Lagrangian Submanifolds and Constrained Variational Calculus

Fernando Jiménez

ICMAT (CSIC-UAM-UCM-UC3M), Madrid, Spain.

Rough Paths and Combinatorics in Control Theory Meeting.

University of California, San Diego
26th July, 2011

Work in collaboration with David Martín de Diego (ICMAT-CSIC) and Manuel de León (ICMAT-CSIC)

“Everything is a Lagrangian Submanifold.”

Alan Weinstein.

Lagrangian Submanifolds are very useful!!

R. Abraham and J.E. Marsden. “Foundations of Mechanics” (1978)

• What is a Lagrangian Submanifold (LS)?

If (M, ω) is a symplectic manifold of finite dimension, $N \subset M$ submanifold, $i_N : N \hookrightarrow M$:

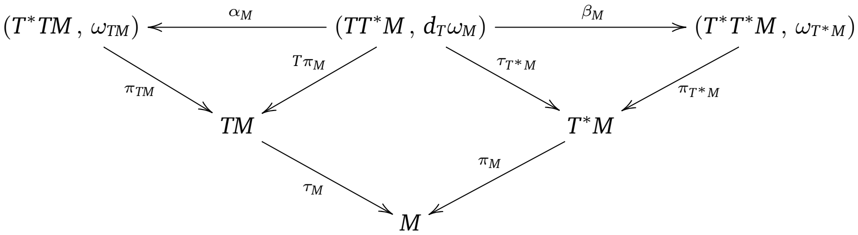
- $i_N^* \omega = 0$
- $\dim N = \frac{1}{2} \dim M$

• Examples:

- $g : M \to M$ is a symplectomorphism, then $\mathrm{Graph}(g) = \{(x, g(x)) \, , \, x \in M\} \subset (M \times M \, , \, \Omega = pr_1^* \omega - pr_0^* \omega) \Rightarrow$ $i_g : \mathrm{Graph}(g) \hookrightarrow M \times M$ is a LS.
- $f : M \to \mathbb{R} \Rightarrow df(M) \subset (T^*M \, , \, \omega_M)$ is a LS. If (q, p) are local coord. for T^*M, by Darboux Theorem $\omega_M = dq \wedge dp$.
- $(T^*M \, , \, \omega_M)$, $H : T^*M \to \mathbb{R}$, $i_{X_H} \omega_M = dH \Rightarrow X_H(T^*M) \subset (TT^*M \, , \, d_T \omega_M)$ is a LS.

Tulczyjew Triple

- Tulczyjew Triple (1976):



- $\beta_M = \flat_{\omega_{T^*M}}, \quad \beta_{\omega_{T^*M}}(v) = i_v\omega_{T^*M}$, where $v \in TT^*M \Rightarrow$

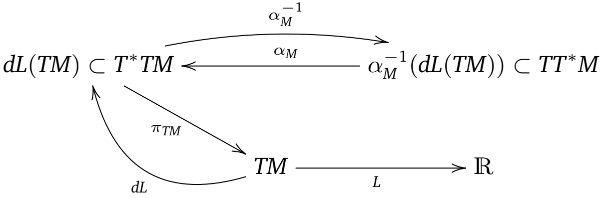
$$\beta_M(q^i, p_i, \dot{q}^i, \dot{p}_i) = (q^i, p_i, -\dot{p}_i, \dot{q}^i).$$

- $\langle \alpha_M(z)\,,\,w\rangle = \langle z\,,\,\kappa_M(w)\rangle$, where $z \in TT^*M$ and $w \in TTM$,

$$\alpha_M(q^i, p_i, \dot{q}^i, \dot{p}_i) = (q^i, \dot{q}^i, \dot{p}_i, p_i).$$

Intrinsic description of Lag. and Ham. Mechanics

• $L : TM \to \mathbb{R}$, then $dL(TM) \subset T^*TM$ is a LS. Moreover, given that α_M is a symplectomorphism, $\alpha_M^{-1}(dL(TM)) \subset TT^*M$ is a LS.



• The solutions of the dynamics of $\alpha_M^{-1}(dL(TM))$ are curves $\gamma : I \subset \mathbb{R} \to T^*M$ s.t. $\frac{d\gamma}{dt} : I \subset \mathbb{R} \to TT^*M$ verifies $\frac{d\gamma}{dt}(I) \subset \alpha_M^{-1}(dL(TM))$. Locally

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = 0.$$

First Question

Is there a way to describe constrained Lagrangian Mechanics through Lagrangian Submanifolds and, moreover, to relate that dynamics with a Hamiltonian system?

Lagrangian Mechanics with Constraints

• Constrained Lagrangian System (CLS): $C \subset TM$ and a Lag. function $L : C \to \mathbb{R}$. Locally C is described by a set of constraints $\phi^\alpha : TM \to \mathbb{R}$.

Theorem (Tulczyjew)

Let M be a smooth manifold, $N \subset M$ a submanifold, and $f : N \to \mathbb{R}$ a smooth function. Then

$$\Sigma_f = \{p \in T^*M \mid \pi_M(p) \in N \text{ and } \langle p, v\rangle = \langle df, v\rangle \\ \text{for all } v \in TN \subset TM \text{ such that } \tau_M(v) = \pi_M(p)\}$$

is a Lagrangian submanifold of T^*M.

• Hence, $\boxed{\Sigma_L \in (T^*TM\,,\,\omega_{TM})}$ is a LS. Moreover, $\boxed{\alpha_M^{-1}(\Sigma_L) \subset (TT^*M\,,\,d_T\omega_M)}$ also is, which means DYNAMICS!. Locally $\mathbb{L} = \tilde{L} + \lambda_\alpha \phi^\alpha : TM \to \mathbb{R}$,

$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial \tilde{L}}{\partial \dot{q}^i} + \lambda_\alpha \frac{\partial \phi^\alpha}{\partial \dot{q}^i}\right) - \frac{\partial \tilde{L}}{\partial q^i} - \lambda_\alpha \frac{\partial \phi^\alpha}{\partial q^i} &= 0 \\ \phi^\alpha(q^i, \dot{q}^i) &= 0 \end{aligned}$$

Constrained Legendre Transformation

$\mathbb{F}L : \Sigma_L \longrightarrow T^*M$ as the mapping $\mathbb{F}L = \tau_{T^*M} \circ (\alpha_M^{-1})|_{\Sigma_L}$.

We will say that (L, C) is *regular* if $\mathbb{F}L$ is a local diffeomorphism and *hyperregular* if $\mathbb{F}L$ is a global diffeomorphism.

- $E_L : \Sigma_L \to \mathbb{R}$, $E_L(\alpha_u) = \langle \alpha_u \, , \, u_u^V \rangle - L(u)$, where $\alpha_u \in \Sigma_L$, $u \in C$ and $u_u^V \in TC \subset TTM$.
- Define $\omega_L = (\mathbb{F}L)^* \omega_M$ on $\Sigma_L \Rightarrow \boxed{i_X \omega_L = dE_L}$.
- If (L, C) is hyperrregular, we can define a Hamiltonian function $H : T^*M \to \mathbb{R}$ by

$$H = E_L \circ (\mathbb{F}L)^{-1}$$

such that $i_{X_H} \omega_M = dH$.

$$\Sigma_L \Rightarrow X_H(T^*M)$$

Constrained Legendre Transformation

$\mathbb{F}L : \Sigma_L \longrightarrow T^*M$ as the mapping $\mathbb{F}L = \tau_{T^*M} \circ (\alpha_M^{-1})|_{\Sigma_L}$.

We will say that (L, C) is *regular* if $\mathbb{F}L$ is a local diffeomorphism and *hyperregular* if $\mathbb{F}L$ is a global diffeomorphism.

- $E_L : \Sigma_L \to \mathbb{R}$, $E_L(\alpha_u) = \langle \alpha_u \, , \, u_u^V \rangle - L(u)$, where $\alpha_u \in \Sigma_L$, $u \in C$ and $u_u^V \in TC \subset TTM$.
- Define $\omega_L = (\mathbb{F}L)^* \omega_M$ on $\Sigma_L \Rightarrow \boxed{i_X \omega_L = dE_L}$.
- If (L, C) is hyperrregular, we can define a Hamiltonian function $H : T^*M \to \mathbb{R}$ by

$$H = E_L \circ (\mathbb{F}L)^{-1}$$

such that $i_{X_H} \omega_M = dH$.

$$\alpha_M^{-1}(\Sigma_L) = X_H(T^*M)$$

- $H : T^*M \to \mathbb{R}$. Since $\pi_M : T^*M \to M$ is a vector bundle, define the dilation vector field $\Delta^* \in \mathfrak{X}(T^*M)$:

$$\Delta^*(\alpha_q) f = \frac{d}{dt}\Big|_{t=0} f(t\alpha_q), \quad \Delta^* = p_i \frac{\partial}{\partial p_i}.$$

- Define the fiber derivative $\mathbb{F}H : T^*M \to TM$

$$\langle \mathbb{F}H(\alpha_q)\,,\,\beta_q\rangle = \frac{d}{dt}\Big|_{t=0} H(\alpha_q + t\beta_q), \;\; \mathbb{F}H(q^i, p_i) = (q^i, \frac{\partial H}{\partial p_i}).$$

- Assume $\mathbb{F}H(T^*M) = C$ is a submanifold $\subset TM \Rightarrow$ Mimicking the Gotay and Nester's definition, we set $L : C \to \mathbb{R}$ such that

$$L \circ \mathbb{F}H = \Delta^* H - H.$$

- $\mathbb{F}H(\alpha_q) = \mathbb{F}H(\beta_q)$ or equivalently $(\Delta^* H - H)(\alpha_q) = (\Delta^* H - H)(\beta_q)$. Extra assumption

Definition

A Hamiltonian $H : T^*M \to \mathbb{R}$ is **almost-regular** if $\mathbb{F}H(T^*M) = C$ is a submanifold of TM and $\mathbb{F}H : T^*M \to C \subset TM$ is a submersion with connected fibers.

- Under this assumption we can consider the infinitesimal condition, i.e.

$$\mathcal{L}_Z(\Delta^* H - H) = 0, \ \forall \ Z \in \ker(\mathbb{F}H_*).$$

- This is the case, hence L is well defined.

Theorem

The following equivalence holds

$$\alpha_M(X_H(T^*M)) = \Sigma_L \ .$$

Sketch of the Poof

For any $W_1 \in X_H(T^*M)$

$$\langle \alpha_M(W_1) \, , \, U \rangle = \langle dL \, , \, U \rangle \ \forall U \in TC.$$

This is equivalent to prove

$$\langle \alpha_M(W_1) \, , \, \mathbb{F}H_*(W_2) \rangle = \langle dL \, , \, \mathbb{F}H_*(W_2) \rangle \ \forall W_2 \in TT^*M.$$

$$\mathbb{F}H^* \alpha_M(W_1) = \mathbb{F}H^* \, dL = d(\Delta^* H - H).$$

From Hamiltonian Mechanics to Constrained Lagrangian Mechanics

$$
\begin{array}{ccccc}
TT^*M \supset X_H(T^*M) & \xrightarrow{\alpha_M} & \alpha_M(X_H(T^*M)) = \Sigma_L & \hookrightarrow & T^*TM \\
\uparrow{\scriptstyle X_H} & & & & \uparrow{\scriptstyle dL} \\
T^*M & & & & C \subset TM \\
\downarrow{\scriptstyle H} & & & & \downarrow{\scriptstyle L} \\
\mathbb{R} & & & & \mathbb{R}
\end{array}
$$

ANY DISCRETE EQUIVALENCE?

J.E. Marsden and M. West: "Discrete Mechanics and Variational Integrators" (2001)

• M is a n-dimensinal smooth manifold (q^i), and TM its tangent bundle $(q^i, \dot{q}^i)$.
$L : TM \to \mathbb{R}$:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i}\right) - \frac{\partial L}{\partial q^i} = 0, \quad 1 \leq i \leq n.$$

• Discrete Mechanics: $TM \Rightarrow M \times M$. Moreover $L_d : M \times M \to \mathbb{R}$:

$$L_d(q_0, q_1) \approx \int_0^h L(q(t), \dot{q}(t))\, dt$$

• Define the sequences $(q_0, ..., q_N) \in M^{N+1}$ and the action sum $S_d = \sum_{k=1}^N L_d(q_{k-1}, q_k)$.

$$D_1 L_d(q_k, q_{k+1}) + D_2 L_d(q_{k-1}, q_k) = 0.$$

This equations define a discrete flow $\phi_{L_d} : M \times M \to M \times M$, $\phi_{L_d}(q_{k-1}, q_k) = (q_k, q_{k+1})$, under regularity assumptions.

- Go into a Hamiltonian Picture through Discrete Legendre Transforms

$$\begin{array}{rcl} \mathbb{F}L_d^- : M\times M & \rightarrow & T^*M \\ (q_0,q_1) & \longmapsto & (q_0, -D_1L_d(q_0,q_1)) \\ \mathbb{F}L_d^+ : M\times M & \rightarrow & T^*M \\ (q_0,q_1) & \longmapsto & (q_1, D_2L_d(q_0,q_1))\ , \end{array}$$

- Pullback the canonical 2-form: $\omega_d = (\mathbb{F}L_d^-)^*\omega_M = (\mathbb{F}L_d^+)^*\omega_M$
- Preservation properties
  - $\phi_{L_d}^*\,\omega_d = \omega_d \Rightarrow$ Symplectic Preservation.
  - The flow ϕ_{L_d} also preserves $J_d : M\times M \rightarrow \mathfrak{g}^*$, defined by

$$\langle J_d(q_k,q_{k+1}),\xi\rangle = \langle D_2L_d(q_k,q_{k+1}),\xi_M(q_{k+1})\rangle$$

  where ξ_M is the fundamental vector field determined by $\xi\in\mathfrak{g} \Rightarrow$ Momentum Map Preservation.

• $H : T^*M \rightarrow \mathbb{R} \Rightarrow$ HAMILTON EQUATIONS $\Rightarrow \phi_t : T^*M \rightarrow T^*M$, $(q(0), p(0)) \in T^*M$

$$\phi_t(q(0), p(0)) = (q(t), p(t))$$

• What if the equations are not easily solvable (or not solvable at all)? $\Rightarrow$ NUMERICAL METHOD $\Rightarrow \phi_h : T^*M \rightarrow T^*M$, $(q_0, p_0) = (q(0), p(0)) \in T^*M$

$$\phi_h(q_0, p_0) = (q_1, p_1),$$

• $(q_1, p_1) \simeq (q(h), p(h))$. We can manage to obtain a symplectic method, i.e.

$$\omega_0 = \omega_1, \quad \text{that is} \quad dq_0 \wedge dp_0 = dq_1 \wedge dp_1.$$

• For instance: Symplectic Euler methods

$$q_1 = q_0 + h\,H_p(q_0, p_1), \quad p_1 = p_0 - h\,H_q(q_0, p_1)$$

Hairer, Lubich, Wanner: "Geometric Numerical Integration: Structure Preserving algorithms for ODEs" (2002)

$$\phi_h : T^*M \to T^*M$$

$$((q_0, p_0), (q_1, p_1)) \in T^*_{q_0}M \times T^*_{q_1}M$$

J.C. Marrero, D. Martín de Diego and A. Stern. "Symplectic groupoids and discrete constrained Lagrangian mechanics" (2011).

• Discrete Constrained Lagrangian (DCL) system can be defined (C_d, L_d), $C_d \subset M \times M$ is a submanifold, $L_d : C_d \to \mathbb{R}$ is the discrete Lagrangian function.

- Thus $\Sigma_{L_d} \subset (T^*(M \times M), \omega_{M\times M})$ is a LS by Tulczyjew's Theorem.

• The proper scenario to describe Hamiltonian symplectic numerical methods is $(T^*M \times T^*M, \Omega)$, where $\Omega = pr_1^*\,\omega_M - pr_0^*\,\omega_M$, and $pr_i : T^*M \times T^*M \to T^*M$

Second Question

Is there a way to relate a discrete (constrained) Lagrangian Submanifold and a general discrete symplectic Hamiltonian dynamics?

$$\Lambda \subset T^*M \times T^*M \sim \Sigma_{L_d} ??$$

Discrete Equivalence

- Symplectomorphism

$$\begin{array}{rcl} \Upsilon : (T^*(M \times M)\,,\,\omega_{M\times M}) & \rightarrow & (T^*M \times T^*M\,,\,\Omega) \\ \gamma_{(q_0,q_1)} \equiv (\gamma_{q_0}\,,\gamma_{q_1}) & \mapsto & (-\gamma_{q_0}\,,\gamma_{q_1}) \end{array}$$

where $(q_0, q_1) \in M \times M$ and $\gamma_{q_i} \in T^*M$.

- We generate $\Upsilon(\Sigma_{L_d})$ LS of $T^*M \times T^*M$.

- Dynamics: $\gamma_{q_0}, \ldots, \gamma_{q_N}$ s.t. $(\gamma_{q_i}, \gamma_{q_{i+1}}) \in \Upsilon(\Sigma_{L_d})$, $0 \leq i \leq N-1$.

$$\gamma_{q_k} \in T^*_{q_k}M \cap pr_0(\Upsilon(\Sigma_{L_d})) \cap pr_1(\Upsilon(\Sigma_{L_d})), 1 \leq k \leq N-1.$$

D. Iglesias, J.C. Marrero, D. Martín de Diego and E. Padrón: “Discrete Dynamics in Implicit Form” (2011).

Discrete Equivalence

Constrained Discrete Legendre Transformations

The mappings $\mathbb{F}L_d^{\pm} : \Sigma_{L_d} \longrightarrow T^*M$ are defined by

$$\begin{aligned}\mathbb{F}L_d^- &= pr_0 \circ \Upsilon\big|_{\Sigma_{L_d}},\\ \mathbb{F}L_d^+ &= pr_1 \circ \Upsilon\big|_{\Sigma_{L_d}}.\end{aligned}$$

We'll say that (L_d, C_d) is *regular* if $\mathbb{F}L_d^-$ is a local diffeomorphism and *hyperregular* if $\mathbb{F}L_d^-$ is a global diffeomorphism.

• Generating function: if N LS $N \subset (M, \omega = d\theta)$, where θ is the Liouville 1-form. Then holds that

$$0 = i_N^*\omega = d(i_N^*\theta),$$

and consequently, by Poincaré's Lemma $i_N^*\theta = dS$. S is the generating function of N.

- Consider Λ LS $\Lambda \subset T^*M \times T^*M$.
- Consider $C_d \subset M \times M$, $L_d : C_d \to \mathbb{R}$, and the LS Σ_{L_d}.
- Consider $\Upsilon^{-1}(\Lambda) \subset (T^*(M \times M), \omega_{M\times M})$.
- Assume $i^*_{\Upsilon^{-1}(\Lambda)}\theta_{M\times M} = dS$, i.e. there exists a generating function S of $\Upsilon^{-1}(\Lambda)$.
- Assume that $\pi_{M\times M}(\Upsilon^{-1}(\Lambda)) = C_d$ is a submersion with connected fibers.

Theorem

Under the previous conditions the function $S : \Upsilon^{-1}(\Lambda) \to \mathbb{R}$ is $(\pi_{M\times M})\big|_{\Upsilon^{-1}(\Lambda)}$-projectable onto a function $L_d : C_d \to \mathbb{R}$. Moreover, the following equation holds

$$\Upsilon^{-1}(\Lambda) = \Sigma_{L_d} \,.$$

$$\Lambda \subset T^*M \times T^*M \sim \Sigma_{L_d} \text{ YES!!}$$

Example: Martinet type sub-Riemannian structure (continuous case)

$$H(q,p) = \frac{1}{2}\left(\left(p_x + p_z\frac{y^2}{2}\right)^2 + \frac{p_y^2}{(1+\beta x)^2}\right),$$

- $q = (x, y.z)^T \in \mathbb{R}^3$ and $p = (p_x, p_y, p_z) \in (\mathbb{R}^3)^* \simeq \mathbb{R}^3$.
- $\mathbb{F}H(x,y,z;p_x,p_y,p_z) = (x,y,z\,;\,(p_x + p_y\frac{y^2}{2}), \frac{p_y}{(1+\beta x)^2}, \left(p_x + p_z\frac{y^2}{2}\right)\,\frac{y^2}{2}) = (x,y,z;\dot{x},\dot{y},\dot{z})$

$$C \subset T\mathbb{R}^3 = \left\{(x,y,z\,;\,\dot{x},\dot{y},\dot{z}) \ \text{ s.t. } \ \dot{z} = \frac{y^2}{2}\,\dot{x}\right\}.$$

- $L \circ \mathbb{F}H = \Delta^* H - H$:

$$L(q,\dot{q}) = \frac{1}{2}\left(\dot{x}^2 + (1+\beta x)^2\dot{y}^2\right).$$

- Corresponds to the sub-Riemannian structure (Δ, g), being $\Delta = \ker\alpha$ for $\alpha = dz - \frac{y^2}{2}dx$ and $g = \mathrm{d}x^2 + (1+\beta x)^2\mathrm{d}y^2$. It is clear that $L(q,\dot{q}) = \frac{1}{2}\,g(\partial/\partial q, \partial/\partial q)$.

Example: Martinet type sub-Riemannian structure (discrete case)

- We apply a symplectic Euler method to the Martinet Hamiltonian

$$q_1 = q_0 + h\, H_p(q_0, p_1), \quad p_1 = p_0 - h\, H_q(q_0, p_1).$$

- We define $\Upsilon^{-1}(\Lambda)$ by means of the generating function $H^+(q_0, p_1) = q_0\, p_1 + h\, H(q_0, p_1)$ i.e.

$$\begin{aligned} x_1 &= x_0 + h\left((p_1)_x + (p_1)_z \frac{y_0^2}{2})\right), \\ y_1 &= y_0 + h\,\frac{(p_1)_y}{(1+\beta\, x_0)^2}, \\ z_1 &= z_0 + h\left((p_1)_x + (p_1)_z \frac{y_0^2}{2})\right)\frac{y_0^2}{2}, \end{aligned}$$

- which determines $C_d \subset M \times M \Rightarrow (z_1 - z_0) = \frac{y_0^2}{2}(x_1 - x_0)$.

- Finally, we find $S(q_0, q_1) = h\left(p_1 \frac{\partial H(q_0, p_1)}{\partial p_1} - H(q_0, p_1)\right)$ which is projectable onto

$$L_d(q_0, q_1) = \frac{1}{2h}\left((x_1 - x_0)^2 + \frac{y_0^2}{2}(y_1 - y_0)^2\right) = h\, L\left(q_0, \frac{q_0 - q_1}{h}\right).$$

- We have shown that given a CLS one can always find a Hamiltonian function. Moreover, we prove that given an arbitrary Hamiltonian system one can always construct a (possibly) CLS that generates the original system.

- We try to get some light over discrete mechanics, which can be interpreted as suitable LS of $T^*M \times T^*M$. We geometrically find when the discrete variational procedure (Σ_{L_d}) matchs a symplectic numerical method for the associated Hamiltonian system (Λ).

THANKS!!