Functional relations for generating functions of higher order moments and cumulants in free probability

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Topological recursion and combinatorics
Online Mini-workshop
5th November 2021

Introduction: free probabilities, combinatorics and moment-cumulants

Free probabilities

Moments $\leftrightarrow$ Cumulants

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Combinatorics

Maps $\leftrightarrow$ fully-simple maps

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Enumerative geometry
Hurwitz numbers

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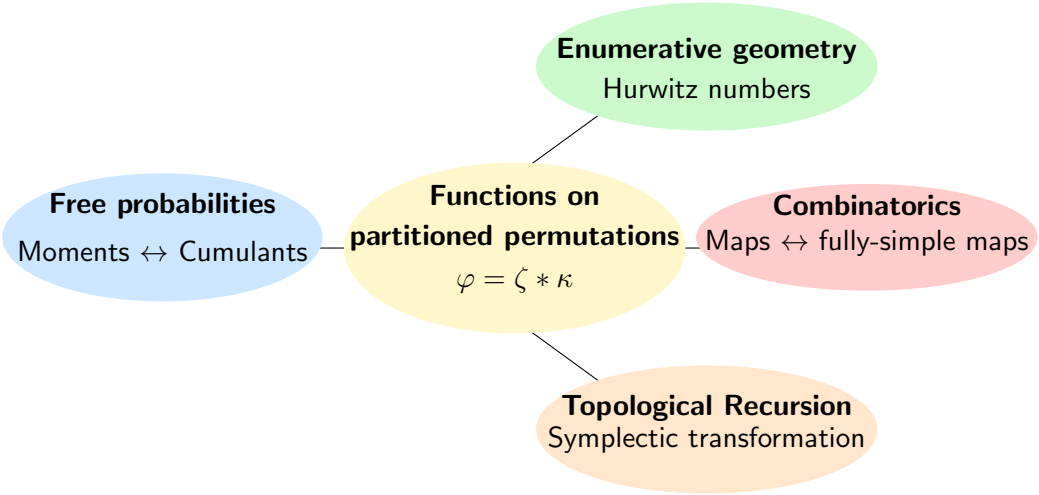

Plan

1. Functions on partition permutations: definitions and problematic
   - Definitions
   - Problematic
   - Functional relations: results
2. Relations with free probabilities, combinatorics and topological recursion
   - Free probabilities: higher order moments and free cumulants
   - Combinatorics: maps and fully-simple maps
   - Topological recursion
3. Functional relations for higher moments and cumulants
   - Formula and example
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Partitioned permutations

Let $n \in \mathbb{Z}_{\geq 0}$. Following [Collins–Mingo–Śniady–Speicher '07], set:

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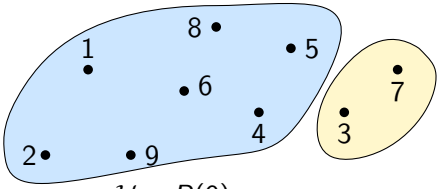

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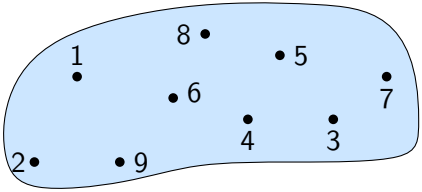

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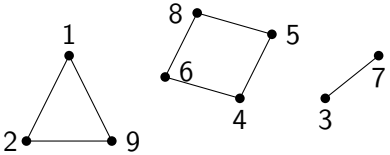
$$\gamma = (1, 2, 9)(4, 5, 8, 6)(3, 7)$$



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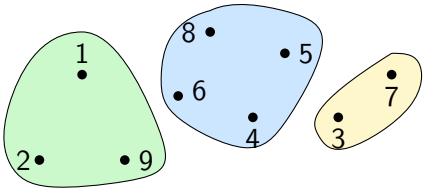

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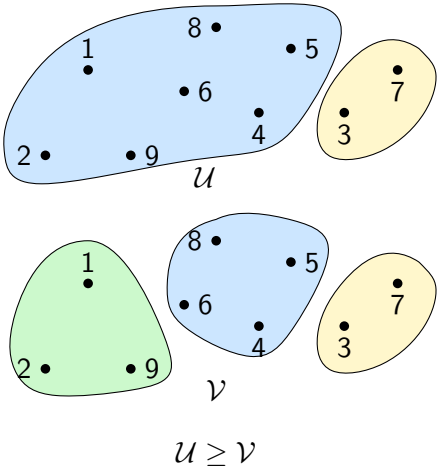

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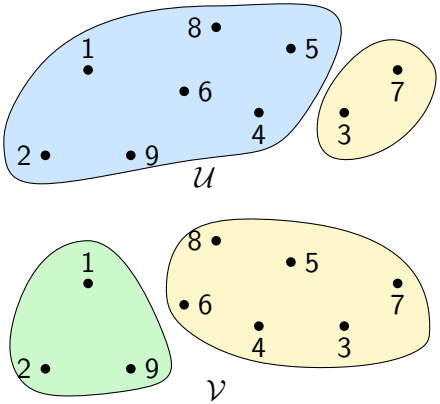

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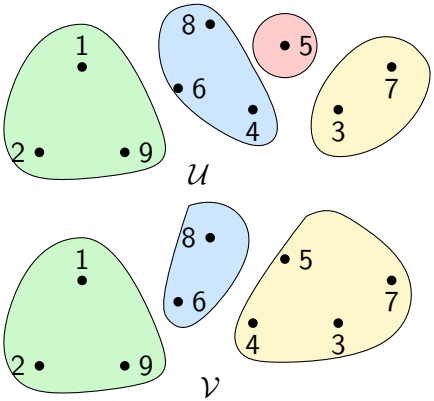

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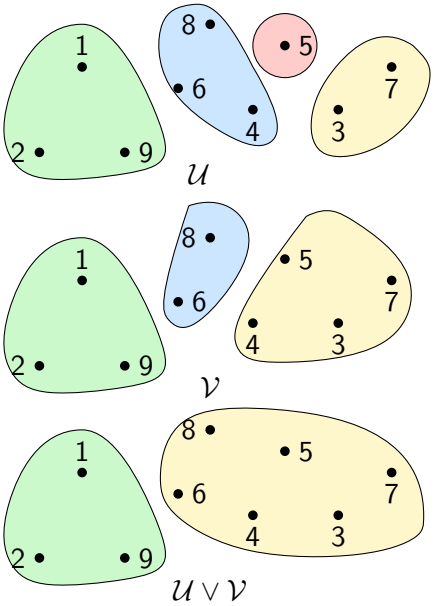

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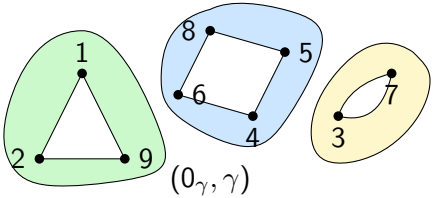
Definition (Partitioned permutations)

$(\mathcal{U}, \gamma)$ is a partitioned permutation on n elements:

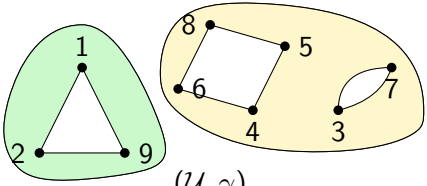
$$\mathcal{U} \in P(n), \quad \gamma \in S(n), \quad \text{and} \quad \mathcal{U} \geq 0_\gamma.$$

$PS(n)$ is the set of partitioned permutations on n elements; $PS := \bigcup_{n \geq 0} PS(n)$

$\gamma = (1, 2, 9)(4, 5, 8, 6)(3, 7)$



$(0_\gamma, \gamma)$



$(\mathcal{U}, \gamma)$

Product on $PS(n)$

The length of $(\mathcal{U}, \gamma) \in PS(n)$ is

$$|(\mathcal{U}, \gamma)| := n + \#\text{cycles of } \gamma - 2\#\text{blocks of } \mathcal{U}.$$

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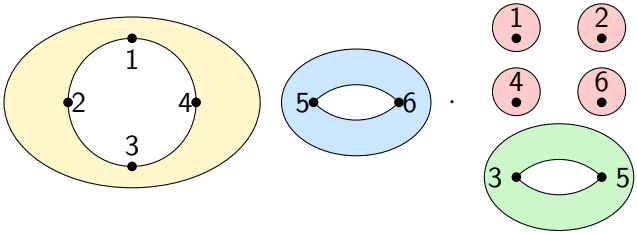

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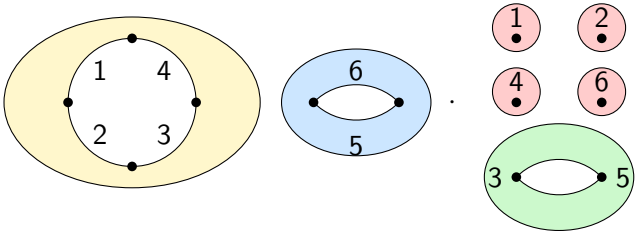

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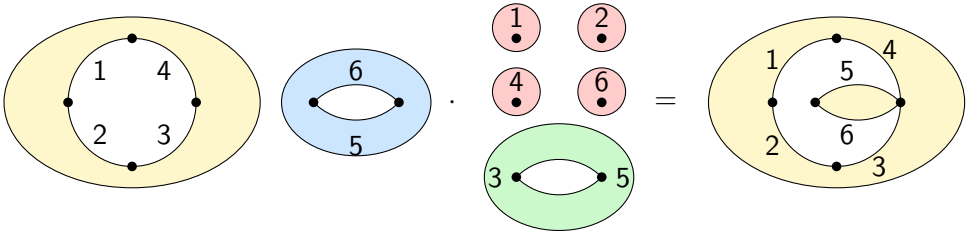

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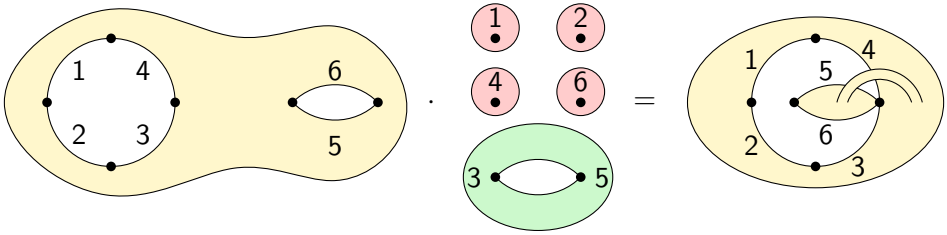

$|(1_6, \gamma)| = 6$ $\qquad$ $|(0_\pi, \pi)| = 1$ $\qquad$ $|(0_{\gamma\pi}, \gamma\pi)| = 5$

Functions on PS

Definition (Convolution of functions)

Let $f, g : PS \to \mathbb{C}$. Their convolution is

$$(f * g)(\mathcal{U}, \gamma) = \sum_{(\mathcal{V},\pi)\cdot(\mathcal{W},\sigma)=(\mathcal{U},\gamma)} f(\mathcal{V}, \pi) g(\mathcal{W}, \sigma).$$

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Convolution with ζ and generating series

Let $f : PS \to \mathbb{C}$ multiplicative. Define $f_{\ell_1,\dots,\ell_n} := f(1_{\ell_1+\dots+\ell_n}, \overbrace{\gamma_1}^{\text{cycle of size } \ell_1} \dots \gamma_n)$.

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Assumption

Let $\varphi, \kappa : PS \to \mathbb{C}$ be multiplicative functions such that $\varphi = \zeta * \kappa$.

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Encode $\varphi_{\ell_1,\dots,\ell_n}$ and $\kappa_{\ell_1,\dots,\ell_n}$ into the generating series:

$$M(x) := 1 + \sum_{\ell \geq 1} \varphi_\ell x^\ell, \qquad C(w) := 1 + \sum_{\ell \geq 1} \kappa_\ell w^\ell,$$

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$$M(x_1,\dots,x_k) := \sum_{\ell_1,\dots,\ell_k \geq 1} \varphi_{\ell_1,\dots,\ell_k} x_1^{\ell_1} \dots x_k^{\ell_k}, \quad C(w_1,\dots,w_k) := \sum_{\ell_1,\dots,\ell_k \geq 1} \kappa_{\ell_1,\dots,\ell_k} w_1^{\ell_1} \dots w_k^{\ell_k}$$

Question

Given $k \geq 1$, is there a functional relation between $M(x_1,\dots,x_k)$ and $C(w_1,\dots,w_k)$?

Functional relation for $k = 1$

Theorem [Speicher '94]

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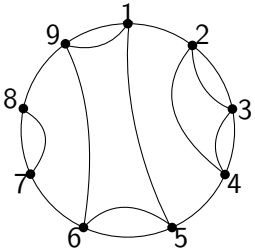

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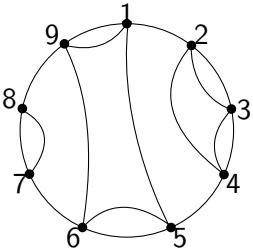
Proof in [Speicher '94] relies on the tree structure of $NC(n)$ (non-crossing permutations):

$$\pi = (1,5,6,9)(2,3,4)(7,8)$$
$$\pi \in NC(9)$$



Functional relation for $k = 1$

Theorem [Speicher '94]

$$C(x\,M(x)) = M(x)$$

Proof. Meaning of $\varphi = \zeta * \kappa$ for $k = 1$:

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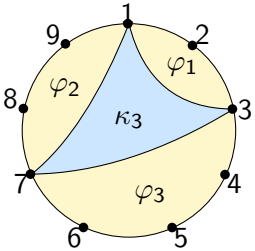

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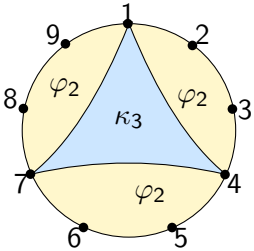

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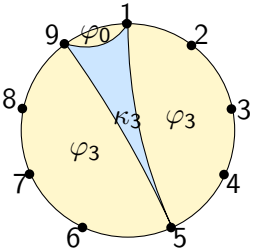

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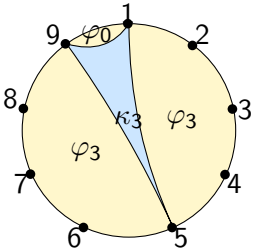

$$= \sum_{r=0}^{\ell} \kappa_r \sum_{\ell_1+\cdots+\ell_r=\ell-r} \varphi_{\ell_1}\cdots\varphi_{\ell_r}$$

At the level of generating functions, this identity translates as $C(xM(x)) = M(x)$. □

Functional relation for $k = 2$

Theorem [Collins–Mingo–Śniady–Speicher '07]

$$M(x_1, x_2) = \left[C(w_1, w_2) - w_1 w_2 \frac{\partial^2}{\partial w_1 \partial w_2} \log \left(\frac{w_1 C(w_1) - w_2 C(w_2)}{w_1 - w_2} \right) \right] \Bigg|_{w_i = x_i M(x_i)}$$
$$\times \frac{1}{M(x_1) M(x_2)} \frac{\partial}{\partial x_1} (x_1 M(x_1)) \frac{\partial}{\partial x_2} (x_2 M(x_2))$$

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Solutions of $(\mathcal{V}, \pi) \cdot (\mathcal{W}, \sigma) = (1_{\ell_1, \ell_2}, \gamma_{\ell_1} \gamma_{\ell_2})$ are "$(1_{\ell_1 + \ell_2}, \gamma_1 \gamma_2)$-non-crossing", they are of 2 types. Sort the sums by handling those 2 types. □

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And for $k \geq 3$?

The number of types of $(1_{\ell_1 + \cdots + \ell_k}, \gamma_1 \ldots \gamma_k)$-non-crossing partitioned permutations grows quickly $\Longrightarrow$ the proof is hard to generalise.

Main result – teasing

Theorem [Borot–C.–Garcia-Failde–Leid–Shadrin +21]

Consider the change of variables $x = w/C(w)$. For $k \geq 3$, we have:

$$M(x_1, \dots, x_k) = \sum_{t \in T_k} \sum_{j_1=0}^{d(v_1)-1} \cdots \sum_{j_k=0}^{d(v_k)-1} \left(\prod_{i=1}^{k} \left(\frac{w_i}{C(w_i)x'(w_i)} \frac{\partial}{\partial w_i} \right)^{j_i} \underbrace{[v_i^{j_i}]}_{\text{Coeff of } v_i^{j_i}} \frac{\left(\frac{\partial}{\partial y} + \frac{v_i}{1+y} \right)^{d(v_i)-1} 1}{C(w_i)x'(w_i)} \Big|_{y=C(w_i)-1} \right) \mathfrak{C}(t)$$

- T_k are trees with multiedges, with k labeled vertices;
- $d(v_i)$ is the degree of the vertex i;
- $\mathfrak{C}(t)$ is expressed in terms of the generating series $C(w_{i_1}, \dots, w_{i_m})$.

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Relation to free probabilities

Definition (HOPS)

$(\mathcal{A}, \varphi)$ is a higher order probability space if:

- $\mathcal{A}$ is a tracial algebra;
- $\varphi = (\varphi_n)_{n \geq 1}$ where $\varphi_n : \mathcal{A}^n \to \mathbb{C}$ is linear and tracial, and s.t. $\varphi_1(1) = 1$, $\varphi_n(1, a_2, \ldots, a_n) = 0$ for $n \geq 2$.

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For $1 \leq j \leq k$, set $L_j = \sum_{i=1}^{j} \ell_j$. The moments are multiplicative functions defined by:

$$\varphi(1_{\ell_1+\cdots+\ell_k}, \gamma_1 \ldots \gamma_k)[a_1, \ldots, a_{\ell_1+\cdots+\ell_k}] := \varphi_k(a_1 \ldots a_{\ell_1}, \ldots, a_{L_{k-1}+1} \ldots a_{L_k})$$

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If $\varphi_n = 0$ for $n \geq 2$: recover first order freeness.

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The functional relations are moment-cumulant functional relations.

Combinatorics: formal hermitian matrix models

$\mathcal{H}_N$: set of $N \times N$ hermitian matrices. Set $V(u) = \frac{u^2}{2} - \sum_{j=3}^{d} \frac{t_j}{j} u^j$ and the measure on $\mathcal{H}_N$:

$$\mathrm{d}\mu(A) = \frac{1}{\mathcal{Z}_0} e^{-\alpha N \mathrm{Tr} V(A)} \, \mathrm{d}A \qquad \text{where} \qquad \mathcal{Z}_0 = \int_{\mathcal{H}_N} \mathrm{d}A e^{-\alpha N \mathrm{Tr} \frac{A^2}{2}}.$$

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From free probabilities to combinatorics *via* matrix models

Matrix model and free probabilities

$$\varphi_{\ell_1,\dots,\ell_k} = \lim_{N\to\infty} N^{k-2} c_k^{(N)}(\operatorname{Tr} A^{\ell_1},\dots,\operatorname{Tr} A^{\ell_k})$$

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Remark: for more general multi-tracial hermitian measures, relation with stuffed maps.

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Free probabilities to maps

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Topological recursion

(see Elba Garcia-Failde's talk)

Conjecture in [Borot–Garcia-Failde '17]:

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Generating series of $\mathrm{Map}^{[g]}_{(\ell_1,\dots,\ell_k)}$ and $\mathrm{FSMap}^{[g]}_{(\ell_1,\dots,\ell_k)}$ satisfy topological recursion with spectral curves $\mathcal{S} = (x, y)$ and $\check{\mathcal{S}} = (y, x)$ respectively.

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As a consequence, one obtains the relation for $k = 3$ in the case of HOPS arising from unitary-invariant hermitian matrix models.
Consider the change of variable $x = \frac{w}{C(w)}$ ($\Leftrightarrow w = xM(x)$):

$$M(x_1, x_2, x_3) = \frac{C(w_1, w_2, w_3)}{\prod_{i=1}^{3} C(w_i)x'(w_i)} + \sum_{i=1}^{3} \frac{1}{C(w_i)x'(w_i)} w_i \frac{\partial}{\partial w_i} \frac{\prod_{j\neq i}\left(C(w_i, w_j) + \frac{w_i w_j}{(w_i - w_j)^2}\right)}{\prod_{m=1}^{3} C(w_m)x'(w_m)}$$

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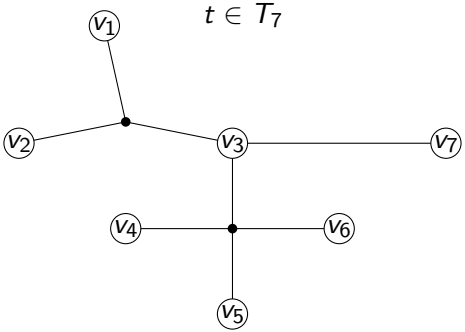
The functional relation: preliminaries

Definition

For $k \geq 1$, T_k denotes the set of trees t with k labeled vertices $v_1, \ldots, v_k$ and with multi-edges $e \in E(t)$.



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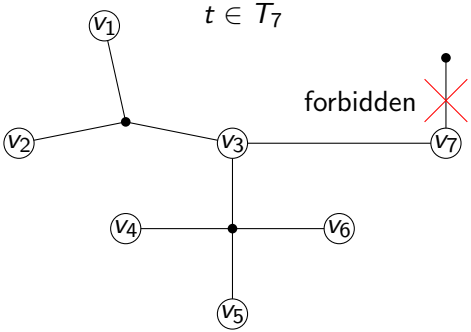

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Let $t \in T_k$ and e be a multi-edge of t, adjacent to the vertices $v_{i_1}, \ldots, v_{i_m}$. The weight $\mathbf{C}(e)$ of e is given by:

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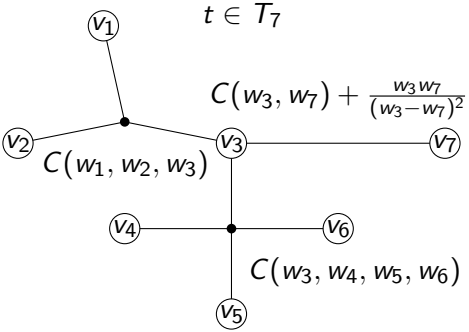

The functional relation: preliminaries

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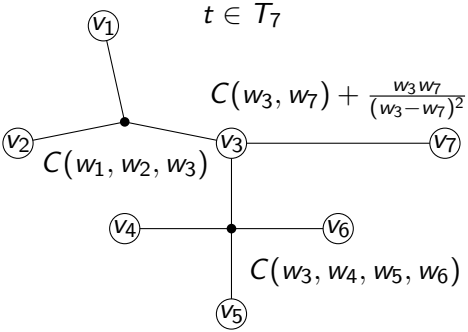
The weight of t is then:

$$\mathfrak{C}(t) := \prod_{e \in E(t)} \mathbf{C}(e)$$

$t \in T_7$



$$\mathfrak{C}(t) = C(w_1, w_2, w_3) C(w_3, w_4, w_5, w_6) \times \left(C(w_3, w_7) + \frac{w_3 w_7}{(w_3 - w_7)^2} \right)$$

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The degree of v_i is noted $d(v_i)$.

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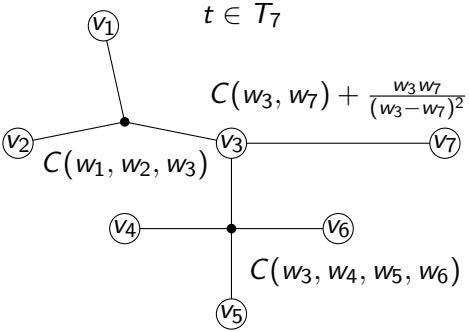

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$$d(v_3) = 3, \ d(v_i) = 1 \ i \neq 3$$

The functional relation: main result

Theorem [Borot–C.–Garcia-Failde–Leid–Shadrin +21]

Consider the change of variables $x = w/C(w)$ ($\Leftrightarrow w = xM(x)$). For $k \geq 3$, we have:

$$M(x_1, \dots, x_k) = \sum_{t \in T_k} \sum_{j_1=0}^{d(v_1)-1} \cdots \sum_{j_k=0}^{d(v_k)-1} \left(\prod_{i=1}^{k} \left(\frac{w_i}{C(w_i)x'(w_i)} \frac{\partial}{\partial w_i} \right)^{j_i} \underbrace{[v_i^{j_i}]}_{\text{Coeff of } v_i^{j_i}} \frac{\left(\frac{\partial}{\partial y} + \frac{v_i}{1+y} \right)^{d(v_i)-1} 1}{C(w_i)x'(w_i)} \Big|_{y=C(w_i)-1} \right) \mathfrak{C}(t)$$

Remark: the sums are finite.

The functional relation: example

For $k = 3$, the l.h.s. is $M(x_1, x_2, x_3)$.

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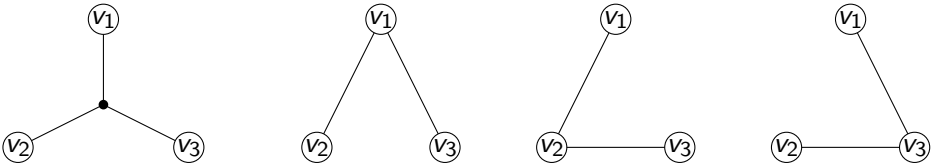
For $k = 3$, the l.h.s. is $M(x_1, x_2, x_3)$. On the r.h.s, T_3 has 4 elements:



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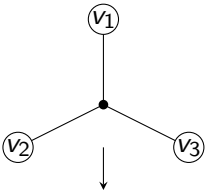
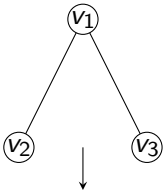


$j_1 = j_2 = j_3 = 0$



$j_1 = 1$
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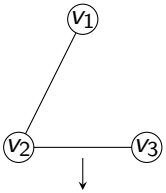

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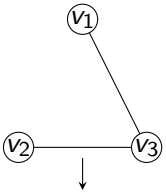

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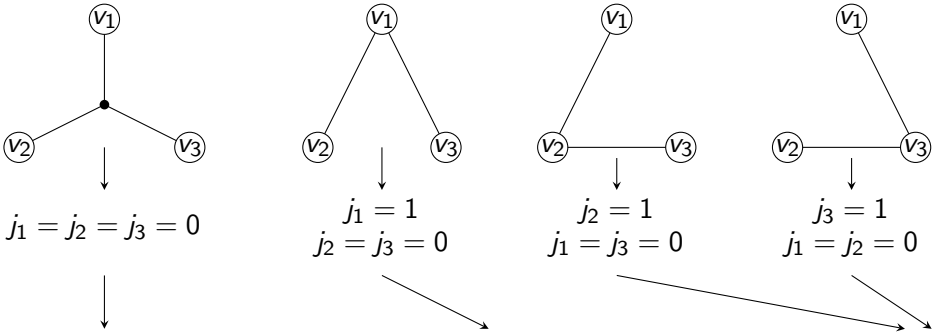

$$\frac{C(w_1,w_2,w_3)}{\prod_{m=1}^{3} C(w_m)x'(w_m)} \qquad \frac{1}{C(w_1)x'(w_1)} w_1 \frac{\partial}{\partial w_1} \frac{\left(C(w_1,w_2)+\frac{w_1 w_2}{(w_1-w_2)^2}\right)\left(C(w_1,w_3)+\frac{w_1 w_3}{(w_1-w_3)^2}\right)}{\prod_{m=1}^{3} C(w_m)x'(w_m)} + \circlearrowleft$$

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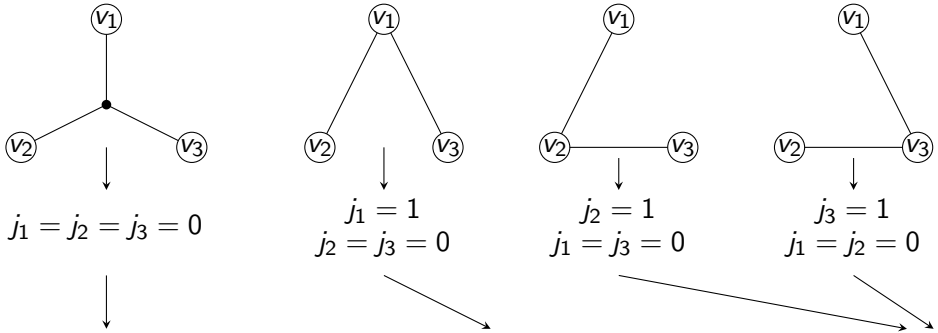

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$$M(x_1, x_2, x_3) = \frac{C(w_1, w_2, w_3)}{\prod_{i=1}^{3} C(w_i)x'(w_i)} + \sum_{i=1}^{3} \frac{1}{C(w_i)x'(w_i)} w_i \frac{\partial}{\partial w_i} \frac{\prod_{j\neq i}\left(C(w_i, w_j) + \frac{w_i w_j}{(w_i-w_j)^2}\right)}{\prod_{m=1}^{3} C(w_m)x'(w_m)}$$

Sketch of the proof

The proof has 2 main steps.

Step 1

There exist vectors Φ, K indexed by partitions of integers with value in $\mathbb{C}[[\hbar]]$ s.t.

$$\varphi = \zeta * \kappa \quad \Leftrightarrow \quad \forall n \geq 1,\ \forall \lambda \vdash n, \quad \Phi_\lambda = z(\lambda) \sum_{\mu \vdash n} [H^<]_{\lambda,\mu} K_\mu.$$

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Step 2

Apply the theorems of Bychkov–Dunin-Barkowski–Kazarian–Shadrin '20 , '21: If $\Phi_\lambda = z(\lambda) \sum_{\mu \vdash n} [H^<]_{\lambda,\mu} K_\mu$ for all $\lambda \vdash n$, then the relations between $M(x_1, \ldots, x_k)$ and $C(w_1, \ldots, w_k)$ hold for $k \geq 3$.

Remark: the techniques employed for the second step involve identities of operators on the bosonic Fock space $\mathcal{F} = \mathbb{C}((\hbar))[[p_1, p_2, p_3, \ldots]]$.

Focus on step 1

Let $n \geq 1$, $\lambda \vdash n$. Define Φ_λ, $K_\lambda \in \mathbb{C}[[\hbar]]$ as:

$$\Phi_\lambda := \sum_{\substack{\mathcal{V} \in P(n) \\ \mathcal{V} \geq \Lambda}} \hbar^{|(\mathcal{V},\gamma_\lambda)|} \varphi(\mathcal{V}, \gamma_\lambda), \qquad K_\lambda := \sum_{\substack{\mathcal{V} \in P(n) \\ \mathcal{V} \geq \Lambda}} \hbar^{|(\mathcal{V},\gamma_\lambda)|} \kappa(\mathcal{V}, \gamma_\lambda)$$

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Given $\lambda, \mu \vdash n$, let $[H_r^<]_{\lambda,\mu}$ be the strictly monotone double Hurwitz number (see Elba Garcia-Failde's talk) and

$$[H^<]_{\lambda,\mu} := \sum_{r=0}^{n-1} \hbar^r [H_r^<]_{\lambda,\mu}.$$

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Lemma [Borot–C.–Garcia-Failde–Leid–Shadrin +21]

$$\varphi = \zeta * \kappa \quad \Leftrightarrow \quad \forall n \geq 1, \forall \lambda \vdash n, \quad \Phi_\lambda = z(\lambda) \sum_{\mu \vdash n} [H^<]_{\lambda,\mu} K_\mu$$

where $z(\lambda) = \frac{n!}{|C(\lambda)|}$.

Focus on step 1: remarks

Remark 1: the inspiration for this lemma comes from combinatorics.

Lemma [Borot–Garcia-Failde '17, Borot–C.–Do–Garcia-Failde '19]

For all $\lambda \vdash n$,

$$\begin{array}{ccc} \mathrm{Map}^{\bullet}_{\lambda} & = z(\lambda) \sum_{\mu \vdash n} [H^{<}]_{\lambda,\mu} & \mathrm{FSMap}^{\bullet}_{\mu} \\ \updownarrow & & \updownarrow \\ \Phi_{\lambda} & = z(\lambda) \sum_{\mu \vdash n} [H^{<}]_{\lambda,\mu} & K_{\mu} \end{array}$$

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Remark 2: the lemma is the genus 0 part of a stronger statement. φ, ζ, κ can be defined as functions on surfaced permutations $(\mathcal{U}, \gamma, g) \in PSG(n)$ with $g : \mathcal{U} \to \mathbb{Z}_{\geq 0}$. The convolution is extended to $\varphi = \zeta \star \kappa$; Φ is changed to:

$$\Phi_{\lambda} = \sum_{\mathcal{V} \geq \Lambda} \sum_{g : \mathcal{V} \to \mathbb{Z}_{\geq 0}} \hbar^{|(\mathcal{V}, \gamma_{\lambda})| + \sum_{v \in \mathcal{V}} 2g(v)} \varphi(\mathcal{V}, \gamma_{\lambda}, g)$$

Theorem [Borot–C.–Garcia-Failde–Leid–Shadrin +21]

$$\varphi = \zeta \star \kappa \quad \Leftrightarrow \quad \forall n \geq 1, \forall \lambda \vdash n, \quad \Phi_{\lambda} = z(\lambda) \sum_{\mu \vdash n} [H^{<}]_{\lambda,\mu} K_{\mu}.$$

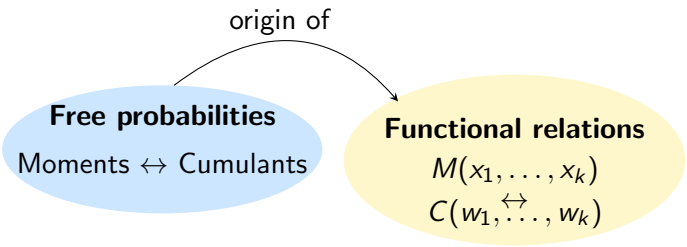
Conclusion

Functional relations

$$M(x_1, \ldots, x_k)$$

$$C(w_1, \overset{\leftrightarrow}{\ldots}, w_k)$$

Conclusion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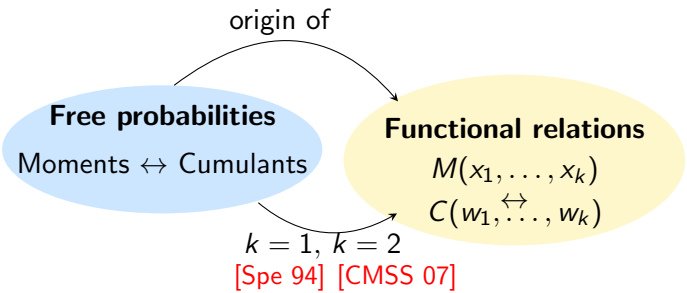
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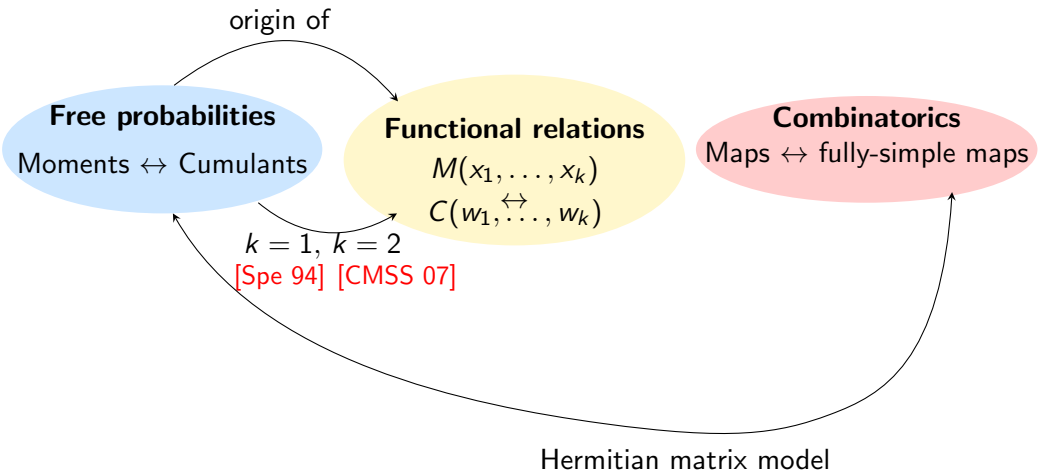
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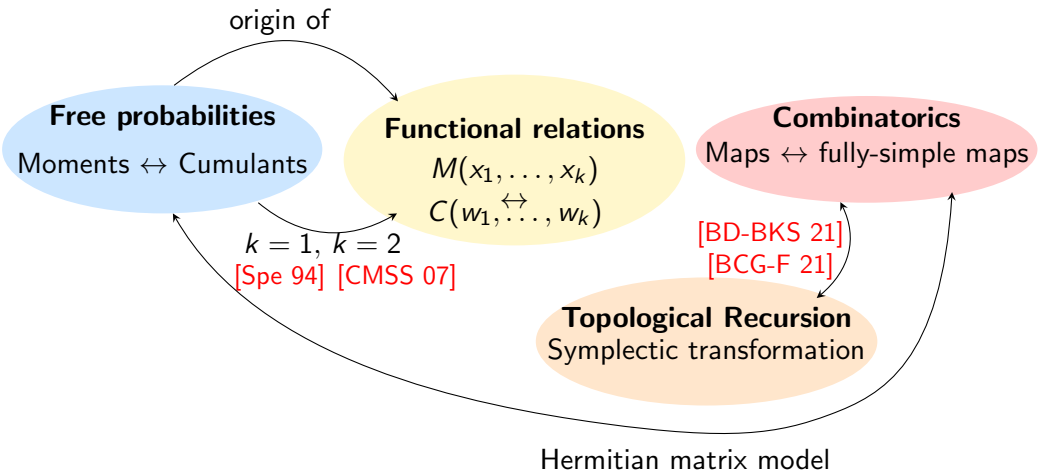
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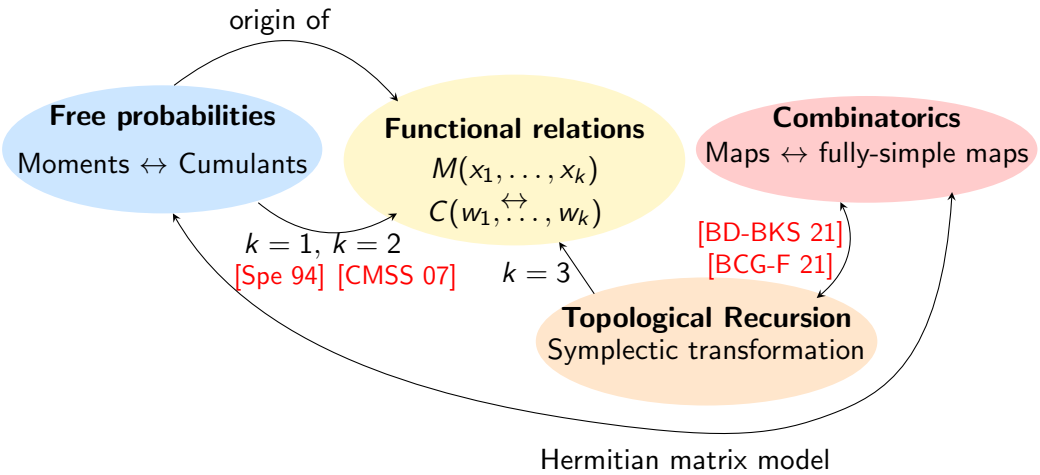
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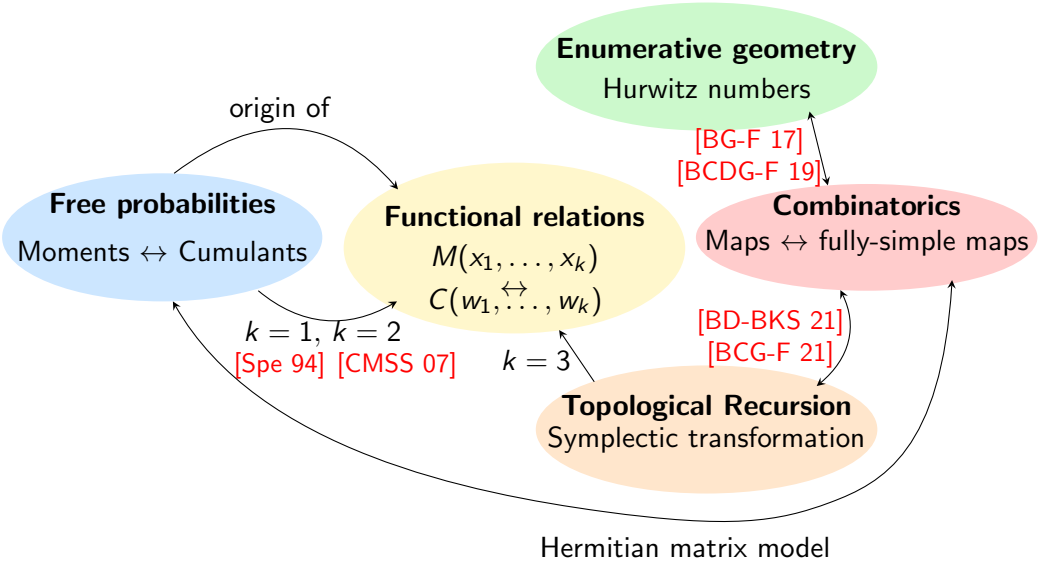
Enumerative geometry
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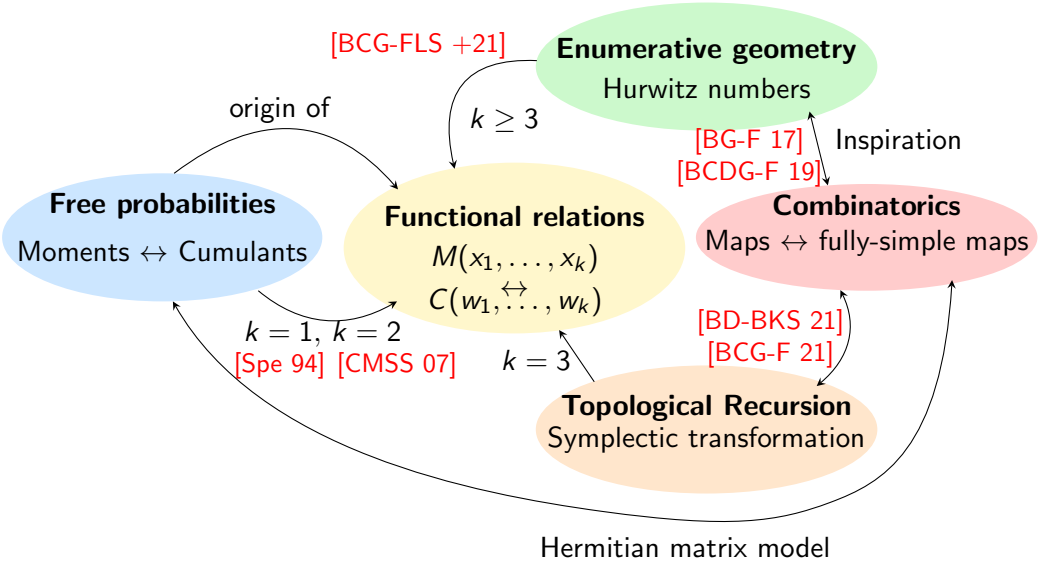
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Future explorations:

- We define a product on $PSG(n)$, a convolution of functions on PSG. Does it pave the way for higher genus freeness?
- At the level of maps, the duality between ordinary and fully-simple maps is related to symplectic invariance of TR for a family of spectral curves. More generally, for moments and free cumulants: is the relation $\varphi = \zeta * \kappa$ deeply connected with symplectic invariance of TR?

Thank you for your attention!