Discussion of “Topological recursion, fully simple maps and Hurwitz numbers”

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TR and counting problems

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- These generating functions can be extended to a unique Riemann surface.
- The base steps are disk and cylinder. These can often be calculated by hand.
- The recursion step is in general hard.

TR for maps

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They are related by $x \leftrightarrow y$ duality, similar to moments and free cumulants.

They are also related by a ‘base change’ given by double monotone Hurwitz numbers.

TR in examples

The formula for topological recursion may seem intimidating:

$$\omega_{g,n+1}(z_0, z_{[n]}) = \sum_{a \in \mathrm{Cr}(x)} \operatorname*{Res}_{z=a} \frac{1}{2} \frac{\int_{\bar{z}}^{z} \omega_{0,2}(\cdot, z_0)}{\omega_{0,1}(z) - \omega_{0,1}(\bar{z}} \Big(\omega_{g-1,n+2}(z, \bar{z}, z_{[n]}) + \sum_{\substack{g_1+g_2=g \\ I \sqcup J = [n]}} \omega_{g_1,|I|+1}(z, z_I) \omega_{g_2,|J|+1}(\bar{z}, z_J) \Big)$$

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For dessins d’enfants, it encodes what happens to the dessin after edge contraction.

For simple Hurwitz numbers, it encodes the cut-and-join equation.

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The ordinary TR formula is said to solve the loop equations.

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Bychkov–Dunin-Barkowski–Kazarian-Shadrin used this in their proof of TR for fully simple maps.

They find a more general ordinary $\leftrightarrow$ fully simple duality, but this is not $x \leftrightarrow y$ duality in general.