Disscussing: Infinitesimal Freeness and Connections to Higher Order Freeness, by J. Mingo

Daniel Perales

University of Waterloo, Canada

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Recap

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- If we want to distinguish, then we consider $\varphi'_N(a_{i_1}^{m_1} b_{j_1}^{n_1} \cdots a_{i_k}^{m_k} b_{j_k}^{n_k}) = N(\varphi_N(a_{i_1}^{m_1} b_{j_1}^{n_1} \cdots a_{i_k}^{m_k} b_{j_k}^{n_k}) - \varphi(a_{i_1}^{m_1} b_{j_1}^{n_1} \cdots a_{i_k}^{m_k} b_{j_k}^{n_k}))$

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- We get a infinitesimal probability space $(\mathcal{A}, \varphi, \varphi')$

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- When working with infinitesimal freeness we ‘differentiate’ the formulas:

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- When computing infinitesimal distributions, non-crossing annular permutations appear

GOE

n	2	4	6	8	10
$\lvert NC_2(n)\rvert$	1	2	5	14	42
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Real Wishart

n	1	2	3	4	5	6
$\lvert S_{NC}(n,-n)\rvert$	0	1	6	29	130	562

Connection to orthogonal polynomials

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$$\mu_p := \frac{1}{N} \sum_{i=1}^{N} \delta_{\lambda_i(p)}.$$

Then, the k-th moment of the polynomial p is the k-th moment of μ_p:

$$m_k(p) := \frac{1}{N} \sum_{i=1}^{N} \lambda_i(p)^k.$$

Connection to orthogonal polynomials

A sequence of polynomials $(p_N)_{N=1}^{\infty}$ has an asymptotic distribution if

$$m_k := \lim_{N\to\infty} m_k(p_N) < \infty \qquad \text{for } k \geq 1$$

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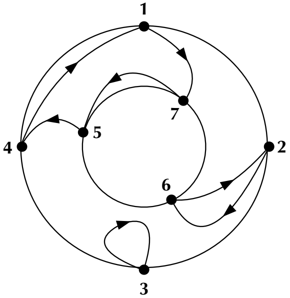
For every degree N there exist natural polynomial convolution $\boxplus_N$ such that we get the free additive convolution on the limit.
There also exist finite free cumulants $(\kappa_j^N)_{j=1}^{\infty}$ that linearize $\boxplus_N$ and tend to the free cumulants.

Asymptotics

Theorem (Terms of order $\Theta(1/d)$)

Let p be a polynomial of degree N and $j \leq N$, then

$$m_j(p) - \sum_{\pi \in NC(j)} \kappa_\pi^N(p) = -\frac{j}{2N} \sum_{\substack{r+s=j \\ \pi \in S_{NC}(r,-s)}} \frac{\kappa_\pi^N(p)}{rs} + O(1/N^2).$$



TR?

Theorem

Let $(u_j)_{j=1}^{\infty}$ and $(v_j)_{j=1}^{\infty}$ be any two sequences of numbers. Then, for any n and any $k = 0, \ldots, n-1$ we have the following decomposition by genus:

$$\frac{(-1)^{n-1}}{(n-1)!} \sum_{\substack{\sigma,\tau \in P(n) \\ \sigma \vee \tau = 1_n \\ \#\sigma + \#\tau = n+1-k}} \mu(0_n, \sigma)\mu(0_n, \tau) u_\sigma v_\tau = (-1)^k \sum_{g=0}^{\lfloor k/2 \rfloor} s_k^{(g)},$$

where

$$s_k^{(g)} = \sum_{\substack{\zeta \vdash n \\ |\zeta| = k+1-2g}} \frac{n}{\prod_{i=1}^{|\zeta|} \zeta_i \prod_{i=1}^{n} t_i^{\zeta}!} \sum_{\alpha \in S_{NC}^{(g)}[\zeta]} u_\alpha v_{\alpha^{-1}\gamma_\zeta}$$

infinitesimal distribution

Theorem (Infinitesimal distributions)

Let μ be a probability measure on $\mathbb{R}$ with compact support and suppose that there is a sequence of polynomials $(p_N)_{N=1}^{\infty}$, such that

$$\kappa_j^N(p_N) = \kappa_j(\mu).$$

Then, the empirical distribution of $(p_N)_{N=1}^{\infty}$ has an infinitesimal asymptotic distribution (μ, μ') with infinitesimal Cauchy transform given by

$$G_{inf}(z) \quad = \quad \frac{G''_\mu(z)}{2G'_\mu(z)} - \frac{G'_\mu(z)}{G_\mu(z)}.$$

or

$$\mu' = \frac{1}{2}(M(M(\mu)) - M(\mu))$$

Main tool: Relate with Second order Cauchy Transform (when $k_{p,q} = 0$)

Lemma

Let μ be a measure with free cumulants $(\kappa_n)_{n=1}^{\infty}$, and let $(\alpha_{r,s})_{r,s\geq 1}$ be the sequence

$$\alpha_{r,s} := \sum_{\pi \in S_{NC}(r,s)} \kappa_\pi .$$

Then the power series $G(z,w) := \frac{1}{zw}\sum_{r,s\geq 1} \alpha_{r,s} z^{-r} w^{-s}$ can be written, at the level of formal power series, in terms of the Cauchy transform of μ as follows

$$G(z,w) = \frac{\partial^2}{\partial z \partial w} \log\left(\frac{G_\mu(w) - G_\mu(z)}{z - w}\right).$$

Example: Hermite polynomials

$$\hat{H}_N(x) := \sum_{k=0}^{\lfloor \frac{N}{2} \rfloor} (-1)^k \frac{(N)_{2k} N^{\frac{N}{2}-k}}{k!2^k} x^{N-2k}.$$

The $\hat{H}_N$ are real-rooted and $\mu_{\hat{H}_N} \xrightarrow[N\to\infty]{} \mu_{sc}$ (semicircle law).

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Moreover, infinitesimal odd moments are 0 are and the even moments are

$$-1, -5, -22, -93, -386, -1586, \ \ldots, \quad m'_{2n} = \frac{1}{2}\left(\binom{2n}{n} - 2^{2n}\right).$$

This closed formula is obtained using annular pair partitions in $S_{NC}(r, -s)$.

Inf. distribution: $$d\mu'(x) = \frac{1}{2}\left(\frac{1}{\pi}\frac{1}{\sqrt{4-x^2}}dx - \frac{\delta_{-2}+\delta_2}{2}\right).$$

Example: Laguerre polynomials

$$L_N^{(\alpha)}(x) := \sum_{k=0}^{N} \frac{(-x)^k (N+\alpha)_{N-k}}{k!(N-k)!}, \qquad \hat{L}_N^{(\lambda)}(x) := L_N^{((\lambda-1)N)}(Nx)$$

The $\hat{L}_N^{(\lambda)}$ are real-rooted and $\hat{L}_N^{(\lambda)} \xrightarrow[N\to\infty]{} \mu_{MP(\lambda)}$ (Marhenko-Pastur).
The infinitesimal moments are

$$m'_n = \frac{1}{2}\left(\sum_{k=0}^{n} \binom{n}{k}^2 \lambda^k - \sum_{k=0}^{n} \binom{2n}{2k} \lambda^k\right) \qquad \forall n \geq 1.$$

For $\lambda = 1$ the infinitesimal moments are

$$-1, -6, -29, -130, -562, -2380, -9949, \ \ldots$$

$$d\mu'(x) = \frac{1}{2}\left(\frac{1}{\pi}\frac{1}{\sqrt{4\lambda - (x-\lambda-1)^2}}dx - \frac{\delta_{(1+\sqrt{\lambda})^2} + \delta_{(1-\sqrt{\lambda})^2}}{2}\right).$$

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Note, the moments only coincide up the infinitesimal distribution:

n	GOE	Hermite polynomial
2	$1 + N^{-1}$	$1 - N^{-1}$
4	$2 + 5N^{-1} + 5N^{-2}$	$2 - 5N^{-1} + 3N^{-2}$
6	$5 + 22N^{-1} + 52N^{-2} + 41N^{-3}$	$5 - 22N^{-1} + 32N^{-2} - 15N^{-3}$

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