Topological recursion, fully simple maps and Hurwitz numbers

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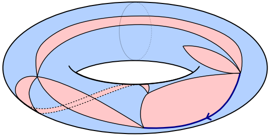

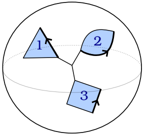

ACPMS mini-workshop, Topological Recursion and Combinatorics, Oslo (online)

November 5, 2021

Based on joint work with...

Main article:

- *Simple maps, Hurwitz numbers and topological recursion*, with G. Borot, Comm. Math. Phys. 380 (2020), 2, math-ph/1710.07851.
- *Relating ordinary and fully simple maps via monotone Hurwitz numbers*, with G. Borot, S. Charbonnier and N. Do, Electron. J. Combin. 26 (2019), 3, math.CO/1904.02267.
- *Topological recursion for fully simple maps from ciliated maps*, with G. Borot and S. Charbonnier, math.CO/2106.09002.

Secondary article:

- *Topological recursion for generalized Kontsevich graphs and r-spin intersection numbers*, with R. Belliard, S. Charbonnier and B. Eynard, math.CO/2105.08035.

Work in progress:

- *From the fully simple duality to the free duality*, with G. Borot, S. Charbonnier, F. Leid and S. Shadrin.

Outline

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Topological recursion (TR, Chekhov–Eynard–Orantin '04-'07)

Goal: "Count surfaces $S_{g,n}$ of genus g with n boundaries (topology (g,n))."

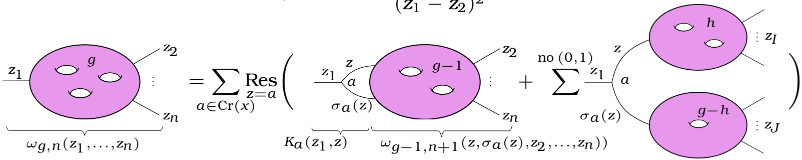
Spectral curve

$$\mathbf{TR}: \begin{cases} \Sigma \text{ Riemann surface} \\ x\colon \Sigma \to \mathbb{CP}^1 \\ \omega_{0,1} = y\,\mathrm{d}x \ \text{1-form (discs)} \\ \omega_{0,2} \ \ (1,1)\text{-form (cylinders)} \end{cases} \rightsquigarrow \text{recursion on } |\chi(S_{g,n})| = 2g-2+n$$

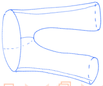
Differential forms $\omega_{g,n}(z_1,\ldots,z_n)$, $z_i \in \Sigma$, $\forall g,n \geq 0$.

- x finitely many simple ramification points $(\mathrm{Cr}(x))$ and y holomorphic around $a \in \mathrm{Cr}(x)$ and $\mathrm{d}y(a) \neq 0 \Rightarrow$ Local involution σ around every ramification point: $x(z) = x(\sigma(z))$.
- $\omega_{0,2}$ symmetric bi-differential on $\Sigma \times \Sigma$ with only double poles along the diagonal and vanishing residues, that is when $z_1 \to z_2$

$$\omega_{0,2}(z_1,z_2) = \frac{\mathrm{d}z_1\mathrm{d}z_2}{(z_1-z_2)^2} + \overbrace{h(z_1,z_2)}^{\text{holomorphic}}.$$



- Terms in correspondence with the ways of cutting a **pair of pants** $(0,3)$ from $S_{g,n}$.



Properties, connections and examples

- Interesting/powerful properties: $\omega_{g,n}$ are symmetric with poles at ramifications points, controlled deformations along families, dilaton equation, symplectic invariance, modularity, integrability...
- Related to many interesting geometric problems: Hurwitz covers, graphs embedded on surfaces, Gromov–Witten invariants, volumes of moduli spaces, statistical physics models, knot invariants, intersection theory of $\overline{\mathcal{M}}_{g,n}$...

- For the Lambert curve $x = ye^{-y}$, TR provides simple Hurwitz numbers.
- For $y = \frac{-\sin(2\pi\sqrt{x})}{2\pi}$, TR gives Mirzakhani's recursion for Weil–Petersson volumes (of the moduli space of bordered hyperbolic surfaces).
- TR on mirror curve of a toric CY3 computes its open Gromov–Witten theory (Bouchard–Klemm–Mariño–Pasquetti, '07), (Fang–Liu–Zong, '16).
- Conjecturally, for the A-polynomial of a knot as a spectral curve, TR computes the colored Jones polynomial of the knot.

Maps

Definition

An *embedded graph* of genus g and n *boundaries* is a connected graph Γ embedded into a closed oriented surface X of genus g such that

$$X \setminus \Gamma \cong \bigsqcup \mathbb{D} \quad \text{(\textit{faces}), with } n \text{ distinguished faces.}$$

A map is an isomorphism class of embedded graphs.

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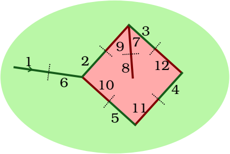

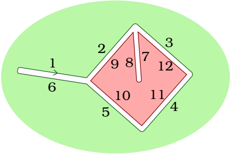

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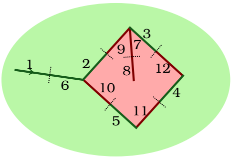

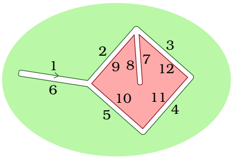

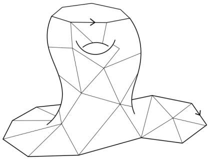

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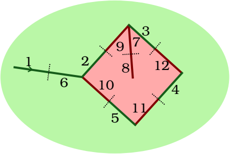

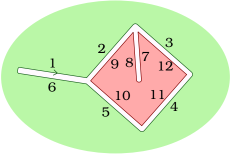

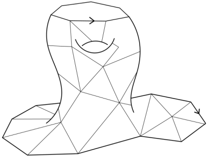

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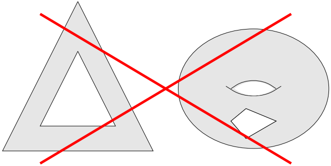

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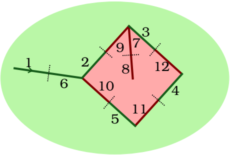

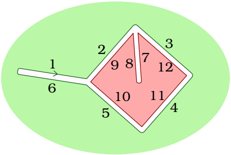

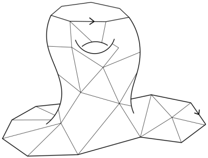

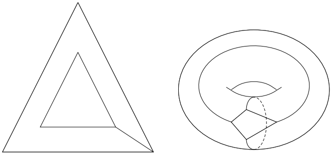

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A bit of history of the origins and context

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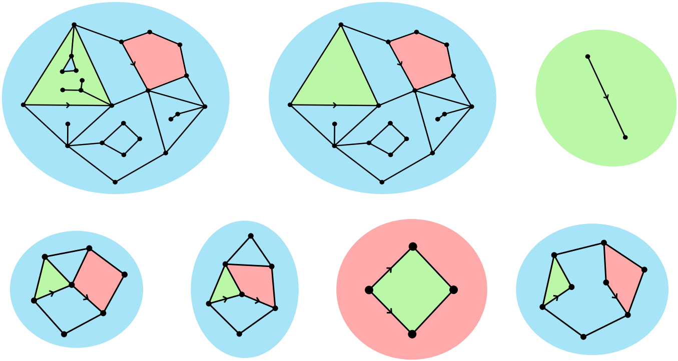
Simple maps

Definition

Simple: Boundaries are simple polygons.

Fully simple: Simple and pairwise disjoint boundaries.



$\mathbb{M}_n^{[g]}(l_1, \ldots, l_n) \rightsquigarrow$ Set of (ordinary) maps of genus g with n boundaries (marked faces) of fixed lengths $l_1, \ldots, l_n$.

Outline

Generating series. Disks and cylinders

Generating series of maps of genus g and n boundaries of lengths $l_1, \dots, l_n$:

$$\mathrm{Map}^{[g]}_{l_1,\dots,l_n} := \sum_{\mathcal{M} \in \mathbb{M}^{[g]}_n(l_1,\dots,l_n)} w(\mathcal{M}), \quad \text{with} \quad w(\mathcal{M}) := \frac{1}{|\mathrm{Aut}\,\mathcal{M}|} \prod_{f \in \mathrm{IFaces}(\mathcal{M})} t_{\mathrm{length}(f)}.$$

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$$W^{[g]}_n(x_1, \ldots, x_n) := \sum_{l_1,\ldots,l_n\geq 0} \frac{\mathrm{Map}^{[g]}_{l_1,\ldots,l_n}}{x_1^{1+l_1}\cdots x_n^{1+l_n}},$$

$$X^{[g]}_n(w_1, \ldots, w_n) := \sum_{k_1,\ldots,k_n\geq 0} \mathrm{FSMap}^{[g]}_{k_1,\ldots,k_n} w_1^{k_1-1}\cdots w_n^{k_n-1}.$$

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Theorem (Borot, G-F, '17)

- ***Disks:*** $X^{[0]}_1(W^{[0]}_1(x)) = x$.
- ***Cylinders:*** *Setting* $x_i = X^{[0]}_1(w_i)$, *or equivalently* $w_i = W^{[0]}_1(x_i)$,

$$\left(W^{[0]}_2(x_1, x_2) + \frac{1}{(x_1 - x_2)^2}\right) dx_1 dx_2 = \left(X^{[0]}_2(w_1, w_2) + \frac{1}{(w_1 - w_2)^2}\right) dw_1 dw_2.$$

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Example (quadrangulations):

$$\mathrm{Map}^{[0]}_4 = 2 + 9t_4 + 54t_4^2 + 378t_4^2 + \dots, \qquad \mathrm{FSMap}^{[0]}_4 = t_4 + 10t_4^2 + 90t_4^3 + \dots$$

Outline

Topological recursion for maps

Initial data:
$$\begin{cases} \Sigma = \mathbb{C}P^1, \\ (x(z) = \alpha + \gamma(z + \frac{1}{z}), y = W_1^{[0]}(x(z))), \\ \omega_{0,1}(z) = y(z)\, \mathrm{d}x(z), \\ \omega_{0,2}(z_1, z_2) = B(z_1, z_2) = \frac{\mathrm{d}z_1 \mathrm{d}z_2}{(z_1 - z_2)^2}. \end{cases}$$

TR output: For $n \geq 1$,

$$\omega_{g,n}(z_1, \dots, z_n) = \operatorname*{Res}_{z \to \pm 1} \frac{\int_{1/z}^{z} \omega_{0,2}(z_1, \cdot)}{2(\omega_{0,1}(z) - \omega_{0,1}(1/z))} \Big(\omega_{g-1,n+1}(z, 1/z, z_{[\![2,n]\!]}) + \sum^{\text{no disk}}_{\substack{0 \leq h \leq g \\ I \sqcup J = [\![2,n]\!]}} \omega_{g-h,1+|I|}(z, z_I) \omega_{h,1+|J|}(1/z, z_J) \Big).$$

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Example ⇝ Quadrangulations: All internal faces are quadrangles, i.e. $t_j = \delta_{j,4} t_j$, with t_4 the weight per internal quadrangle. Spectral curve:

$$x(z) = \gamma\left(z + \frac{1}{z}\right), \quad y(z) = W_1^{[0]}(x(z)) = \frac{1}{\gamma z} - \frac{t_4 \gamma^3}{z^3},$$

with $\gamma = \sqrt{\frac{1 - \sqrt{1 - 12t_4}}{6t_4}} = 1 + \frac{3t_4}{2} + \frac{63}{8} t_4^2 + \frac{891}{16} t_4^3 + \frac{57915}{128} t_4^4 + O(t_4^4).$

Topological recursion for maps

Initial data: $\begin{cases}(x(z)=\alpha+\gamma(z+\frac{1}{z}), y=W_1^{[0]}(x(z))),\ \omega_{0,1}(z)=y(z)\,\mathrm{d}x(z),\\ \omega_{0,2}(z_1,z_2)=B(z_1,z_2)=\frac{\mathrm{d}z_1\mathrm{d}z_2}{(z_1-z_2)^2}.\end{cases}$

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with $\gamma=\sqrt{\frac{1-\sqrt{1-12t_4}}{6t_4}}=1+\frac{3t_4}{2}+\frac{63}{8}t_4^2+\frac{891}{16}t_4^3+\frac{57915}{128}t_4^4+O(t_4^4)$. The zeros of $x'(z)$ are at $z=\pm1$ and the deck transformation is $\sigma(z)=\frac{1}{z}$. One can compute, for example:

$$\omega_{1,1}(z)=\frac{z^3(t_4\gamma^4z^4+z^2(1-5t_4\gamma^4)+t_4\gamma^4)}{\gamma(z^2-1)^5(1-3t_4\gamma^4)^2}\mathrm{d}z=W_1^{[1]}(x(z))dx(z)\Rightarrow$$

$$W_1^{[1]}(x_1)=(t_4+15t_4^2+198t_4^3+\dots)\frac{1}{x_1^3}+(1+15t_4+198t_4^2+2511t_4^3\dots)\frac{1}{x_1^5}$$

$$+(10+150t_4+1980t_4^2+25110t_4^3\dots)\frac{1}{x_1^7}+(70+1190t_4+16590t_4^2+216720t_4^3\dots)\frac{1}{x_1^9}+\dots$$

Symplectic invariance

$$(\Sigma, (x, y)) \overset{\mathsf{TR}}{\rightsquigarrow} \omega_{g,n}(z_1, \ldots, z_n) \; (\omega_{g,0} = \mathfrak{F}_g \in \mathbb{C})$$

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$\Phi = \mathcal{E} \colon (x, y) \mapsto (y, x)$
not well understood.

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Φ preserving $|\mathrm{d}x \wedge \mathrm{d}y|$ $\downarrow$ ... $\|$?

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$\Phi = \mathcal{E}\colon (x,y) \mapsto (y,x)$
not well understood.

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$$(\mathbb{CP}^1, (x, y = W_1^{[0]}(x)), \omega_{0,2} = B)$$

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Maps

$\overset{\mathcal{E}}{\longleftrightarrow}$

Symplectic invariance

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Maps

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Fully simple maps

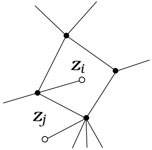
- Our proof: combinatorial, via ciliated maps.
- Proof by Bychkov–Dunin-Barkowsi–Kazarian–Shadrin, '21: via semi-infinite wedge formalism.

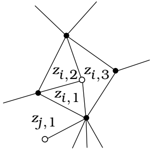
Sketch of the proof (Borot–Charbonnier–G-F, '21)

Work with Belliard–Charbonnier–Eynard, '21: introduce ciliated maps as Feynman graphs for a generalised Kontsevich model and prove TR for them (diagonal correlators of the matrix model with external field).
Motivation: r-spin intersection numbers.

Ciliated and multi-ciliated maps: Degree of black vertices $v \rightsquigarrow 3 \leq d_v \leq r+1$. Max one white vertex per boundary satisfying star constraint (corners around a white vertex belong to pairwise distinct faces). $\{\lambda_1, \ldots, \lambda_N\} \rightsquigarrow$ internal faces.

$\mathcal{C}^{[r]}_{g,n}(z_1, \ldots, z_n) = \mathcal{S}^{[r]}_{g,(1,\ldots,1)}(z_1, \ldots, z_n)$ $\qquad$ $\mathcal{S}^{[r]}_{g,(k_1,\ldots,k_n)}(S_1, \ldots, S_n)$





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simple $\Leftrightarrow$ star constraint; fully $\Leftrightarrow$ uniqueness constraint.

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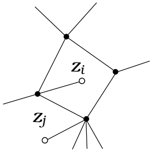

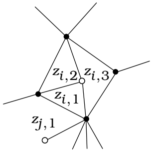

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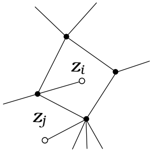

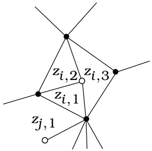

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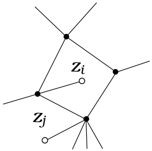

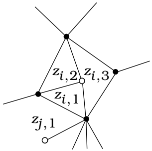

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TR for fully simple maps and motivation

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3-fold motivation:

- In **TR**, fully simple maps implement the **symplectic invariance dual** of ordinary maps.
- Relation to **free probability**.
- TR solves the **enumeration of fully simple maps** (so far only achieved for planar triangulations (Krikun, '07), planar quadrangulations with even boundary lengths (Bernardi–Fusy, '18)).

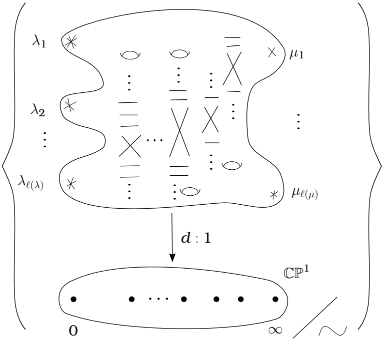
Outline

Double monotone Hurwitz numbers

Definition

Let $k \in \mathbb{Z}_{\geq 0}$. The *double Hurwitz number* $[H_k]_{\lambda,\mu}$ is the number of possibly disconnected coverings of the sphere with ramification profile λ over 0, μ over ∞, and simply ramified over k points in $\mathbb{P}^1 \setminus \{0, \infty\}$, weighted by the size of their automorphism group.

$[H_k]_{\lambda,\mu} := \#$



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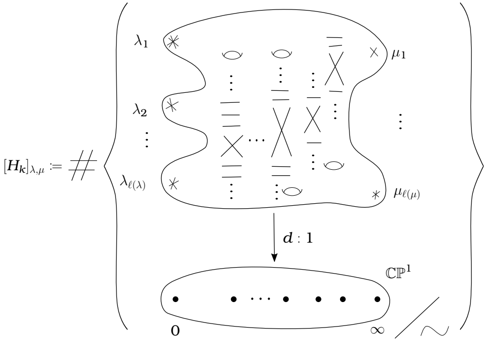

- $C_\lambda \rightsquigarrow$ Conjugacy class in $\mathfrak{S}_d$ of elements of cycle type $\lambda \vdash d$.

$$[H_k]_{\lambda,\mu} = \frac{1}{d!} \left| \{(\sigma, \tau_1, \ldots, \tau_k) \mid \sigma \in C_\lambda, \tau_i \in C_{(2,1\ldots,1)}, \sigma\tau_1 \cdots \tau_k \in C_\mu\} \right| .$$

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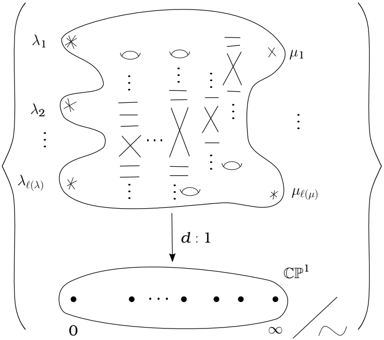

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Consider the transpositions τ_i written as $(a_i\ b_i)$, with $a_i < b_i$, $i = 1, \ldots, k$.

- $b_i \leq b_{i+1} \rightsquigarrow$ *Weakly monotone:* $[H_k^{\leq}]_{\lambda,\mu}$ (Goulden–Guay-Paquet–Novak,'11).
- $b_i < b_{i+1} \rightsquigarrow$ *Strictly monotone:* $[H_k^{<}]_{\lambda,\mu}$.

$$[H^{<}]_{\lambda,\mu} = \sum_{k \geq 0} [H_k^{<}]_{\lambda,\mu} N^{-k} \qquad \text{and} \qquad [H^{\leq}]_{\lambda,\mu} = \sum_{k \geq 0} [H_k^{\leq}]_{\lambda,\mu} N^{-k}.$$

From ordinary to fully simple via Hurwitz numbers

Let $\lambda \vdash d$. Generating series of possibly disconnected maps with $n := \ell(\lambda)$ boundaries of lengths $\lambda_1, \ldots, \lambda_n$:

$$\mathrm{Map}^{\bullet}_{\lambda} := \sum_{\substack{\mathcal{M} \text{ possibly disconnected map} \\ \text{with boundary lengths given by } \lambda}} N^{\chi(\mathcal{M})} w(\mathcal{M}).$$

Analogously for possibly disconnected fully simple maps: $\mathrm{FSMap}^{\bullet}_{\lambda}$.

- $z(\lambda) = \frac{d!}{|C_\lambda|} = \prod_{i=1}^{\ell(\lambda)} \lambda_i \prod_{j \geq 1} m_j(\lambda)!, \quad m_j(\lambda) := |\{j \mid \exists i \in [\![\ell(\lambda)]\!] \text{ such that } \lambda_i = j\}|.$

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Theorem (Borot–G-F, '17, Borot–Charbonnier–Do–G-F, '19)

$$\mathrm{FSMap}^{\bullet}_{\mu} = z(\mu) \sum_{\lambda \vdash d} [H^{\leq}]_{\mu,\lambda}\big|_{N=-N} \mathrm{Map}^{\bullet}_{\lambda}, \tag{1}$$

$$\mathrm{Map}^{\bullet}_{\lambda} = z(\lambda) \sum_{\mu \vdash d} [H^{<}]_{\lambda,\mu} \mathrm{FSMap}^{\bullet}_{\mu}. \tag{2}$$

Proof via matrix models

- $\mathrm{d}\nu \rightsquigarrow$ Unitary invariant measure on the space $\mathcal{H}(N)$ of $N \times N$ hermitian matrices.
- $\gamma = (c_1\ c_2\ \ldots\ c_{\ell(\gamma)})$ cycle in $\mathfrak{S}_N \rightsquigarrow \mathcal{P}_\gamma(M) := \prod_{i=1}^{\ell(\gamma)} M_{c_i,\gamma(c_i)}$.

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Generalization: Any unitary invariant measure $\mathrm{d}\nu$ (*stuffed maps* $\rightsquigarrow$ maps whose internal faces are allowed to have any topology, not only the disk topology).

Proofs via bijective combinatorics (joint work with G. Borot, S. Charbonnier and N. Do)

Definition

A *dessin d'enfant* is a map in which each edge is adjacent to one boundary face and one internal face. In this context, we refer to the boundary faces as blue faces and the internal faces as red faces.

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$D_k(\lambda, \mu) \rightsquigarrow$ number of (possibly disconnected) dessins d'enfant with blue face degrees prescribed by λ and red face degrees by μ, and with k more edges than vertices.

$$D_k(\lambda, \mu) = z(\lambda)[H_k^{<}]_{\lambda,\mu}.$$

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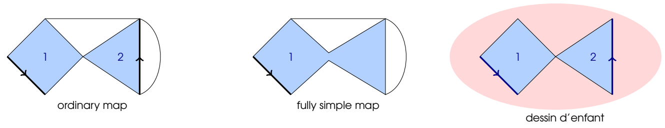

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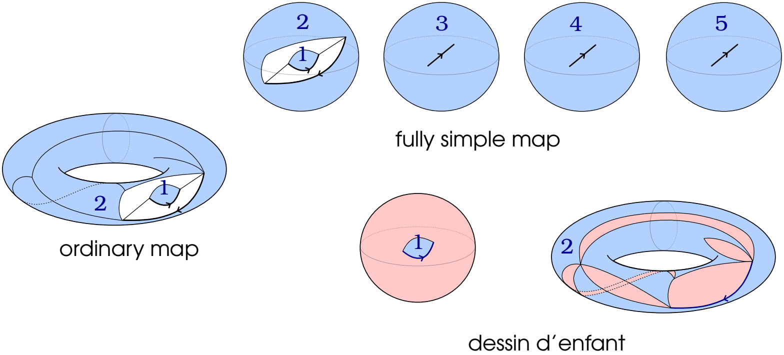

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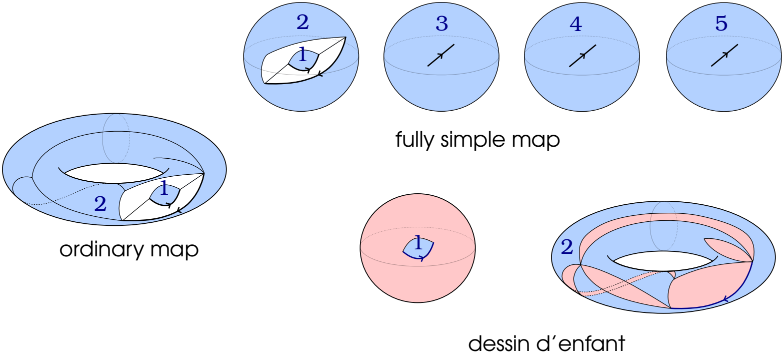

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Generalization: Hypermaps (bicolored maps; marked blue faces ⇝ boundaries, blue faces ⇝ internal faces, red faces ⇝ hyperedges).

Outline

Enumerative interpretation of higher order free cumulants

n-th order moments	$\leftrightarrow$	Ordinary maps
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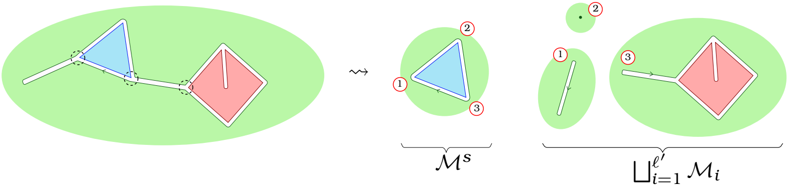
Spoilers of Séverin Charbonnier's talk
(ongoing work with Borot–Charbonnier–Leid–Shadrin):

- R-transform machinery for $n \geq 3$?
- Universality of TR $\rightsquigarrow$ Universal theory of approximate higher order free cumulants taking into account **higher genus** corrections?
- Relation between the definition of (higher order) free cumulants in terms of moments via (non-crossing) partitioned permutations and our formula relating fully simple maps to ordinary maps through (strictly) monotone Hurwitz numbers?

Thank you for your attention!



Tusen takk for oppmerksomheten!