq C(n,p)(\lambda'_k)^{-\frac{n}{p}}.$$

This prevents u_{k_j} from converging to a positive function in $L^p(\Omega)$. Indeed, the sets

$$\Omega^+ = \limsup \Omega_{k_j}^+, \quad \Omega^- = \limsup \Omega_{k_j}^-$$

have positive measure by a well-known "Selection Lemma". We may assume that $u = \lim u_{k_j}$ a.e. in Ω. Passing to suitable subsequences we can show that $u \geq 0$ a.e. in Ω^+ and $u \leq 0$ a.e. in Ω^-. This is a contradiction. □

There are many more open problems about the spectrum of the p-Laplacian than those that have been mentioned here, in passing. To

[8] For *smooth* domains this is credited to Anane, cf. [2].

mention two more: Is every eigenvalue of finite multiplicity? What about multiplicity in the situation with general boundary values? Consider the equation $\operatorname{div}\left(|\nabla u|^{p-2}\,\nabla u\right) + \lambda\,|u|^{p-2}\,u = 0$ with given boundary values, say φ. This has always at least one solution. Does it have several solutions, if λ happens to be an eigenvalue? In the linear case one just adds solutions to see this.

6. THE ASYMPTOTIC CASE

It is instructive to see what happens when $p \to \infty$. Arcane phenomena occur in this fascinating case. Let

$$\lambda(p) = \inf_{\varphi} \frac{\displaystyle\int_{\Omega} |\nabla\varphi|^p dx}{\displaystyle\int_{\Omega} |\varphi|^p dx} = \inf_{\varphi} \frac{\|\nabla\varphi\|_p^p}{\|\varphi\|_p^p}$$

denote the principal frequency and write

$$\Lambda_\infty = \inf_{\varphi} \frac{\|\nabla\varphi\|_\infty}{\|\varphi\|_\infty}, \tag{6.1}$$

where $\varphi \in C_0^\infty(\Omega)$. It turns out that the distance function

$$\delta(x) = \operatorname{dist}(x, \partial\Omega)$$

"solves" the minimization problem:

$$\Lambda_\infty = \frac{\|\nabla\delta\|_\infty}{\|\delta\|_\infty}. \tag{6.2}$$

To see this, notice that

$$|\varphi(x)| \le \|\nabla\varphi\|_\infty \delta(x)$$

by the Mean Value Theorem. Hence

$$\frac{\|\nabla\varphi\|_\infty}{|\varphi(x)|} \ge \frac{1}{\delta(x)} \ge \frac{1}{\|\delta\|_\infty} = \frac{\|\nabla\delta\|_\infty}{\|\delta\|^\infty},$$

since$|\nabla\delta(x)| = 1$ a.e. in Ω. Thus we conclude that

$$\frac{\|\nabla\varphi\|_\infty}{\|\varphi\|_\infty} \ge \frac{\|\nabla\delta\|_\infty}{\|\delta\|_\infty}$$

for each admissible φ. This proves (6.2).

However, the minimization problem often has too many solutions in $W^{1,\infty}(\Omega) \bigcap C(\overline{\Omega})$ with boundary values 0. In order to define the genuine ∞-eigenfunctions, one has to find the limit equation of

$$\operatorname{div}(|\nabla u|^{p-2}\nabla u) + \lambda(p)|u|^{p-2}u = 0$$

as $p \to \infty$. It is shown in [27] that the limit equation is

$$\max\left\{\Lambda_\infty - \frac{|\nabla u(x)|}{u(x)}, \Delta_\infty u(x)\right\} = 0 \tag{6.3}$$

for positive solutions. (At each point x in Ω the larger of the two quantities is equal to zero.) Here

$$\Delta_\infty u = \sum_{i,j=1}^{n} \frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}\frac{\partial^2 u}{\partial x_i \partial x_j} \tag{6.4}$$

is the so called ∞-Laplacian. Unfortunately, the second derivatives of the solutions do not always exist. The above equation has to be interpreted *in the viscosity sense*, because it does not have any weak formulation with test-functions under the integral sign. We refer to [27] about all this.

DEFINITION 10 *Let $u \geq 0$ and $u \in C(\Omega)$. We say that u is a viscosity solution of the equation (6.3), if*

(*i*) *whenever $x_0 \in \Omega$ and $\varphi \in C^2(\Omega)$ are such that $u(x_0) = \varphi(x_0)$ and $u(x) < \varphi(x)$, when $x \neq x_0$, then*

$$\Lambda_\infty - \frac{|\nabla\varphi(x_0)|}{\varphi(x_0)} \geq 0 \quad \textit{or} \quad \Delta_\infty\varphi(x_0) \geq 0.$$

(*ii*) *whenever $x_0 \in \Omega$ and $\varphi \in C^2(\Omega)$ are such that $u(x_0) = \varphi(x_0)$ and $u(x) > \varphi(x)$, when $x \neq x_0$, then*

$$\Lambda_\infty - \frac{|\nabla\varphi(x_0)|}{\varphi(x_0)} \leq 0 \quad \textit{and} \quad \Delta_\infty\varphi(x_0) \leq 0.$$

Notice that each point requires its own family of test-functions.

The essential feature is that the difference $u(x) - \varphi(x)$ attains its extremum at the touching point x_0, where the derivatives of the test-function are to be evaluated.

For example, when Ω is the ball $|x| < 1$, the infinity ground state is

$$u(x) = 1 - |x|.$$

We have $\Delta_\infty u(x) = 0$, when $x \neq 0$. The origin is the important point. Here $\Lambda_\infty = 1$ is determined. Since there are no test-functions touching from below at $x_0 = 0$, condition (ii) is automatically regarded as fullfilled. If the function

$$\varphi(x) = 1 + \langle a, x\rangle + 0(|x|^2)$$

touches from above, we must have

$$1 + \langle a, x\rangle \geq 1 - |x|$$

as $x \to 0$. Hence $|a| \leq 1$ and so

$$\Lambda_\infty - \frac{|\nabla\varphi(0)|}{\varphi(0)} = 1 - \frac{|a|}{1} \geq 0,$$

that is, condition (i) holds.

In passing, let me mention that in a square (cube) the distance function δ does not solve the equation. This means that it is not the limit of the ground states u_p, as $p \to \infty$. Recall (6.1) and (6.2).

As we observed

$$\Lambda_\infty = \frac{1}{\max\limits_{x\in\Omega} \operatorname{dist}(x, \partial\Omega)}. \tag{6.5}$$

Thus the principal frequency can be detected from the geometry: it is the reciprocal number of the radius of the largest ball that can be inscribed in the domain Ω. This is an advantage. For example, if Ω is the punctured ball $0 < |x| < 1$, then $\Lambda_\infty = 1/2$. We should point out that *all boundary points are regular in the case* $p = \infty$. The solution is zero even at isolated boundary points! The equation

$$\max\left\{\Lambda - \frac{|\nabla u(x)|}{u(x)}, \Delta_\infty u(x)\right\} = 0$$

has a positive solution with zero boundary values only when $\Lambda = \Lambda_\infty$. No other Λ will do. In this respect we have a typical eigenvalue.

Let us consider the formula in VI, Section 4.

Lemma 11 $\quad \Lambda_\infty = \lim\limits_{p\to\infty} \sqrt[p]{\lambda(p)}.$

Proof: Using the distance function as test-function in the Rayleigh quotient, we have

$$\sqrt[p]{\lambda(p)} \leq \frac{\|\nabla\delta\|_p}{\|\delta\|_p}$$

and hence

$$\limsup_{p\to\infty} \sqrt[p]{\lambda(p)} \leq \frac{\|\nabla\delta\|_\infty}{\|\delta\|_\infty} = \Lambda_\infty.$$

To achieve the inequality

$$\liminf_{p\to\infty} \sqrt[p]{\lambda(p))} \geq \Lambda_\infty$$

we use a compactness argument for the eigenfunctions u_p. For p large enough

$$\sqrt[p]{\lambda(p)} = \frac{\|\nabla u_p\|_p}{\|u_p\|_p} < \Lambda_\infty + 1.$$

With the normalization $\|u_p\|_p = 1$ the norms $\|\nabla u_p\|_m$ are uniformly bounded, when $p \geq m$. Using a diagonalization procedure, we can select a subsequence u_{p_j} that converges weakly in each $W^{1,q}(\Omega)$, $q < \infty$, and uniformly in each $C^{\alpha}(\Omega)$, $\alpha < 1$, to a function denoted by u_∞. By the weak lower semicontinuity

$$\frac{\|\nabla u_\infty\|_q}{\|u_\infty\|_q} \leq \liminf_{j\to\infty} \frac{\|\nabla u_{p_j}\|_q}{\|u_{p_j}\|_q} \leq \liminf_{j\to\infty} \frac{\|\nabla u_{p_j}\|_{p_j} |\Omega|^{\frac{1}{q}-\frac{1}{p_j}}}{\|u_{p_j}\|_q}$$

$$= \liminf_{j\to\infty} \frac{\lambda(p_j)^{1/p_j} |\Omega|^{\frac{1}{q}-\frac{1}{p_j}}}{\|u_{p_j}\|_q} \leq \frac{|\Omega|^{\frac{1}{q}}}{\|u_\infty\|_q} \liminf_{j\to\infty} \lambda(p_j)^{1/p_j}$$

Taking the normalization into account and letting $q \to \infty$, we obtain

$$\frac{\|\nabla u_\infty\|_\infty}{\|u_\infty\|_\infty} \leq \liminf_{j\to\infty} \lambda(p_j)^{1/p_j}.$$

The left-hand side is $\geq \Lambda_\infty$, because u_∞ is admissible in the quotient. The right-hand side can be replaced by $\liminf \lambda(p)^{1/p}$, since we can begin the construction with an arbitrary sequence of p's. $\square$

Much more is known but there are also challenging open problems in the case $p = \infty$. The interested reader can find some pieces of information in P. Juutinen, P. Lindqvist & J. Manfredi: *The infinity Laplacian: examples and observations*, Institut Mitag-Leffler, Report 26, 1999/2000.

References

[1] G. ALESSANDRINI: On Courant's nodal domain theorem, Preprint (Univ. Trieste, Quaderni Matematici, II Serie, **331**) 1994.

[2] A. ANANE: Simplicité et isolation de la premiére valeur propre du p-laplacien avec poids, C. R. Acad. Sci. Paris **305**, Sér. I Math., 1987, pp. 725–728.

[3] W. ALLEGRETTO & Y. HUANG: A Picone's identity for the p-Laplacian and applications, Nonlinear Analysis **32**, 1998, pp. 819–830.

[4] A. ANANE & N. TSOULI: Sur la seconde valeur propre du p-Laplacien, Université Mohammed I Oujda, Manuscript 1994.

[5] J. GARCÍA AZORERO & I. PERAL ALONSO: Existence and nonuniqueness for the p-Laplacian: Nonlinear eigenvalues, Communications in Partial Differential Equations **12**, 1987, pp. 1389–1430.

[6] J. GARCIA AZORERO & I. PERAL ALONSO: Comportement asymptotique des valeurs propres du p-laplacien, C.R. Acad. Sci. Paris **307**, Série I, 1988, pp. 75–78.

[7] T. BHATTACHARYA: Some results concerning the eigenvalue problem for the p-Laplacian, Annales Academiæ Scientarum Fennicæ, Series A.I. Mathematica, **14**, 1989, pp. 325–343.

[8] T. BHATTACHARYA: A proof of the Faber–Krahn inequality for the first eigenvalue of the p-Laplacian, Annali di Matematica pura ed applicata (IV) **177**, 1999, pp. 225–240.

[9] P. BLANCHARD & E. BRÜNING: Variational Methods in Mathematical Physics, Springer-Verlag, New York 1992.

[10] P. BINDING & P. DRÁBEK & Y. HUANG: On the Fredholm alternative for the p-Laplacian, Proceedings of the American Mathematical Society **125** (1997), pp. 3555–3559.

[11] M. BELLONI & B. KAWOHL: A direct uniqueness proof for equations involving the p-Laplace operator, Manuscripta Mathematica **109** (2002), pp. 229–231.

[12] I. CHAVEL: Eigenvalues in Riemannian Geometry, Academis Press, New York 1984.

[13] L. CAFFARELLI & X. CABRÉ: Fully Nonlinear Elliptic Equations, Colloquium Publications **43**, American Mathematical Society, Providence 1995.

[14] R. COURANT & D. HILBERT: Methoden der Mathematischen Physik, Erster Band, Berlin 1931.

[15] R. COURANT & D. HILBERT: Methods of Mathematical Physics, Volume I, New York 1953.

[16] M. CRANDALL: Viscosity solutions: A primer. In Lecture Notes in Mathematics 1660, Springer-Verlag 1997, pp. 1–43.

[17] E. DIBENEDETTO: $C^{1+\alpha}$ local regularity of weak solutions of degenerate elliptic equations, Nonlinear Analysis **7**, 1983, pp. 827–850.

[18] L. EVANS: Quasiconvexity and partial regularity in the calculus of variations, Archive for Rational Mechanics and Analysis **95**, 1986, pp. 227–252.

[19] L. EVANS: Weak Convergence Methods for Nonlinear Partial Differential Equations, CBMS, Regional Conference Series in Mathematics 74, American Mathematical Society 1990.

[20] L. EVANS & R. GARIEPY: Measure Theory and Fine Properties of Functions, CRC Press, Boca Raton 1992.

[21] L. FRIEDLANDER: Asymptotic behavior of the eigenvalues of the p-Laplacian, Communications in Partial Differential Equations **14**, 1989, pp. 1059–1069.

[22] R. GARIEPY & W. ZIEMER: A regularity condition at the boundary for solutions of quasilinear elliptic equations, Archive for Rational Mechanics and Analysis **67**, 1977, pp. 25–89.

[23] M. GIAQUINTA: Introduction to Regularity Theory for Nonlinear Elliptic Systems, Birkhäuser Verlag, Basel 1993.

[24] D. GILBARG & N. TRUDINGER: Elliptic Partial Differential Equations of Second Order, 2^{nd} Edition, Springer-Verlag, Berlin 1983.

[25] E. GIUSTI: Metodi Diretti nel Calcolo Variazione,Unione Matematica Italiana, Bologna 1994.

[26] L. HEDBERG & T. KILPELÄINEN: On the stability of Sobolev spaces with zero boundary values, Mathematica Scandinavica **85**, 1999, pp. 245–258.

[27] P. JUUTINEN & P. LINDQVIST & J. MANFREDI: The ∞-eigenvalue problem, Archive for Rational Mechanics and Analysis **148**, 1999, pp. 89–105.

[28] B. KAWOHL: Rearrangements and Convexity of Level Sets in PDE (Lecture Notes in Mathematics 1150), Springer-Verlag, Heidelberg 1985.

[29] T. KILPELÄINEN & P. LINDQVIST: Nonlinear ground states in irregular domains. Indiana University Mathematics Journal **49**, 2000, pp. 325–331.

[30] O. LADYZHENSKAYA & N. URAL'TSEVA: Linear and Quasilinear Elliptic Equations, Academic Press, New York 1968.

[31] E. LIEB: On the lowest eigenvalue of the Laplacian for the intersection of two domains, Inventiones Mathematicae **74**, 1983, pp. 441-448.

[32] E. LIEB: The stability of matter: from atoms to stars, Bulletin of the American Mathematical Society **22**, no. 1, 1990, pp. 1–49.

[33] P. LINDQVIST: On the equation div $\left(|\nabla u|^{p-2}\,\nabla u\right) + \lambda\,|u|^{p-2}\,u = 0$, Proceedings of the American Mathematical Society **109**, Number 1, 1990, pp. 157–164. Addendum, ibidem **116**, Number 2, 1992, pp. 583–584.

[34] P. LINDQVIST: Note on a nonlinear eigenvalue problem, Rocky Mountain Journal of Mathematics **23**, 1993, pp. 281–288.

[35] P. LINDQVIST: On non-linear Rayleigh quotients, Potential Analysis **2**, 1993, pp. 199–218.

[36] P. LINDQVIST: A note on the nonlinear Rayleigh quotient, Proceedings of the twenty-first Nordic congress of mathematicians, Lecture notes in pure and applied mathematics, Marcel Dekker, Inc., New York 1994.

[37] P. LINDQVIST: On a nonlinear eigenvalue problem, Berichte Univ. Jyväskylä Math. Inst. **68** (1995), pp. 33–54.

[38] P. LINDQVIST & J. MANFREDI & E. SAKSMAN: Superharmonicity of nonlinear ground states. Revista Matematica Iberoamericana **16**, 2000, pp. 17–28.

[39] V. MAZ'JA: On the continuity at a boundary point of solutions of quasi-linear elliptic equations, Vestnik Leningradskogo Universiteta **25**, 1970, pp. 42–55 (in Russian). Vestnik Leningrad Univ. Math. **3**, 1976, pp. 225–242 (English translation).

[40] M. MARCUS & V. MIZEL & Y. PINCHOVER: On the best constant for Hardy's inequality in $\mathbb{R}^n$, Transactions of the American Mathematical Society **350**, 1998, pp. 3237–3255.

[41] R. OSSERMAN: Isoperimetric inequalities and eigenvalues of the Laplacian, Proceedings of the International Congress of Mathematicians, Helsinki 1978, Volume I, pp. 435–442.

[42] M. ÔTANI: A remark on certain nonlinear elliptic equations, Proceedings of the Faculty of Science, Tokai University **19**, 1984, pp. 23–28.

[43] D. PASSASEO: Nonlinear elliptic equations involving critical Sobolev exponents. Lecture Notes, Padova 2000.

[44] G. PÓLYA: On the characteristic frequences of a symmetric membrane, Math. Z. **63**, 1955, pp. 331–337.

[45] G. PÓLYA & G. SZEGÖ: Isoperimetric Inequalities in Mathematical Physics, Princeton University Press, Princeton 1951.

[46] P. RABINOWITZ: Some aspects of nonlinear eigenvalue problems, Rocky Mountain Journal of Mathematics **3**, 1973, pp. 161–202.

[47] F. RIESZ & B. SZ.-NAGY: Vorlesungen über Funktionalanalysis, Deutscher Verlag der Wissenschaften, Berlin 1956.

[48] S. SAKAGUCHI: Concavity properties of solutions to some degenerate quasilinear elliptic Dirichlet problems, Annali della Scuola Normale Superiore de Pisa, Serie IV (Classe di Scienze) **14**, 1987, pp. 403–421.

[49] S. SOBOLEV: Applications of Functional Analysis in Mathematical Physics, American Mathematical Society, Providence, RI 1963.

[50] M. STRUWE: Variational Methods (Applications to Nonlinear Partial Differential Equations and Hamiltonian Systems), Springer-Verlag, Berlin 1990.

[51] F. de THÉLIN: Quelques résultats d'existence et de non-existence pour une E.D.P. elliptique non linéaire, C.R. Acad. Sci. Par **299**, Série I. Math. 1984, pp. 911–914.

[52] F. de THÉLIN: Sur l'espace propre associé à la première valeur propre du pseudo-laplacien, C.R. Acad. Sci. Paris **303**, Série I. Math. 1986, pp. 355–358.

[53] P. TOLKSDORF: Regularity for a more general class of quasi-linear elliptic equations, Journal of Differential Equations **51**, 1984, pp. 126–150.

[54] A. TOUZANI: Quelques résultats sur le A_p-laplacien avec poids indéfini, Universite Libre de Bruxelles 1991–92. (Thèse.)

[55] N. TRUDINGER: On Harnack type inequalities and their application to quasi-linear elliptic equations, Communications on Pure and Applied Mathematics **20**, 1967, 721–747.

[56] L. VERON & M. GUEDDA: Quasilinear elliptic equations involving critical Sobolev exponents, Nonlinear Analysis TMA **13**, no 8, 1989, pp. 879–902.

[57] W. ZIEMER: Weakly Differentiable Functions, Springer-Verlag, New York 1989.