HAMILTONIAN VECTOR FIELDS, OBSERVABLES AND LIE SERIES

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PLAN

- Formal power series and a realization theorem
- Lie series and Hamiltonian realizations
- Algebraic criteria for existence of Hamiltonian realizations
- Global realization theorems

CONTROLLED AND OBSERVED SYSTEMS

System:

$$\Sigma: \quad \dot{x} = f(x,u) = f_u(x), \qquad y_v = h_v(x),$$

where:
$x(t) \in M$ – state space,
$u(t) \in U$ – input space (a set, e.g. finite set),
$y_v(t) \in \mathbb{R}$,
$v \in V$ – enumerates output components (observables).

The system is represented by $\Gamma = \{M, \{f_u\}_{u\in U}, \{h_u\}_{v\in V}\}$, where:

M – real analytic manifold of dimension n;
$\{f_u\}_{u\in U}$ – a family of C^ω vector fields on M;
$\{h_u\}_{v\in V}$ – a family of C^ω functions on M;

U and V will be assumed finite, $\sharp(U) \geq 2$, $\sharp(V) \geq 1$.

FORMAL POWER SERIES OF Σ

A controlled and observed system is represented by a triple

$$\Gamma = \{M, \{f_u\}_{u\in U}, \{h_v\}_{v\in V}\}.$$

For a given $x_0 \in M$, the system Γ defines a family of formal power series in noncommuting formal variables $u \in U$:

$$S^v = \sum_{w\in U^*} S^v_w w, \qquad v \in V,$$

where:

. $w = u_1 \cdots u_k$ are words in the alphabet U,

. U^* consists of all words, including empty word, and

$$S^v_w = S^v_{u_1\cdots u_k} := (f_{u_1} \cdots f_{u_k} h_v)(x_0),$$

are numbers (iterated derivatives at x_0 of h_v along vect. fields $f_{u_k}, \ldots, f_{u_1}$).

Question: Does the family $\{S^v\}_{v\in V}$ represent "completely" system Γ?

REALIZATION PROBLEM

This question can be stated as a realization problem:

- Given a family of formal power series $S = \{S^v\}_{v \in V}$, does there exist a controlled and observed system $\Gamma = \{M, \{f_u\}_{u \in U}, \{h_u\}_{v \in V}\}$ and a point $x_0 \in M$ such that its series at x_o coincide with the given ones?

- If so, in what sense is $\Gamma = \{M, \{f_u\}_{u \in U}, \{h_u\}_{v \in V}\}$ unique?

REALIZATION THEOREM

We impose two conditions on the family of formal power series $S = \{S^v\}_{v \in V}$:

Convergence condition: $\exists\ C > 0,\ R > 0$ such that, for any word $w = u_1 \cdots u_k$,

$$|S^v_{u_1 \cdots u_k}| \ \leq\ CR^k k!. \qquad (C)$$

Rank condition:

$$\text{rank}_L S \ <\ \infty. \qquad (R)$$

THM Existence. A family

$$S = \{S^v\}_{v \in V}$$

of formal power series corresponds to a local analytic system

$$\Gamma = \{M, \{f_u\}_{u \in U}, \{h_u\}_{v \in V}\}$$

at a point $x_0 \in M$ iff it satisfies conditions (C) and (R).
Then there exists Γ with $\dim M = \text{rank}_L S$.

Uniqueness. If two systems Γ and $\tilde{\Gamma}$ of dimension $n = \text{rank}_L S$ correspond to the same family S then they are related by a local C^ω-diffeomorphism.

Remark Statement of a similar THM: M. Fliess, Inv. Math. 1983.

Proofs: [J86a] (see also [J00]), Sussmann 1989 (?), unpublished.

Global versions: [J80] and [J86c].

THE LIE RANK

The Lie rank used in the theorem is (Fliess 83):

$$\text{rank}_L S = \sup \text{rank}\,(S^{v_j}_{L_i w_j})^k_{i,j=1}$$

where the supremum of ranks of $k \times k$ matrices is taken:
over all $k \geq 1$,
over all Lie polynomials $L_1, \ldots, L_k \in Lie\{U\}$,
over all words $w_1, \ldots, w_k \in U^*$,
and over all elements $v_1, \ldots, v_k \in V$.

PART II:

HAMILTONIAN REALIZATION PROBLEM

SYMPLECTIC AND POISSON STRUCTURES

Let (M, ω) – symplectic manifold, with ω – closed, nondegenerate 2-form $\omega \in \Lambda^2(T^*M)$.

ω defines Poisson bracket: for $\phi \in C^\infty(M)$, $\psi \in C^\infty(M)$,

$$\{\phi, \psi\} = P(d\phi, d\psi) = \sum P_{k\ell} \frac{\partial \phi}{\partial x^k} \frac{\partial \psi}{\partial x^\ell},$$

where $P = \omega^{-1} \in \Lambda^2(TM)$ is the Poisson tensor corresponding to ω:

$$\omega = \sum a_{ij} dx^i \wedge dx^j,$$

$$P = \sum P_{k\ell} \frac{\partial}{\partial x^k} \wedge \frac{\partial}{\partial x^\ell},$$

where $(P_{k\ell}) = (a_{ij})^{-1}$. Poisson bracket is antisymmetric and satisfies

$$\{\phi, \{\psi, \gamma\}\} + \{\psi, \{\gamma, \phi\}\} + \{\gamma, \{\phi, \psi\}\} = 0, \qquad \text{(JACOBI)}$$

$$\{\phi, \{\psi, \gamma\}\} = \{\{\phi, \psi\}, \gamma\} + \{\psi, \{\phi, \gamma\}\}. \qquad \text{(LEIBNIZ)}$$

Poisson structure is defined in the same way by any antisymmetric tensor $P \in \Lambda^2(TM)$ so that the corresponding Poisson bracket satisfies (JACOBI).

HAMILTONIAN VECTOR FIELDS

Given a symplectic form ω on M, or a Poisson tensor P, X is a Hamiltonian vector field on M if, locally, there is a function H on M such that

$$\omega(\cdot, X) = dH,$$

or (equivalently),

$$X = P\, dH,$$

where we treat P as a linear operator $T^*M \to TM$.

Thus, any function $H : M \to \mathbb{R}$ defines a Hamiltonian vector field

$$\vec{H} = P\, dH.$$

Locally,

$$\vec{H} = \sum_{i,j} P_{ij} \frac{\partial H}{\partial x^j} \frac{\partial}{\partial x^i}\,.$$

HAMILTONIAN CONTROLLED AND OBSERVED SYSTEM

We take the input and output alphabets equal: $U = V$.

Def A system

$$\Gamma = \{M, \{f_u\}_{u \in U}, \{h_u\}_{u \in U}\}$$

is Hamiltonian if $\exists$ Poisson tensor P on M such that

$$f_u = P\, dh_u, \quad u \in U,$$

i.e., vector fields f_u are Hamiltonian, with Hamiltonians h_u.

Remarks:

- In physics h_u would be called observables and f_u – the corresponding infinitesimal symmetries.
- In control theory Hamiltonian systems appear e.g. in conservative electric cirquits (A. Van der Schaft, P. Crouch), with a slightly changed definition.

THE BRACKETING MAP

Let $\mathbb{R}\langle U\rangle$ denote the free algebra generated by U (the algebra of formal polynomials in noncommuting variables $u \in U$). There is also a Lie algebra structure in $\mathbb{R}\langle U\rangle$, with the commutator product $[P, Q] = PQ - QP$.

Let $Lie\{U\} \subset \mathbb{R}\langle U\rangle$ be the free Lie algebra generated by U (the smallest Lie subalgebra in $\mathbb{R}\langle U\rangle$ generated by the variables $u \in U \subset \mathbb{R}\langle U\rangle$).

There is a canonical linear map

$$[\ \ ] : \mathbb{R}\langle U\rangle \to Lie\{U\},$$

called here bracketing map, defined on words $w = u_1 \cdots u_k$ by

$$[u_1 u_2 \ldots u_{k-1} u_k] = [u_1, [u_2, \cdots [u_{k-1}, u_k]]]$$

and extended to the free algebra $\mathbb{R}\langle U\rangle$ by linearity.

This map is "onto". We shall later use its kernel $S = \ker[\ \ ]$.

A well known criterion says that a homogeneous polynomial $W \in Lie\{U\}$ of degree k is a Lie polynomial iff

$$[W] = kW.$$

HAMILTONIAN C-O SYSTEM DEFINES A LIE SERIES

Let $\Gamma = \{M, \{f_u\}_{u\in U}, \{h_u\}_{v\in V}\}$ be given. Denote

$$h_{u_1\cdots u_{k-1}u_k} = f_{u_1}\cdots f_{u_{k-1}} h_{u_k}.$$

If Γ is Hamiltonian then we also have, for $w = u_1\cdots u_k$,

$$h_w = h_{u_1\cdots u_{k-1}u_k} = \{h_{u_1}, \{h_{u_2}, \cdots, \{h_{u_{k-1}}, h_{u_k}\}\cdots\}\}.$$

We can extend the definition of h_w from words $w = u_1\cdots u_k \in U^*$ to polynomials $W = \sum_{w\in U^*} \lambda_w w$ by linearity:

$$h_W = \sum_{w\in U^*} \lambda_w h_w.$$

Let $x \in M$ be fixed. For any word $w = u_1\cdots u_k$ we define

$$L_x([w]) = h_w(x).$$

This map extends by linearity to a unique linear function

$$L_x : Lie\{U\} \ \rightarrow \ \mathbb{R}.$$

L_x can be identified with a Lie series in noncommuting formal variables $u \in U$.

We call L_x the Lie series of Γ at x.

DOES A LIE SERIES DEFINE A HAMILTONIAN SYSTEM?

Consider now a linear function

$$L : Lie\{U\} \ \rightarrow \ \mathbb{R},$$

which we call Lie series because such a function can be identified with a Lie series in noncommuting formal variables $u \in U$.

Question 1. When a Lie series L corresponds to a Hamiltonian system?

Question 2. When a formal power series $S : \mathbb{R}\langle U\rangle \rightarrow \mathbb{R}$ has a realization

$$\{M, \{f_u\}_{u\in U}, \{h_u\}_{u\in U}, x_0\}$$

which admits a Hamiltonian structure, i.e., $\exists$ a Poisson tensor P such that $f_u = Ph_u$?

Question 3. When $\Gamma = \{M, \{f_u\}_{u\in U}, \{h_u\}_{v\in V}\}$ admits a Hamiltonian structure?

ANSWER TO Q1

THM (2011) Existence. A Lie series $L : Lie\{U\} \to \mathbb{R}$ corresponds to a C^ω Hamiltonian system $\{M, \{f_u\}_{u\in U}, \{h_u\}_{u\in U}, x_0\}$ iff

$$|L([u_1 \cdots u_k])| \ \leq \ C(R)^k k!, \qquad (A)$$

for some $C > 0$, $R > 0$, and

$$\text{rank}_K L \ < \ \infty, \qquad (R)$$

where $\text{rank}_K L$ is the rank of the bilinear map

$$Lie\{U\} \times Lie\{U\} \to \mathbb{R}$$

defined by

$$(X, Y) \mapsto L([X, Y]).$$

Then $\exists$ such a system with $\dim M = \text{rank}_K L$ and M symplectic.

Uniqueness. If two symplectic Hamiltonian systems of dimension $n = \text{rank}_K L$ correspond to the same Lie series L then they are related by a symplectomorphism.

REMARKS

- The rank $\text{rank}_K L$ corresponds to Kirillov's rank in the "method of orbits" in representation theory and geometric quantization (Souriau, Kostant, Kirillov).

- In a global version of the above theorem a group acts on the dual free Lie algebra $(Lie\{U\})^*$ (the space of Lie series). Finiteness of the rank means that $Orb(L)$ is a finite dimensional "submanifold" in $(Lie\{U\})^*$ and

$$M = Orb(L).$$

The natural symplectic structure on M corresponds to the symplectic structure in the method of orbits.

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PART III:

When a controlled and observed system admits a Hamiltonian structure?

CRITERIA FOR Γ TO ADMIT A HAMILTONIAN STRUCTURE

Consider a system $\Gamma = \{M, \{f_u\}_{u \in U}, \{h_u\}_{v \in V}\}$.

Question. When Γ admits a Hamiltonian structure, i.e., ∃ Poisson tensor P such that $f_u = P\, dh_u$, for any $u \in U$?

Define functions

$$h_{u_1 \cdots u_{k-1} u_k} = f_{u_1} \cdots f_{u_{k-1}} h_{u_k},$$

for $k \geq 1$ and $u_1, \ldots, u_k \in U$. By linearity we extend the definition to

$$h_W := \sum \lambda_w h_w, \quad \text{for} \quad W = \sum_w \lambda_w w \in \mathbb{R}\langle U \rangle.$$

THM [J86d] Equivalent:

(a) Γ admits a Hamiltonian structure.

(b) $h_W = 0$ for any $W \in \mathbb{R}\langle U \rangle$ such that $[W] = 0$.

(c) $h_{[u_1 \cdots u_k]} = k h_{u_1 \cdots u_k}$, for any $k \geq 2$, $u_1, \ldots, u_k \in U$.

CRITERIA FOR Γ TO ADMIT A HAMILTONIAN STRUCTURE

THM (repeated) Equivalent:

(a) Γ admits a Hamiltonian structure.

(b) $h_W = 0$ for any $W \in \mathbb{R}\langle U\rangle$ such that $[W] = 0$.

(c) $h_{[u_1 \cdots u_k]} = k h_{u_1 \cdots u_k}$, for any $k \geq 2$, $u_1, \ldots, u_k \in U$.

(d) The linear map $S : \mathbb{R}\langle U\rangle \to C^\infty(M)$ given by

$$W \to h_W$$

factorizes through a map $M : Lie\{U\} \to C^\infty(M)$
via the bracketing map $[\ \ ] : \mathbb{R}\langle U\rangle \to Lie\{U\}$, i.e.,

$$S(W) = M([W]).$$

REMARKS

- The map $M_x : Lie\{U\} \to \mathbb{R}$ defined by

$$M_x(W) = (M(W))(x)$$

can be regarded as a momentum map at x for the Hamiltonian system $\Gamma = \{M, \{f_u\}_{u\in U}, \{h_u\}_{v\in V}\}$. More precisely, the family of Hamiltonian vector fields

$$\{f_u\}_{u\in U}$$

defines a Poisson (symplectic) action of a ∞-dimensional group (pseudogroup) generated by the (local) flows of all f_u.

- The momentum M_x can be identified with the Lie series of the system $\{M, \{f_u\}_{u\in U}, \{h_u\}_{v\in V}\}$ at x, i.e.,

$$L_x = M_x.$$

COMBINATORICS: CONDITION (B) MADE EFFECTIVE

Condition (b) for system $\Gamma = \{M, \{f_u\}_{u \in U}, \{h_u\}_{v \in V}\}$ to admit a Hamiltonian structure says:

(b) $h_W = 0$ for any $W \in \mathbb{R}\langle U \rangle$ such that $[W] = 0$.

It can be replaced by:
(b') $h_W = 0$ for any homogeneous polynomial $W \in \mathbb{R}\langle U \rangle$ such that $[W] = 0$ and $deg(W) \leq 3N$, if Γ is observable of order N (i.e., 1-forms dh_w with words $w \in U^*$ of length at most N span T^*M at each x).

Denote:

$S_k =$ space of homogeneous polynomials W of deg. k such that $[W] = 0$.

Proposition [J86d] If Γ is observable of order N then the number of independent conditions in (b') is $p_2 + p_3 + \cdots + p_{3N}$, with

$$p_k = \frac{r}{k-1} \sum_{d|k-1} \mu(d) r^{(k-1)/d} - \frac{1}{k} \sum_{d|k} \mu(d) r^{k/d},$$

where μ is the Möbius function on the positive integers: $\mu(d) = 0$ if d has multiple divisors and $\mu(d) = (-1)^s$, where s is the number of prime divisors.

NUMBER OF INDEPENDENT CONDITIONS

If $r = \sharp(U) = 2$, then:

k :	1	2	3	4	5	6	7	8
$\dim S_k$:	0	3	6	13	26	55	110	226
p_k :	0	3	0	1	0	3	0	6

and the independent conditions are given by the words W in the alphabet $U = \{u, v\}$:

$$uu,\ \ vv,\ \ uv + vu,\ \ [uv]uv,$$

$$[uuv]uuv,\ \ [vvu]vvu,\ \ [uv]uvuv + [uvuv]uv,$$

$$[uuuv]uuuv,\ \ [vvvu]vvvu,\ \ [uuv]vvvvu + [vvvvu]uuv,$$

$$[vvu]uuuuv + [uuuuv]vvu,\ \ [uv]uvuvuv + [uvuvuv]uv,\ \ [uvuv]uvuv.$$

PART IV:

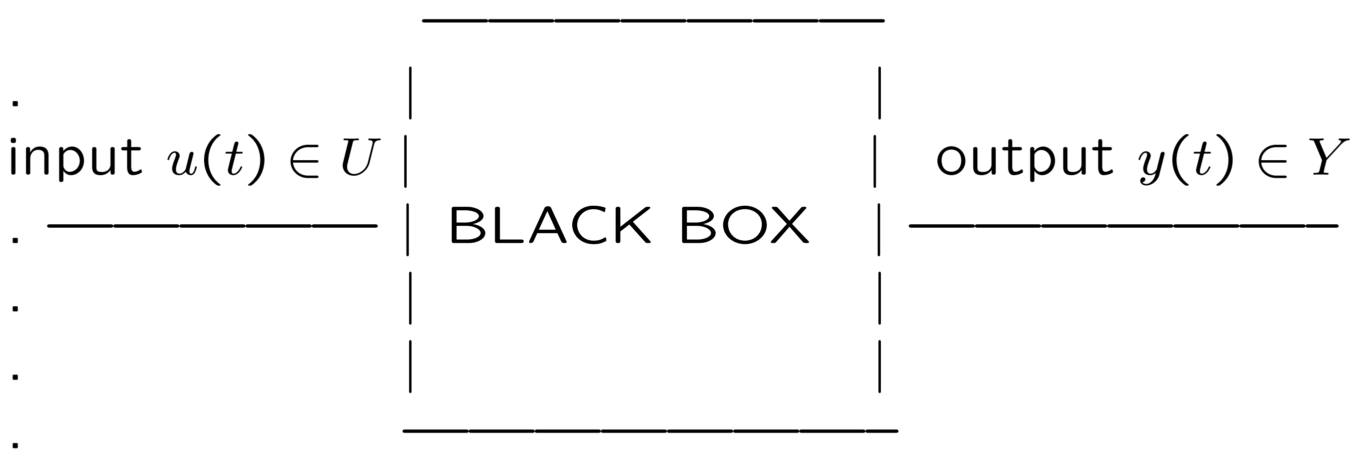
A global realization theorem

A black box system

U − input space (a set or $\mathbb{R}^m$)

Y − output space (e.g. $Y = \mathbb{R}^p$). For simplicity we take $Y = \mathbb{R}$.

$t \in [0, T)$ − time



Input-output map

Convention: the black box maps input signals into output signals

$$u(\cdot) \longmapsto y(\cdot).$$

This map is nonlinear and non-anticipating (causal), i.e.,

$y(t)$ depends only on $u|_{[0,t)}$.

Such a map is called input-output map and may be explicitly represented by

- causal functionals on a semigroup of inputs
- Volterra series
- Chen-Fliess series
- formal power series of noncommuting variables

Semigroup of piecewise constant inputs

Let (t, u) denote the constant function, equal to $u \in U$ on $[0, t)$.

Using concatenation, piecewise constant functions $a : [0, T_a) \to U$ can be written as

$$a = (t_1, u_1)(t_2, u_2) \cdots (t_k, u_k),$$

$k \geq 0$, $t_i > 0$, $u_i \in U$, with $T_a = t_1 + \cdots + t_k$ and

$$a(t) = u_i, \quad \text{for} \quad t \in [T_{i-1}, T_i),$$

where $T_0 = 0$ and $T_i = t_1 + \cdots + t_i$.

They form a semigroup S_U, with multiplication = concatenation. For $k = 0$ we get the empty domain function e (neutral element).

Causal function (input-output function)

Given semigroup of inputs

$$S_U = \{ (t_1, u_1) \cdots (t_k, u_k), \quad k \geq 0, \ \ t_i > 0, \ \ u_i \in U \},$$

a function

$$F : S_U \to \mathbb{R}$$

is called a causal function(al) or input-output function.

F is one of possible representations of the black box behavior.

In the analytic case it is equivalent to representation by a formal power series $S = \sum_{w \in U^*} S_w w$ (see [J86b]), where

$$S_w = S_{u_1 \cdots u_k} = \frac{d}{dt_1} \cdots \frac{d}{dt_k} F((t_1, u_1) \cdots (t_k, u_k))|_{t_1 = \cdots = t_k = 0}.$$

Controlled and observed system

Consider an analytic control system

$$\Sigma : \quad \dot{x} = f(x,u), \quad x(0) = x_0, \quad y = h(x),$$

where $x(t) \in M$ – a real analytic, connected manifold,

$u(t) \in U$ – a set (possibly infinite),

$y(t) \in Y = \mathbb{R}$,

$h : M \to \mathbb{R}$ is an analytic function (observable).

Assume, the vector fields $f_u = f(\cdot, u)$ are complete.

Input-output function of controlled and observed system

$$\Sigma : \quad \dot{x} = f(x,u), \quad x(0) = x_0, \quad y = h(x).$$

Given a piecewise constant control $u = a : [0, T_a) \to U$ in S_U, let

$$\Phi_a(x_0) = x_a(T_a)$$

denote the final point, at $t = T_a$, of the corresponding trajectory of Σ. Put

$$y = h(\Phi_a(x_0)).$$

The map $F_\Sigma : S_U \to \mathbb{R}$ defined by

$$F_\Sigma : \; a \mapsto h(\Phi_a(x_0))$$

is a causal function, called input-output function of Σ.

The realization problem

Each analytic, complete, controlled and observed system

$$\Sigma: \quad \dot{x} = f(x,u), \quad x(0) = x_0, \quad y = h(x).$$

defines a causal function $F_\Sigma : S_U \to \mathbb{R}$.

Question: When a causal function $F : S_U \to \mathbb{R}$ comes from a controlled and observed system Σ?

Note that, given the input space U and the output space $Y = \mathbb{R}$, the system Σ is defined by the 4-tuple $\Sigma = (M, f, h, x_0)$.
Thus, in order to construct the system from a causal function $F : S_U \to \mathbb{R}$ we have to construct this 4-tuple.

The most difficult to construct is the manifold M. For Σ minimal (i.e., transitive and observable), if it exists it is unique up to a diffeomorphism (Sussmann 77) .

A realization theorem

THM (B.J., SIAM J. Control & Optim. 1980)

A causal function

$$F : S_U \rightarrow \mathbb{R}$$

has an analytic, complete realization $\Sigma = (M, f, h, x_0)$ iff

(A) all functions

$$(t_1, \ldots, t_k) \longmapsto F((t_1, u_1) \cdots (t_k, u_k)) \in \mathbb{R}$$

are analytic on $\mathbb{R}^k_+$ and have analytic extensions to $\mathbb{R}^k$, where $R_+ = [0, \infty)$;

(B) $\operatorname{rank} F < \infty$.

rank F will be defined below.

Extendability and the input group

In the construction of the realization we use the input group:

$$G_U = \{\, (t_1, u_1) \cdots (t_k, u_k), \quad k \geq 0, \quad t_i \in \mathbb{R}, \quad u_i \in U \,\} / \sim,$$

which is the free semigroup of words, with alphabet $\mathbb{R} \times U$, considered up to identifications

$$(0, u) \sim e \;\; \text{(empty word)}, \quad (t_1, u)(t_2, u) \sim (t_1 + t_2, u),$$

and the inverse

$$((t_1, u_1) \cdots (t_k, u_k))^{-1} = (-t_k, u_k) \cdots (-t_1, u_1).$$

The extendability requirement in condition (A) is equivalent to the fact that $F : S_U \to \mathbb{R}$ has a (unique) extension to G_U.

Remarks

- If U is finite then one can remove the completeness requirement and analytic extendability in (A) ([J86a], F. Celle and J.-P. Gauthier 87).

- A similar result holds in the differentiable category [J80].

- The theorem can be extended to bounded measurable inputs [J80].

THE RANK

We define ([J80]):

$$\text{rank}\, F = \sup \text{rank} \left(\frac{\partial}{\partial t_i} F(ab_j) \right)_{i,j=1}^{k}$$

where the supremum is taken over all

$$a = (t_1, u_1) \cdots (t_k, u_k) \in S_U, \quad b_1, \ldots, b_k \in S_U, \quad \text{and} \quad k \geq 1.$$

The rank on the right is the usual rank of a $k \times k$ matrix.

In the analytic case the above rank and the Lie rank used earlier are equivalent ([J86b], [J00]).

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