# ROUGH PATHS
# SIGNATURES OF PATHS

TERRY LYONS

# NOTES I

The main original paper

- http://dmle.cindoc.csic.es/pdf/MATEMATICAIBEROAMERICANA_1998_14_02_01.pdf
- is not so bad, and contains a proof of the "neo-classical" factorial estimate and the uniform continuity and extension theorems.
- On the topic of the neoclassical inequality I proved it with non-optimal constants and only conjectured the best constant. More recently E.R. Love improved it, but it was really settled only recently by Hino and Hara "Fractional order Taylor's series and the neo-classical inequality" see
- http://arxiv.org/PS_cache/arxiv/pdf/1001/1001.1775v1.pdf

# NOTES II

**But I think for a first read, the St Flour notes are a good starting point**

- T. Lyons, M. Caruana, T. Lévy, Differential Equations Driven by Rough Paths Ecole d'Eté de Probabilités de Saint-Flour XXXIV-2004, Lecture Notes in Mathematics, Vol. 1908
- as a great deal of care was taken to make it readable and ensure the details of the main theorem were 100% transparent.
- Errata: http://www.math.ens.fr/~levy/errata.pdf
- but this book does not have the details for the two issues mentioned above which are also important at a basic level.

# NOTES III

Friz and Victoir

- Multidimensional stochastic processes as rough paths. Theory and Applications, Cambridge Studies in Advanced Mathematics (CUP, 2009), P. Friz with N. Victoir
- http://www.statslab.cam.ac.uk/~peter/RoughPathsBook/TOC.pdf
- A "Dunford and Schwartz" covers many technical issues with elegance (approximation of paths etc.) that are needed. It also covers the absolutely essential results of A.M.Davie which among other things show what is sharp etc. and prove Peano's theorem (to my Picard).
- But the book is not dimension independent for pedagogic reasons - I think this will be a significant drawback in the end - as the theory really is uniform and works in Banach spaces.

Lyons and Qian; Lejay; two other books with different contributions.

# EXAMPLES AND APPLICATIONS I

**Classical Brownian motion as a rough path**

- A path-wise view of solutions of stochastic differential equations EM Sipiläinen - PhD thesis, University of Edinburgh

**Diffusion sample paths as rough paths**

- Extending the Wong-Zakai theorem to reversible Markov processes RF Bass, BM Hambly, TJ Lyons - Journal of the European Mathematical ..., 2002 - Springer

**Fractional Brownian motion as a rough path**

- Stochastic analysis, rough path analysis and fractional Brownian motions L Coutin, Z Qian - Probability Theory and Related Fields, 2002 - Springer

# EXAMPLES AND APPLICATIONS II

**Extending general paths to be rough paths**

- An extension theorem to rough paths T Lyons, N Victoir - Annales de l'Institut Henri Poincare (C) Non Linear ..., 2007 - Elsevier

**Free Brownian motion as a rough path**

- The Lévy Area Process for the Free Brownian Motion* M Capitaine, C Donati-Martin - Journal of Functional Analysis, 2001 - Elsevier

**Infinite Dimensional BM as a rough path**

- Lévy area of Wiener processes in Banach spaces M Ledoux, T Lyons, Z Qian - The Annals of Probability, 2002 - projecteuclid.org

**Martingale inequalities for rough paths**

- Peter Friz* and Nicolas Victoir, The Burkholder-Davis-Gundy Inequality for Enhanced Martingales

# EXAMPLES AND APPLICATIONS III

**Processes with jumps**

- Path-wise solutions of stochastic differential equations driven by Lévy processes DRE Williams - Revista matemática iberoamericana, 2001

**Rough Paths coming from homogenisation**

- On the importance of the Lévy area for systems controlled by converging stochastic processes. Application to homogenization A Lejay, TJ Lyons - New Trend in Potential Theory

# EXAMPLES AND APPLICATIONS IV

**Large deviations**

- Large deviations and support theorem for diffusion processes via rough paths M Ledoux, Z Qian, T Zhang - Stochastic processes and their applications, 2002 - Elsevier
- Large deviations for rough paths of the fractional Brownian motion A Millet, M Sanz-Solé - Annales de l'Institut Henri Poincare (B) Probability ..., 2006 - Elsevier
- Approximations of the Brownian rough path with applications to stochastic analysis P Friz, N Victoir - Annales de l'Institut Henri Poincare (B) Probability and ..., 2005 - Elsevier
- Large deviations for heat kernel measures on loop spaces via rough paths Y Inahama, H Kawabi - Journal of the London ..., 2006
- Good rough path sequences and applications to anticipating stochastic calculus L Coutin, P Friz, N Victoir - Annals of probability, 2007
- Small deviations in p-variation for multidimensional Lévy processes T Simon - JOURNAL OF MATHEMATICS-KYOTO UNIVERSITY, 2003

# EXAMPLES AND APPLICATIONS V

**Support Theorem**

- Lévy's area under conditioning P Friz, T Lyons, D Stroock Annales de l'Institut Henri Poincare (B) …, 2006

**Hormander type theorems:**

- Densities for rough differential equations under Hoermander's condition T Cass, P Friz Annals of Mathematics, 171 (2010), 2115–2141

**Signatures of Paths:**

- On the radius of convergence of the logarithmic signature TJ Lyons, N Sidorova - Illinois Journal of Mathematics, 2006
- Uniqueness for the signature of a path of bounded variation and the reduced path group B Hambly, T Lyons Annals of Mathematics, 171 (2010), 109–167

# EXAMPLES AND APPLICATIONS VI

Numerical methods:

- Cubature on Wiener space T Lyons, N Victoir - ... of the Royal Society of London. ..., 2004 And several other papers
- Note that in different language, Kusuoka was doing the very similar things - and I think the method is best referred to as the KLV or Kusuoka-Lyons-Victoir method although the language adds value.
- It only works really well with recombination.
- Christian Litterer and Terry Lyons: High order recombination and an application to cubature on Wiener space http://arxiv.org/abs/1008.4942

Japanese School, stationary phase, spectral gaps on loop spaces,.... Aida,...

· Rough Paths – new tools for

understanding streams

Recall:

If $\mu$ is a probability measure on $\mathbb{R}$ then its characteristic function $\varphi(\lambda) = \mathbb{E}_\mu(e^{2\pi i x})$ characterises $\mu$ (Powerful tool: central limit theorem etc.)

The laplace transform useful if measure $\mu$ has a sharp tail. $m_k = \mathbb{E}_\mu(x^k)$ known and $\sum \lambda^k m_k/k!$ has a domain of convergence.

# Stochastic Analysis

Is about understanding measures $\mu$ on infinite dimensional spaces - particularly path spaces.

A key tool, introduced by Varadhan and developed by Stroock and Varadhan - is the martingale problem.

In effect computing the Laplace transform for Markov Measures on Paths

# General Measures on Paths

There are many measures on paths that are not markov (lerw, a fluid motion over an interval [0,T] in phase space).

It is difficult to use these methods.

But there are new tools - we need an integral calculus to make them work for us.

/ en

are many er types of path! Sequential streamed data - order - ...

$\gamma : [0,1] \longrightarrow (E, d)$ finite length, cts, ... ?

Can we do calculus?

Can we make sense of

$$dy_t = f(y_t)\, d\gamma_t, \quad y_0 = a ?$$

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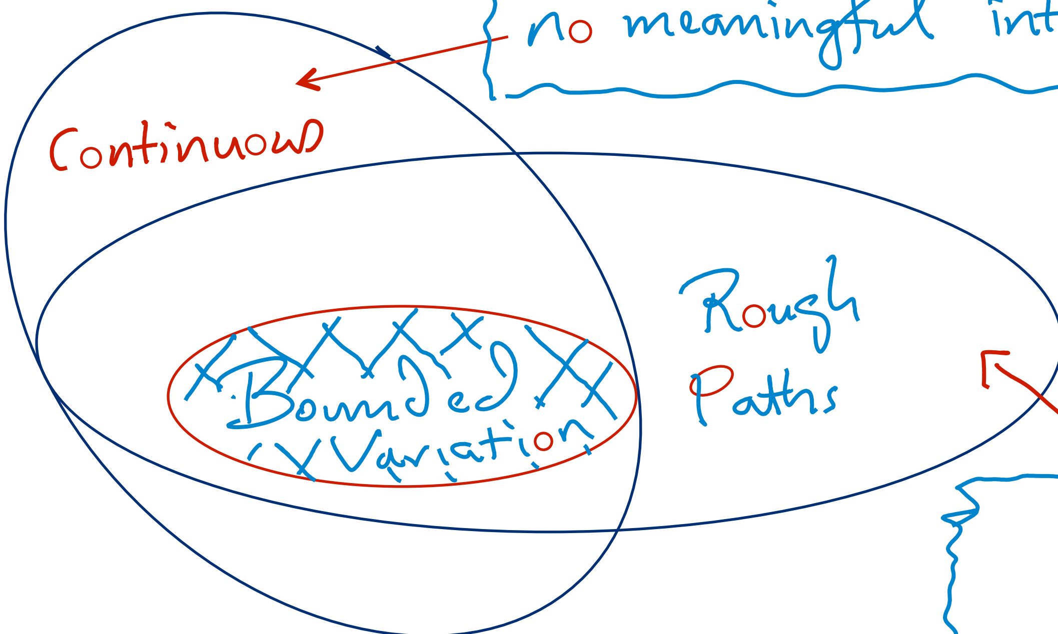
# Rough Paths

no meaningful interpretation if $E = \mathbb{R}^d$, $d \geq 2$




"complete" interpretation providing $f$ smooth enough

# Rough Metrics

$1 \leq p < \infty$ : a metric $d_p$ on paths of finite length in a Banach Space. So that if $f \in \mathrm{Lip}\, p'$ $(p' > p)$ then

$$\gamma_\cdot \to y_1 : \quad dy = f(y)\,d\gamma \quad y_0 = a$$

is uniformly continuous

The completion of BV in $d_p$ is tractable and is known as the space of geometric rough paths.

# Probability

Sippallainen: The Lévy piecewise linear dyadic approximations to Brownian motion in $\mathbb{R}^d$ converge in $d_p$ for all $p>2$.

This embeds BM (+Levy Area) $\hookrightarrow \Omega G_p(\mathbb{R}^d)$ $p>2$

This is a canonical, but not a unique embedding. In fact there is no unique way to understand the effect of BM. One needs an extra decision.

# Probability

Same approach (via dyadic piecewise approximation) works for fBM (Qian & Coutin) and reversible markov processes (Lyons Hambly) providing the paths have p variation $< 4$.

So one can drive stochastic differential equations by either noise, the Wong Zakai theorem applies.

Neither is (close to) a semi-martingale or Itô theory

## A transform from stream space to effect space.

## The signature of a path

Suppose $\gamma$ is a path constant outside $[S,T]$ and

$dy_t = f(y_t)\,d\gamma_t, \quad y_{-\infty} = a$. Then $y_{+\infty} = \pi_f(a)$

"$\pi_f$" describes the effect of $\gamma$ on $a$.

It is often the case that we are only interested in $\gamma$ for $\pi$

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# A different way to describe $\gamma$

Recall how we described $x$ in terms of $\lambda \to e^{2\pi i x \lambda}$.

Describe $\gamma : [S,T] \longrightarrow E$ by solving the differential equation

$$dS_u = S_u \cdot d\gamma_u \qquad S_S = I$$

$S_T$ is then the Signature of the control $\gamma$ over $[S,T]$

The fully non-commutative exponential.

# The tensor algebra

$A = \{a_1, \ldots a_n\}$ an alphabet

$E$ the vector space spanned by letters $\sum_1^n \lambda_i a_i$

$E^k$ ” ” ” ” words of length $k$

$E^0 = \mathbb{R}$.

$$\begin{cases} T((A)) = E^0 \oplus E^1 \oplus \cdots & \otimes, + \\ \| \sum \lambda_w w \| = \sum |\lambda_w| & \end{cases}$$

$dS^1 = S^1 \otimes d\gamma$ is BDD LINEAR on $T((E))$.

# Iterated Integrals

Natural parameterisation invariant functions of $\gamma$

$$S^1_{ST} := \int_{S<u<T} d\gamma = \gamma_T - \gamma_S$$ "the increment"

$$S^2_{ST} := \int_{S<u_1<u_2<T} d\gamma_{u_1}\, d\gamma_{u_2} = \tfrac{1}{2}(\gamma_T-\gamma_S)\otimes(\gamma_T-\gamma_S)$$

+ "the area"

$$= \tfrac{1}{2}(\text{inc})^2 + \tfrac{1}{2}\sum A^{ij}\,[a_i, a_j]$$

etc....

# An expression for the signature

$$S_{ST}(\gamma) = 1 + \int_{S<u<T} d\gamma + \cdots \int \cdots \int_{S<u_1<\cdots<u_n<T} d\gamma_{u_1} \cdots d\gamma_{u_n} \cdots$$

$$= \left( S_{ST}^{0} + S_{ST}^{1} + \cdots S_{ST}^{n} + \cdots \right)$$

Is a transform of paths into effect space.

uniform: $\left\| S_{ST}^{k} \right\| \leqslant \dfrac{|||\gamma|||_p^k}{\beta_p (k/p)!}$

# Main Properties of $S$.

The signature is a homomorphism from paths with concatenation into the algebra $T((E))$ that is parameterisation invariant and, up to a specific treelike equivalence, injective on BV paths (Hambly, Lyons, Ann. Math 2010)

Linear $dy = A^i y \, d\gamma^i \Rightarrow y_T = \sum_{k=0}^{\infty} \overbrace{A \ldots A}^{k} S^k_{ST} \, y_S$

So uniform estimates on convergence. $S$ top down of $\gamma$!!!

# Random Paths, Expected Signatures

The expected signature makes sense because signatures live in a vector space.

Suppose $S(\gamma)$ compactly supported, then

$\mathbb{E}(S_\gamma)$ determines the law of $S(\gamma)$

c.f. Laplace Transforms.

BM on $[0,1]$ $\rightsquigarrow$ $\exp\left(\frac{1}{2}\left(\sum^{d} e_i \otimes e_i\right)\right)$

Basis of many numerical methods

# Other Measures on Paths?

Many new problems.

Solve PDEs for their effect.

Wave eqn - earthquakes

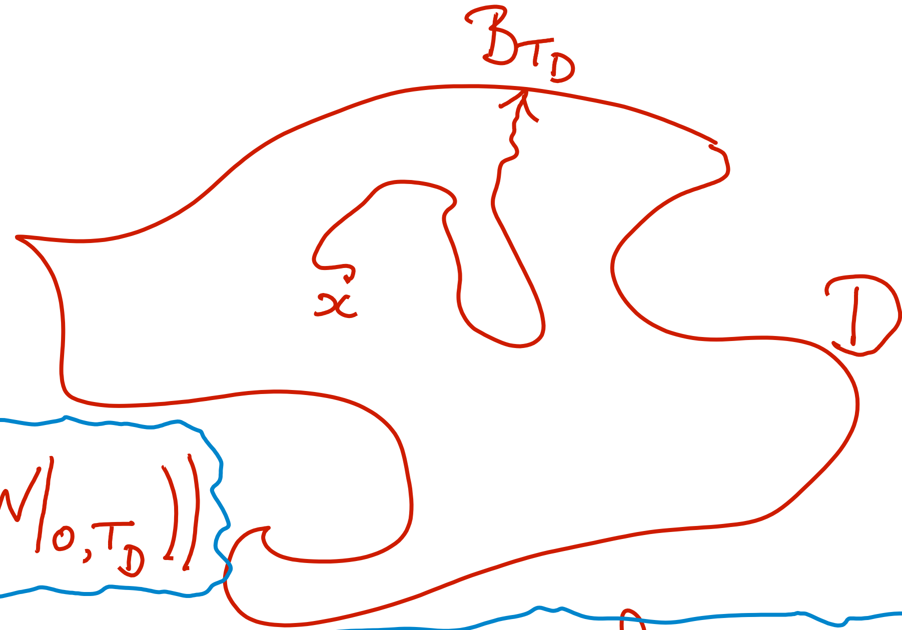
Fluids - to capture + describe turbulence

Characterise LERW...

But lets start with something more Modest

# BROWNIAN MOTION STOPPED ON EXIT

Ni + Ly



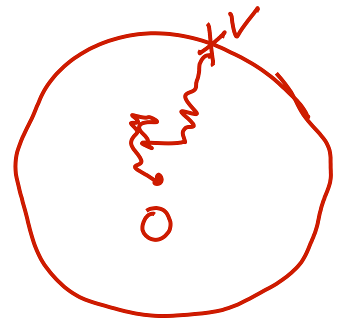

$$\varphi(x) = \mathbb{E}_x\left(S\left(W|_{0,T_D}\right)\right)$$

martingale property

$$\varphi(x) = 1 + 0 + ? + ? + \dots \quad \in T((\mathbb{R}^d))$$

# How to compute $\varphi$?

Introduce an auxilliary fn.

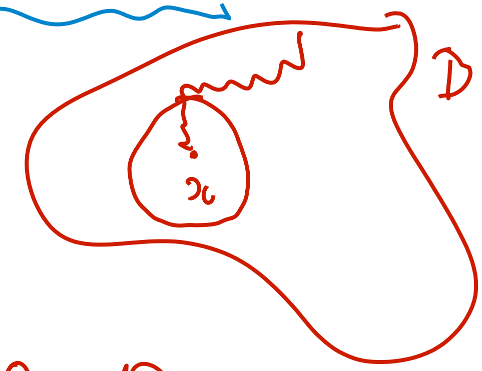



$$\psi(v) = \mathbb{E}_0\left(S(\gamma) \,\middle|\, \gamma_{T_{B(|v|)}} = v\right) = 1 + v + \frac{1}{2} v \otimes v + O(|v|^3)$$

$$\Delta \psi(0) = \sum_{1}^{d} e_i \otimes e_i$$

$$\nabla \psi(0) = \sum_{j=1}^{d} \delta_{ij}\, e_j$$

# Strong Markov + Multiplicative




$\forall x \in D$

$$\varphi(x) = \frac{1}{2\pi}\int_0^{2\pi} \psi(re^{i\theta})\,\varphi(x+re^{i\theta})\,d\theta$$

or

$$\Delta\left(\psi(v)\,\varphi(x+v)\right)\Big|_{v=0} = 0$$

$$\Delta \psi \varphi = \Delta\psi\,\varphi + \psi\Delta\phi + 2\nabla\psi\nabla\varphi.$$

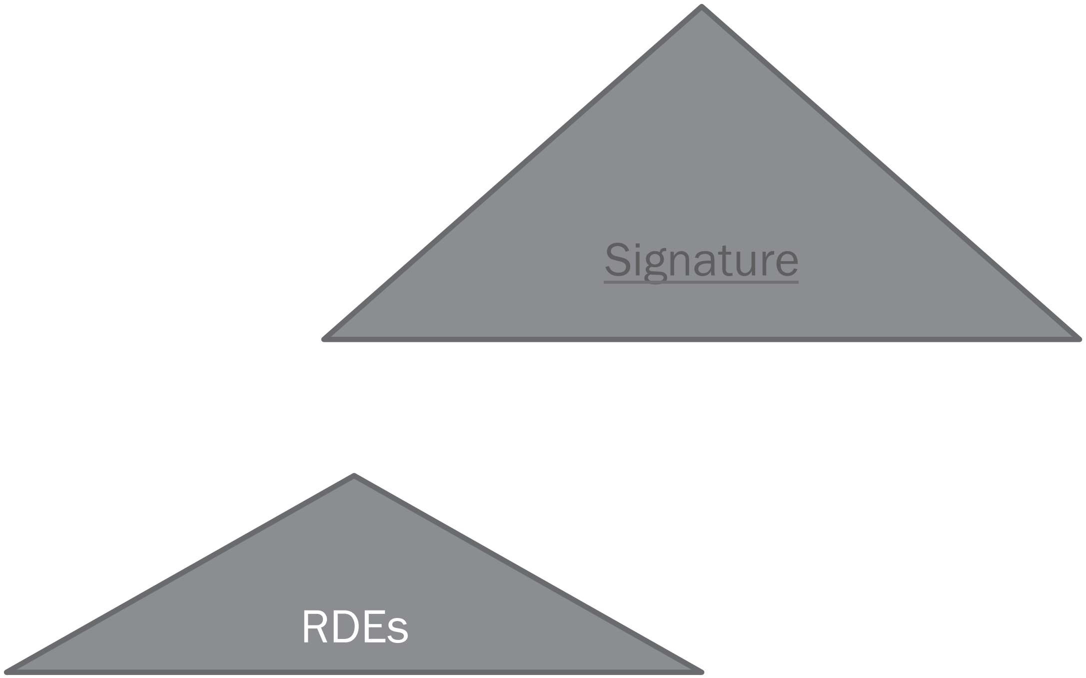
# A coupled family of PDEs

$$\Delta \varphi + 2 e_i \otimes \frac{\partial}{\partial e_i} \varphi + \sum_1^d (e_i \otimes e_i) \varphi = 0$$

Dirichlet boundary data for $\varphi_k$, $k > 0$.

Recurrence relation $\Delta \varphi_k = -2 e_i \otimes \frac{\partial}{\partial e_i} \varphi_{k-1} - \sum e_i \otimes e_i \cdot \varphi_{k-2}$

Solution exists, & for unit disk $\varphi_k = \mathrm{Poly} \cdot (1-(x^2+y^2))$


Signature
RDEs