On the Radius of Convergence of Interconnected Analytic Nonlinear Systems*

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*See `www.ece.odu.edu/~sgray/RPCCT2011/thitsaslides.pdf`

1. Introduction

- For each $c \in \mathbb{R}^{\ell}\langle\langle X\rangle\rangle$, one can associate an m-input, ℓ-output operator F_c in the following manner:

  - With $t_0, T \in \mathbb{R}$ fixed and $T > 0$, define recursively for each $\eta \in X^*$ the mapping $E_\eta : L_1^m[t_0, t_0 + T] \to \mathcal{C}[t_0, t_0 + T]$ by

$$E_{x_i\bar{\eta}}[u](t, t_0) = \int_{t_0}^{t} u_i(\tau)E_{\bar{\eta}}[u](\tau, t_0)\, d\tau,$$

    where $E_\emptyset = 1$, $x_i \in X$, $\bar{\eta} \in X^*$ and $u_0(t) \equiv 1$.

  - The input-output operator corresponding to c is the Fliess operator

$$y = F_c[u](t) = \sum_{\eta\in X^*} (c, \eta)\, E_\eta[u](t, t_0).$$

- If there exist real numbers $K_c, M_c > 0$ such that

$$|(c,\eta)| \leq K_c M_c^{|\eta|} |\eta|!, \quad \forall\, \eta \in X^*, \tag{1}$$

where $|\eta|$ denotes the number of symbols in η, then c is said to be locally convergent. The set of all such series is denoted by $\mathbb{R}^{\ell}_{LC}\langle\langle X \rangle\rangle$.

- If $c \in \mathbb{R}^{\ell}_{LC}\langle\langle X \rangle\rangle$ then

$$F_c : B^m_{\mathfrak{p}}(R)[t_0, t_0+T] \to B^{\ell}_{\mathfrak{q}}(S)[t_0, t_0+T]$$

for sufficiently small $R, T > 0$, where the numbers $\mathfrak{p}, \mathfrak{q} \in [1, \infty]$ are conjugate exponents, i.e., $1/\mathfrak{p} + 1/\mathfrak{q} = 1$ (Gray and Wang, 2002).

- In particular, when $\mathfrak{p} = 1$, the series defining $y = F_c[u]$ converges provided

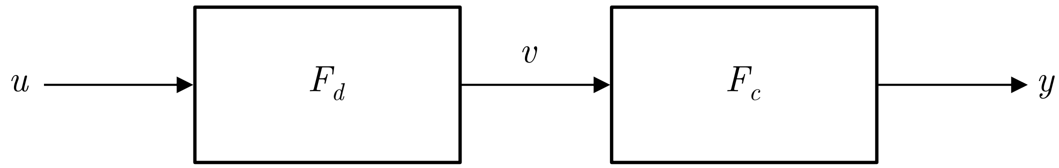
$$\max\{R, T\} < \frac{1}{M_c(m+1)}$$

- Let $\pi : \mathbb{R}^{\ell}_{LC}\langle\langle X \rangle\rangle \to \mathbb{R}^+$ take c to the smallest possible geometric growth constant M_c satisfying (1).

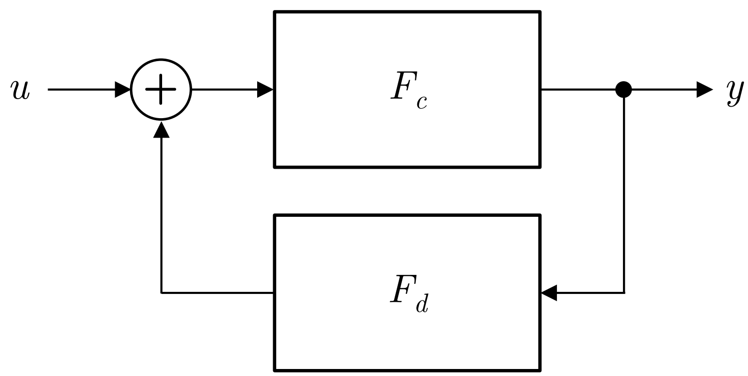
- In this case, $\mathbb{R}^{\ell}_{LC}\langle\langle X\rangle\rangle$ can be partitioned into equivalence classes, and the number $1/M_c(m+1)$ will be referred to as the radius of convergence for the class $\pi^{-1}(M_c)$.
- For example, $c = \sum_{n\geq 0} K_c M_c^n n!\, x_1^n$, $\bar{c} = \sum_{\eta\in X^*} K_c M_c^{|\eta|} |\eta|!\, \eta$ are in the same equivalence class.
- This definition is in contrast to the usual situation where a radius of convergence is assigned to individual series.
- In practice, it is not difficult to estimate the minimal M_c for many series, in which case, the radius of convergence for $\pi^{-1}(M_c)$ can be easily computed.
- If there exist real numbers $K_c, M_c > 0$ such that

$$|(c,\eta)| \leq K_c M_c^{|\eta|}, \;\; \forall\, \eta \in X^*,$$

then c is said to be globally convergent. The set of all such series is denoted by $\mathbb{R}^{\ell}_{GC}\langle\langle X\rangle\rangle$.



(a) cascade connection



(b) feedback connection

Fig. 1 The cascade and feedback interconnections

- It is known that the cascade connection of two locally convergent Fliess operators always yields another locally convergent Fliess operator (Gray and Li, 2005).

- Every self-excited feedback interconnection ($u = 0$) of two locally convergent Fliess operators has a locally convergent Fliess operator representation (Gray and Li, 2005).

- Lower bounds on the radius of convergence were given by Gray and Li (2005) for the cascade and self-excited feedback connections.

Problem Statement

Compute the radius of convergence of the

- cascade
- self-excited feedback
- and unity feedback interconnection

of two input-output systems represented as locally convergent Fliess operators.

Remarks:

- The Lambert W-function plays the key role throughout the computations.
- The unity feedback system has the same generating series as the Faa di Bruno compositional inverse, i.e., $c@\delta = (-c)^{-1}$.

2. Mathematical Preliminaries

Definition 1: (Fliess, 1981) A series $c \in \mathbb{R}^{\ell}\langle\langle X\rangle\rangle$ is said to be exchangeable if for arbitrary $\eta, \xi \in X^*$

$$|\eta|_{x_i} = |\xi|_{x_i}\,,\ i = 0, 1, \ldots, m \quad \Rightarrow \quad (c, \eta) = (c, \xi).$$

Theorem 1: If $c \in \mathbb{R}^{\ell}\langle\langle X\rangle\rangle$ is an exchangeable series and $d \in \mathbb{R}^{m}\langle\langle X\rangle\rangle$ is arbitrary then the composition product can be written in the form

$$c \circ d \quad = \quad \sum_{k=0}^{\infty} \sum_{\substack{r_0,\ldots,r_m \geq 0 \\ r_0+\cdots+r_m=k}} (c, x_0^{r_0} \cdots x_m^{r_m})\, D_{x_0}^{r_0}(1) \sqcup\!\sqcup \cdots \sqcup\!\sqcup D_{x_m}^{r_m}(1).$$

Definition 2: A series $\bar{c} \in \mathbb{R}^{\ell}_{LC}\langle\langle X \rangle\rangle$ is said to be a locally maximal series with growth constants $K_c, M_c > 0$ if each component of $(\bar{c}, \eta)$ is $K_c M_c^{|\eta|} |\eta|!$, $\eta \in X^*$. An analougus definition holds when $\bar{c} \in \mathbb{R}^{\ell}_{GC}\langle\langle X \rangle\rangle$.

Theorem 2: (Wilf, 1994) Let $f(z) = \sum_{n \geq 0} a_n/n!\, z^n$ be analytic in some neighborhood of the origin in the complex plane. Suppose a singularity of $f(z)$ of smallest modulus be at a point $z_0 \neq 0$, and let $\epsilon > 0$ be given. Then there exists N such that for all $n > N$,

$$|a_n| < (1/|z_0| + \epsilon)^n \, n!.$$

Furthermore, for infinitely many n,

$$|a_n| > (1/|z_0| - \epsilon)^n \, n!.$$

3. Radius of Convergence

3.1 The Cascade Connection

Theorem 3: Let $X = \{x_0, x_1, \ldots, x_m\}$. Let $c \in \mathbb{R}^{\ell}_{LC}\langle\langle X\rangle\rangle$ and $d \in \mathbb{R}^{m}_{LC}\langle\langle X\rangle\rangle$ with growth constants $K_c, M_c > 0$ and $K_d, M_d > 0$, respectively. If $b = c \circ d$ then

$$|(b,\nu)| \leq K_b M_b^{|\nu|} |\nu|!, \quad \nu \in X^* \tag{2}$$

for some $K_b > 0$, where

$$M_b = \frac{M_d}{1 - mK_d W\left(\frac{1}{mK_d}\exp\left(\frac{M_c - M_d}{mM_cK_d}\right)\right)},$$

where W denotes the Lambert W-function, namely, the inverse of the function

$$g(W) = W\exp(W).$$

Furthermore, no smaller geometric growth constant can satisfy (2).

Two lemmas are needed for the proof of Theorem 3. The following lemma can be proved inductively.

Lemma 1: Let $X = \{x_0, x_1, \ldots, x_m\}$ and $c, d \in \mathbb{R}^\ell\langle\langle X\rangle\rangle$ such that $|c| \leq d$, where $|c| := \sum_{\eta\in X^*} |(c,\eta)|\,\eta$. Then for any fixed $\xi \in X^*$ it follows that $|\xi \circ c| \leq \xi \circ d$.

Remark: If $\bar{c}$ and $\bar{d}$ are maximal series with growth constants K_c, M_c and K_d, M_d, respectively, it can be shown through the left linearity of the composition product and Lemma 1 that $|c \circ d| \leq \bar{c} \circ \bar{d}$.

Lemma 2: Let $X = \{x_0, x_1, \ldots, x_m\}$. Let $\bar{c} \in \mathbb{R}^{\ell}_{LC}\langle\langle X\rangle\rangle$ and $\bar{d} \in \mathbb{R}^{m}_{LC}\langle\langle X\rangle\rangle$ be locally maximal series with growth constants $K_c, M_c > 0$ and $K_d, M_d > 0$, respectively. If $\bar{b} = \bar{c} \circ \bar{d}$, then the sequence $(\bar{b}_i, x_0^k)$, $k \geq 0$ has the exponential generating function

$$f(x_0) = \frac{K_c}{1 - M_c x_0 + (mM_cK_d/M_d)\ln(1 - M_d x_0)}$$

for any $i = 1, 2, \ldots, \ell$. Moreover, the smallest possible geometric growth constant for $\bar{b}$ is

$$M_b = \frac{M_d}{1 - mK_dW\left(\frac{1}{mK_d}\exp\left(\frac{M_c - M_d}{mM_cK_d}\right)\right)}.$$

Proof of Lemma 2 (outline): There is no loss of generality in assuming $\ell = 1$. First observe that $\bar{c}$ is exchangeable, and thus, from Theorem 1 it follows that

$$\begin{aligned}
\bar{b} &= \sum_{k=0}^{\infty} K_c M_c^k \sum_{\substack{r_0,\ldots,r_m \geq 0 \\ r_0+\cdots+r_m=k}} k! \, \frac{x_0^{\,\sqcup\!\sqcup\, r_0}}{r_0!} \sqcup\!\sqcup \ldots \sqcup\!\sqcup \frac{(x_m \circ \bar{d})^{\,\sqcup\!\sqcup\, r_m}}{r_m!} \\
&= \sum_{k=0}^{\infty} K_c \left(M_c(x_0 + m x_0 \bar{d}_1) \right)^{\sqcup\!\sqcup\, k},
\end{aligned}$$

from which the following shuffle equation is obtained

$$\bar{b} = K_c + M_c[\bar{b} \sqcup\!\sqcup (x_0 + m x_0 \bar{d}_1)]. \tag{3}$$

Let $b_n := \max\{(\bar{b}, \nu) : \nu \in X^n\}$. Then it can be shown using (3) that b_n satisfies the following recursive formula

$$b_n = M_c \sum_{i=0}^{n-2} b_i m K_d M_d^{(n-i-1)} (n-i-1)! \binom{n}{i} + b_{n-1} M_c (1 + mK_d) n, \quad (4)$$

$n \geq 2$, where $b_0 = K_c$ and $b_1 = K_c M_c (1 + mK_d)$.

Remark: When all the growth constants and m are unity, b_n, $n \geq 0$ is the integer sequence shown in Table 1.

Table 1: Sequence satisfying (4) with all constants set to unity

sequence	OEIS number	$n = 0, 1, 2, \ldots$
b_n	A052820	$1, 2, 9, 62, 572, 6604, 91526, \ldots$

It is easily verified that the sequence $b_n, n \geq 0$ has the exponential generating function

$$f(x_0) = \frac{K_c}{1 - M_c x_0 + (mM_cK_d/M_d)\ln(1 - M_d x_0)}.$$

Since f is analytic at $z_0 = 0$, by Theorem 2 the smallest geometric growth constant is $M_b = 1/|x_0'|$, where x_0' is the singularity nearest to the origin

$$x_0' = \frac{1}{M_d}\left[1 - mK_dW\left(\frac{1}{mK_d}\exp\left(\frac{M_c - M_d}{mM_cK_d}\right)\right)\right].$$

Thus, the lemma is proved.

Remark: The proof of Theorem 3 follows directly from Lemmas 1 and 2.

Theorem 4: Let $X = \{x_0, x_1, \ldots, x_m\}$. Let $c \in \mathbb{R}^{\ell}_{GC}\langle\langle X \rangle\rangle$ and $d \in \mathbb{R}^{m}_{GC}\langle\langle X \rangle\rangle$ with growth constants $K_c, M_c > 0$ and $K_d, M_d > 0$, respectively. Assume $\bar{c}$ and $\bar{d}$ are globally maximal series with growth constants $K_c, M_c > 0$ and $K_d, M_d > 0$, respectively . If $b = c \circ d$ and $\bar{b} = \bar{c} \circ \bar{d}$ then

$$|(b, \nu)| \leq (\bar{b}_i, x_0^{|\nu|}), \ \nu \in X^*, \ i = 1, 2, \ldots, \ell,$$

where the sequence $(\bar{b}_i, x_0^k)$, $k \geq 0$ has the exponential generating function

$$f(x_0) = K_c \exp\left(\frac{mK_d \exp(M_d x_0) + M_d x_0 - mK_d}{M_d/M_c}\right).$$

Therefore, the radius of convergence is infinity.

Remark: Consistent with the known fact that global convergence is not preserved under the cascade connection.

3.2 Self-excited Feedback Connection

Theorem 5: Let $X = \{x_0, x_1, \ldots, x_m\}$ and $c \in \mathbb{R}^m_{LC}\langle\langle X \rangle\rangle$ with growth constants $K_c, M_c > 0$. If $e \in \mathbb{R}^m_{LC}\langle\langle X_0 \rangle\rangle$ satisfies $e = c \circ e$ then

$$|(e, x_0^n)| \leq K_e \left(\mathcal{A}(K_c) M_c\right)^n n!, \quad n \geq 0,$$

for some $K_e > 0$ and

$$\mathcal{A}(K_c) = \frac{1}{1 - mK_c \ln\left(1 + 1/mK_c\right)}.$$

Furthermore, no smaller geometric growth constant can satisfy the inequality above.

Two lemmas are needed for the proof of Theorem 5. The following lemma is proved by induction.

Lemma 3: Let $X = \{x_0, x_1, \ldots, x_m\}$. Suppose $c, \bar{c} \in \mathbb{R}^m_{LC}\langle\langle X\rangle\rangle$ have growth constants $K_c, M_c > 0$, where $\bar{c}$ is locally maximal. If $e, \bar{e} \in \mathbb{R}^m\langle\langle X_0\rangle\rangle$ satisfy, respectively, $e = c \circ e$ and $\bar{e} = \bar{c} \circ \bar{e}$ then $|e_i| \leq \bar{e}_i$, $i = 1, 2, \ldots, m$.

Remark: Therefore, the radius of convergence of this interconnection is determined by $\bar{e}$.

Lemma 4: Let $X = \{x_0, x_1, \ldots, x_m\}$. Suppose $\bar{c} \in \mathbb{R}^m_{LC}\langle\langle X \rangle\rangle$ is a locally maximal series with growth constants $K_c, M_c > 0$. Then each component of the solution $\bar{e} \in \mathbb{R}^m_{LC}\langle\langle X_0 \rangle\rangle$ of the self-excited unity feedback equation $\bar{e} = \bar{c} \circ \bar{e}$ has the exponential generating function

$$f(x_0) = \frac{-1}{m\left[1 + W\left(-\frac{1+mK_c}{mK_c}\exp\left[\frac{M_c x_0 - (1+mK_c)}{mK_c}\right]\right)\right]}.$$

In addition, the smallest possible geometric growth constant for $\bar{e}$ is

$$M_e = \frac{M_c}{1 - mK_c \ln(1 + 1/mK_c)}.$$

Proof of Lemma 4 (outline): It is not hard to show that $\bar{e}$ has the following realization

$$\begin{aligned} \dot{z} &= \frac{M_c}{K_c}(z^2 + z^3), \quad z(0) = K_c \\ y &= z. \end{aligned}$$

Thus,

$$z(t) = \frac{-1}{m\left[1 + W\left(-\frac{1+mK_c}{mK_c}\exp\left[\frac{M_c t - (1+mK_c)}{mK_c}\right]\right)\right]}.$$

However, z is the exponential generating function of the sequence $(\bar{e}, x_0^n)$, $n \geq 0$. Therefore, the smallest geometric constant is given by

$$M_e = \frac{M_c}{1 - mK_c \ln(1 + 1/mK_c)}.$$

Thus, the lemma is proved.

Remark: The proof of Theorem 5 follows directly from lemmas 3 and 4.

Theorem 6: Let $X = \{x_0, x_1, \ldots, x_m\}$ and $c \in \mathbb{R}^m_{GC}\langle\langle X \rangle\rangle$ with growth constants $K_c, M_c > 0$. If $e \in \mathbb{R}^m\langle\langle X_0 \rangle\rangle$ satisfies $e = c \circ e$ then

$$|(e, x_0^n)| \leq K_e \left(\mathcal{B}(K_c) M_c\right)^n n!, \quad n \geq 0, \tag{5}$$

for some $K_e > 0$ and

$$\mathcal{B}(K_c) = \frac{1}{\ln\left(1 + 1/mK_c\right)}.$$

Furthermore, no geometric growth constant smaller than $\mathcal{B}(K_c)M_c$ can satisfy (5), and thus the radius of convergence is $1/(\mathcal{B}(K_c)M_c)$.

Remark: Consistent with the known fact that global convergence is not preserved under the self-excited feedback connection.

Remark: The growth functions in Theorems 5 and 6 have series expansion about $K_c = \infty$:

$$\begin{aligned} \text{local case}: \quad \mathcal{A}(K_c) &= \frac{4}{3} + 2K_c + O\left(\frac{1}{K_c}\right) \\ \text{global case}: \quad \mathcal{B}(K_c) &= \frac{1}{2} + K_c + O\left(\frac{1}{K_c}\right). \end{aligned}$$

Thus, the radius of convergence for the global case is about twice that for the local case when $K_c \gg 0$.

3.3 The Unity Feedback Connection

Theorem 7: Let $X = \{x_0, x_1, \ldots, x_m\}$ and $c \in \mathbb{R}^m_{LC}\langle\langle X\rangle\rangle$ with growth constants $K_c, M_c > 0$. If $e \in \mathbb{R}^m\langle\langle X\rangle\rangle$ satisfies $e = c\tilde{\circ}e$ then

$$|(e,\eta)| \leq K_e(\mathcal{A}(K_c)M_c)^{|\eta|}|\eta|!, \quad \eta \in X^*, \tag{6}$$

for some $K_e > 0$, where

$$\mathcal{A}(K_c) = \frac{1}{1 - mK_c \ln{(1 + 1/mK_c)}}.$$

Furthermore, no smaller geometric growth constant can satisfy the inequality above.

First the following lemma can be proven by an inductive argument.

Lemma 5: Let $X = \{x_0, x_1, \ldots, x_m\}$. Suppose $c, \bar{c} \in \mathbb{R}^m_{LC}\langle\langle X \rangle\rangle$ have growth constants $K_c, M_c > 0$, where $\bar{c}$ is locally maximal. If $e, \bar{e}$ satisfy, respectively, $e = c\tilde{\circ}e$ and $\bar{e} = \bar{c}\tilde{\circ}\bar{e}$ then $|e_i| \leq \bar{e}_i$, $i = 1, 2, \ldots, m$.

Proof of Theorem 7 (outline): The proof has the following steps:

1. The Fliess operator $F_{\bar{e}}$ is shown to have the realization

$$\begin{aligned} \dot{z} &= \frac{M_c}{K_c}\left(z^2 + mz^3 + z^2\sum_{i=1}^{m} u_i\right), \quad z(0) = K_c, \\ y &= z. \end{aligned}$$

2. The coefficients of $\bar{e}$ can be computed by taking the Lie derivatives of $h(z) = z$ with respect to the vector fields

$$\begin{aligned} g_0(z) &= \frac{M_c}{K_c}(z^2 + mz^3) \\ g_i(z) &= \frac{M_c}{K_c}z^2, \quad i = 1, 2, \ldots, m, \end{aligned}$$

That is,

$$(\bar{e}, \eta) = L_{g_\eta} h(z_0), \;\; \eta \in X^*.$$

3. The series $\bar{e}$ has coefficients satisfying

$$0 < (\bar{e}, \eta) \leq \left(\bar{e}, x_0^{|\eta|}\right), \;\; \eta \in X^*.$$

4. The growth rate of $(\bar{e}, x_0^{|\eta|})$ is obtained by Theorem 4. Thus, the result follows.

Example 1: Suppose e satisfies $e = c \circ e$ with $c = \sum_{\eta \in X^*} |\eta|!\, \eta$. Clearly c is an exchangeable locally convergent series with $K_c = M_c = 1$. Therefore, $M_e = 1/(1 - \ln(2))$.

This self-excited unity feedback system has state space model

$$\begin{aligned} \dot{z} &= z^2(1+z), \quad z(0) = 1 \\ y &= z. \end{aligned}$$

Remark: The singularity nearest to the origin of the generating function formed by the self-excited feedback connection of a maximal series is real and positive. Therefore, a finite escape time is observed.

The finite escape time should be $t_{esc} = 1/M_e = 1 - \ln(2) \approx 0.3069$.

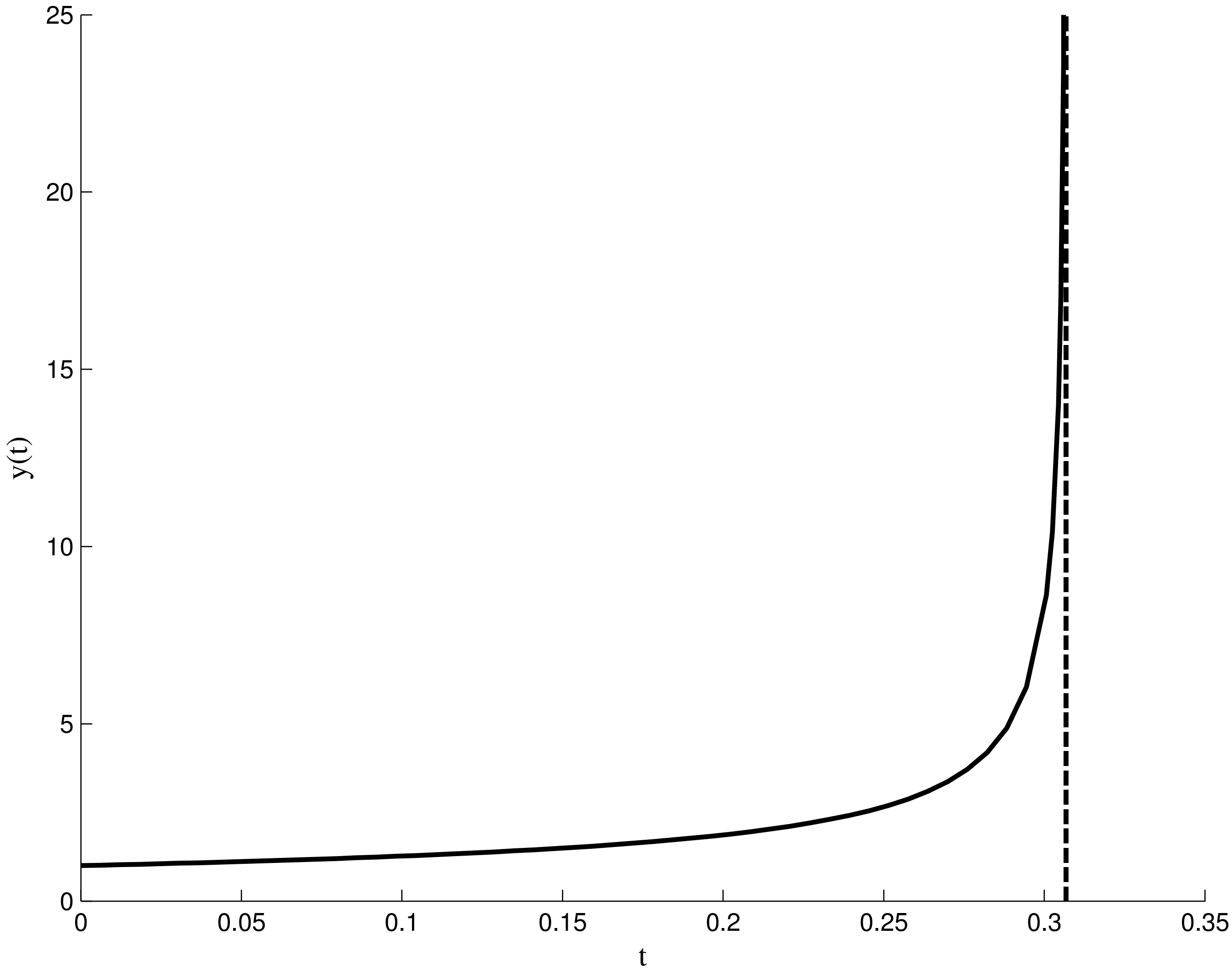


Fig. 2: The output of the self-excited loop in Example 1.

4. Conclusions and Future Research

- The radius of convergence for the cascade, self-excited feedback and unity feedback connections of two convergent Fliess operators were computed.

- It was found that the Lambert-W function plays a central role in computing the radii of convergence for these connections. This suggests some relationship to the combinatorics of rooted nonplanar labeled trees (Corless, 1996; Flajolet and Sedgewick, 2009).

- Perhaps this might provide a more natural combinatoric interpretation of the composition and feedback products of formal power series.