Tulczyjew's Triple in Classical Field Theories: Lagrangian submanifolds of premultisymplectic manifolds.

E. Guzmán

ICMAT- University of La Laguna
e-mail: eguzman@ull.es

"Workshop on Rough Paths and Combinatorics in Control Theory"
University of California, San Diego
Center for Computational Mathematics
Department of Mathematics
July 25-27, 2011.

Joint work with C. Campos and J.C. Marrero

Motivation

W. Tulczyjew (1976) succeeded in formulating Classical Mechanics in terms of Lagrangian submanifolds in special symplectic manifolds. This approach permits to deal with singular Lagrangians.

Question: Is it possible to formulate a similar result in Classical Field Theory?

In the last years, some attempts have been made in this direction. However, although accurate and interesting, they all exhibit some defects, when comparing with Tulczyjew's original work.

In our approach, we propose a solution to the previous problems and deficiencies. In a natural way, the symplectic structures in Tulczyjew's work are replaced by premultisymplectic structures.

The Lagrangian Formalism

Consider a mechanical system moving in a configuration space Q whose tangent bundle TQ describes the states – positions and velocities – of the system. The dynamics of the system is typically governed by a function $L : TQ \longrightarrow \mathbb{R}$ of the form

$$L(v_q) = \frac{1}{2} g(v_q, v_q) - U(q).$$

Variational formulation:

$$S_L(c(.)) = \int_{t_0}^{t_1} L(c^i(t), \dot{c}^i(t))dt.$$

The extremals of the S_L are characterized by the equations

$$\frac{\partial L}{\partial q^i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i} = 0.$$

Symplectic formulation:

$$X \in \mathcal{X}(TQ) \quad SODE, \quad i_X \Omega_L = dE_L.$$

In the case, L is regular (that is $\left(\frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j}\right)$ is a regular matrix), then Ω_L is symplectic over TQ. Since,

$$\exists! \ \xi_L \in \mathcal{X}(TQ) \text{ SODE and } i_{\xi_L}\Omega_L = dE_L.$$

The Hamiltonian Formalism

The motion of the previous system is governed by a function H on the phase space of the system - positions and momentas -, $H : T^*Q \longrightarrow \mathbb{R}$. The Hamiltonian function is the total energy of the system, typically,

$$H(p_q) = K(p_q) + U(q).$$

Equations of motion

Given a Hamiltonian vector field X_H. That is, $X_H \in \mathcal{X}(T^*Q)$ such that

$$i_{X_H}\Omega_Q = dH.$$

The integral curves of X_H are characterized by the equations

$$\dot{q}^i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q^i}.$$

The Legendre transformation and equivalence

Given a Lagrangian $L : TQ \longrightarrow \mathbb{R}$, the Legendre transformation is the fibered map $leg_L : TQ \longrightarrow T^*Q$ given by

$$< leg_L(v), w >:= \frac{d}{dt} L(v + tw)_{|t=0},$$

$$leg_L(q^i, \dot{q}^i) = \left(q^i, p_i = \frac{\partial L}{\partial \dot{q}^i} \right).$$

If L is regular, then leg_L is a local diffeomorphism.

If L is hiper-regular (that is, leg_L is a global diffeomorphism), we may define $h := E_L \circ leg_L^{-1}$.

Then, ξ_L and X_H are leg_L- related.

Tulczyjew's Triple for Classical Mechanics

W. Tulczyjew (1976) succeeded in formulating classical mechanics in terms of Lagrangian submanifolds in special symplectic manifolds.

Submanifold of what manifold? and Lagrangian with respect to what structure?

$$(T(T^*Q), \Omega_Q^c)$$

where Ω_Q^c is the complete lift of the canonical symplectic structure Ω_Q of T^*Q.

$$S_L = A_Q^{-1}(dL(TQ)) \qquad S_H = (b_{\Omega_Q})^{-1}(dH(T^*Q))$$

$$\begin{array}{ccccc} T^*(TQ) & \xleftarrow{\;A_Q\;} & T(T^*Q) & \xrightarrow{\;b_{\Omega_Q}\;} & T^*(T^*Q) \\ \nwarrow {\scriptstyle dL} & & & & {\scriptstyle dH} \nearrow \\ & TQ & & T^*Q & \end{array}$$

The isomorphisms A_Q and b_{Ω_Q}

$A_Q : T(T^*Q) \longrightarrow T^*(TQ)$ is defined by

- The natural pairing $< -, - >$: $T^*Q \times TQ \longrightarrow Q \times \mathbb{R}$.
- The involution map $\mathcal{K} : T(TQ) \longrightarrow T(TQ)$. This isomorphism relates the two different vector bundle structures of $T(TQ)$ over TQ.

$b_{\Omega_Q} : T(T^*Q) \longrightarrow T^*(T^*Q)$ is defined by

- The canonical symplectic structure of T^*Q.

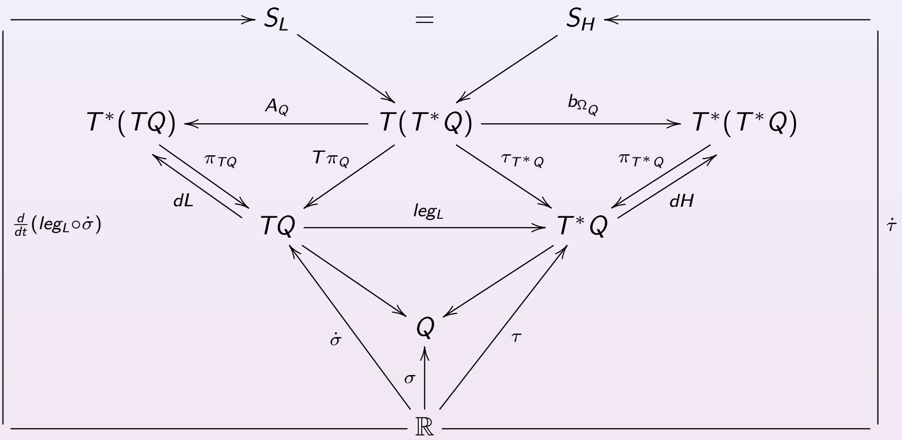
$A_Q(q^i, p_i, \dot{q}^i, \dot{p}_i) = (q^i, \dot{q}^i, \dot{p}_i, p_i)$ is a symplectomorphism

$b_{\Omega_Q}(q^i, p_i, \dot{q}^i, \dot{p}_i) = (q^i, p_i, -\dot{p}_i, \dot{q}^i)$ is an anti-symplectomorphism

$$A_Q \equiv \text{the Tulczyjew canonical diffeomorphism}$$

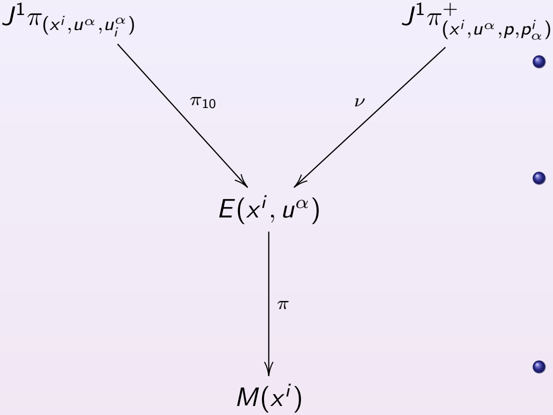
Tulczyjew's triple for Classical Mechanics

$$S_L = A_\pi^{-1}(dL(TQ)) \qquad S_h = b_{\Omega_Q}^{-1}(dH(T^*Q))$$



WHAT HAPPENS IN CLASSICAL FIELD THEORY?

Setting



- Base manifold: oriented manifold M, $dimM = m$, with fixed volume form η.
- Bundle configuration space: fibre bundle $\pi : E \longrightarrow M$, $\dim E = n + m$.
- Space of "velocities": the first jet manifold $J^1\pi = \{j^1_x\phi\}$ where $\phi \in \Gamma(\pi)$ s.t. $T\pi \circ T\phi = Id_{TM}$.
- Space of "momenta": the dual jet manifold $J^1\pi^+ \cong \Lambda^m_2 E$.

The canonical multisymplectic form on $\Lambda^m_2 E$

Remark: A multisymplectic form Ω on a vector space V is a closed $(m+1)$-form on V such that it is non-degenerate. That is the linear mapping

$$v \in V \longrightarrow i_v\Omega \in \Lambda^m V^*$$

is injective.

$\Lambda^m_2 E$ **admits a canonical (m+1) multisymplectic structure.**

$$\widetilde{\Theta}(\omega)(X_1, \ldots, X_m) = \omega(T_\omega\nu(X_1), \ldots, T_\omega\nu(X_m))$$

where $\omega \in \Lambda^m_2 E, \quad X_i \in \mathcal{X}(\Lambda^m_2 E)$ and $\nu : \Lambda^m_2 E \longrightarrow E$.

$$\widetilde{\Omega} := -d\widetilde{\Theta}.$$

Let $\mathcal{L} : J^1\pi \longrightarrow \Lambda^m M$ be a fibered mapping, i.e. Lagrangian density, $\mathcal{L} = L\eta$ where $L : J^1\pi \longrightarrow \mathbb{R}$.

Variational formulation: Let $\sigma \in \Gamma(\pi)$

$$S_{\mathcal{L}}(j^1\sigma) = \int_U \mathcal{L}(x^i, u^\alpha, u_i^\alpha).$$

The extremals of $S_{\mathcal{L}}$ are characterized by the equations

$$\frac{\partial L}{\partial u^\alpha} - \frac{d}{dx^i}\left(\frac{\partial L}{\partial u_i^\alpha}\right) = 0.$$

Multisymplectic formulation

$$(j^1\sigma)^*(i_X\Omega_L) = 0, \quad \forall X \in \mathcal{X}(J^1\pi).$$

The Hamiltonian formalism

Consider the canonical projection

$$\mu : (J^1\pi)^+ \cong \Lambda_2^m E \longrightarrow \frac{\Lambda_2^m E}{\Lambda_1^m E} = J^1\pi^*.$$

μ is a principal $\mathbb{R}$-bundle. We have a one to one correspondence between

$$h : J^1\pi^* \longrightarrow (J^1\pi)^+$$
$$h(x^i, u^\alpha, p^i_\alpha) = (x^i, u^\alpha, -H(x^i, u^\alpha, p^i_\alpha), p^i_\alpha)$$

$$F_h : (J^1\pi)^+ \longrightarrow \mathbb{R}$$
$$\lambda \longrightarrow \lambda - h(u(\alpha))$$
$$F_h(x^i, u^\alpha, p^i_\alpha) = p + H(x^i, u^\alpha, p^i_\alpha).$$

Equations of motion:
$\tau : M \longrightarrow J^1\pi^*$ a section of $\pi_1^* : J^1\pi^* \longrightarrow M$ is a solution of the Hamilton equations if

$$i_{T(h\circ\tau)(X_n)}\widetilde{\Omega} = (-1)^{m+1} dF_h$$

$$\frac{\partial u^\alpha}{\partial x^i} = \frac{\partial H}{\partial p^i_\alpha}; \quad \frac{\partial p^i_\alpha}{\partial x^i} = -\frac{\partial H}{\partial u^\alpha}$$

Equivalence

The extended Legendre map

$$Leg_L : J^1\pi \longrightarrow J^1\pi^+$$

$$Leg_L(j^1_x\phi)(X_1, \ldots, X_m) = (\Theta_L)_{j^1_x\phi}(\widetilde{X_1}, \ldots, \widetilde{X_m})$$

where $j^1\phi \in J^1\pi$ and $X_i \in T_{\phi(x)}E$, where $\widetilde{X}_i \in T_{j^1_x\phi}J^1\pi$ are such that $(T\pi_{10})(\widetilde{X}_i) = X_i$.

The Legendre transformation

$$leg_L : J^1\pi \longrightarrow J^1\pi^*$$

$$leg_L = \mu \circ Leg_L$$

$$Leg_L(x^i, u^\alpha, u^\alpha_i) = \left(x^i, u^\alpha, L - \frac{\partial L}{\partial u^\alpha_i} u^\alpha_i, \frac{\partial L}{\partial u^\alpha}\right).$$

$$leg_L(x^i, u^\alpha, u^\alpha_i) = \left(x^i, u^\alpha, \frac{\partial L}{\partial u^\alpha}\right).$$

Equivalence

The Legendre transformation leg_L is a local diffeomorphism, if and only if the Lagrangian function L is regular (that is, the Hessian matrix $\left(\frac{\partial^2 L}{\partial u_i^\alpha \partial u_j^\beta}\right)$ is regular.)

Whenever leg_L is a global diffeomorphism, we say that the Lagrangian L is hyper-regular. In this case, we may define the Hamiltonian section

$$h = Leg_L \circ (Leg_L)^{-1} .$$

Tulczyjew's triple for Classical Field Theory

In this work, we describe the Classical Field Theory of first order in terms of Lagrangian submanifolds in a premultisymplectic manifold.

A premultisymplectic structure Ω on M is a closed $(m+1)$-form on M.

This premultisymplectic manifold will be the first jet of the fibration

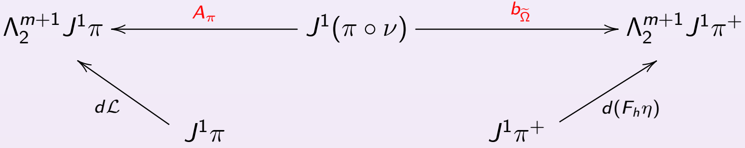
$$(\pi \circ \nu) : J^1\pi^+ \cong \Lambda_2^m E \longrightarrow E \longrightarrow M.$$

$$J^1(\pi \circ \nu)$$

$$(x^i, u^\alpha, p, p^i_\alpha, u^\alpha_j, p_j, p^i_{\alpha j}).$$

How are the Lagrangian submanifolds defined?

$$S_L = (A_\pi)^{-1}(d\mathcal{L}(J^1\pi)) \qquad S_H = (b_{\widetilde{\Omega}})^{-1}(d(F_h\eta)(J^1\pi^+))$$



Construction of the mapping A_π

$A_\pi : J^1(\pi \circ \nu) \longrightarrow \Lambda_2^{m+1} J^1\pi$ is defined by

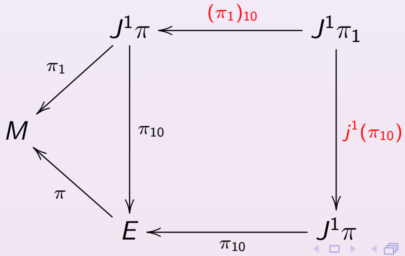
- The natural pairing $< -, - >: J^1\pi \times J^1\pi^+ \longrightarrow M \times \mathbb{R}$.

$$(j_x^1\phi)^*(\lambda) = < \lambda, j_x^1\phi > \eta_x.$$

- An involution map: Consider the first jet prolongation of the fibration $\pi_1 : J^1\pi \longrightarrow M$.

$$J^1\pi_1(x^i, u^\alpha, u_i^\alpha, \bar{u}_j^\alpha, u_{ij}^\alpha).$$

$J^1\pi_1$ admits two different affine structures over $J^1\pi$.



Construction of the mapping A_π

Let ∇ be a linear connection on M.

$$ex_\nabla(x^i, u^\alpha, u_i^\alpha, \bar{u}_j^\alpha, u_{ij}^\alpha) = (x^i, u^\alpha, \bar{u}_i^\alpha, u_j^\alpha, u_{ji}^\alpha - \Lambda_{ji}^l(u_l^\alpha - \bar{u}_l^\alpha))$$

After some operations

$$A_\pi : J^1(\pi \circ \nu) \longrightarrow \Lambda_2^{m+1} J^1\pi$$

$$A_\pi(x^i, u^\alpha, p, p_\alpha^i, u_j^\alpha, p_j, p_{\alpha j}^i) = (p_{\alpha i}^i du^\alpha + p_\alpha^i du_i^\alpha) \wedge \eta$$

Construction of the mapping $\beta_{\widetilde{\Omega}}$

$$\beta_{\widetilde{\Omega}} : J^1(\pi \circ \nu) \longrightarrow \Lambda_2^{m+1} J^1\pi^+$$

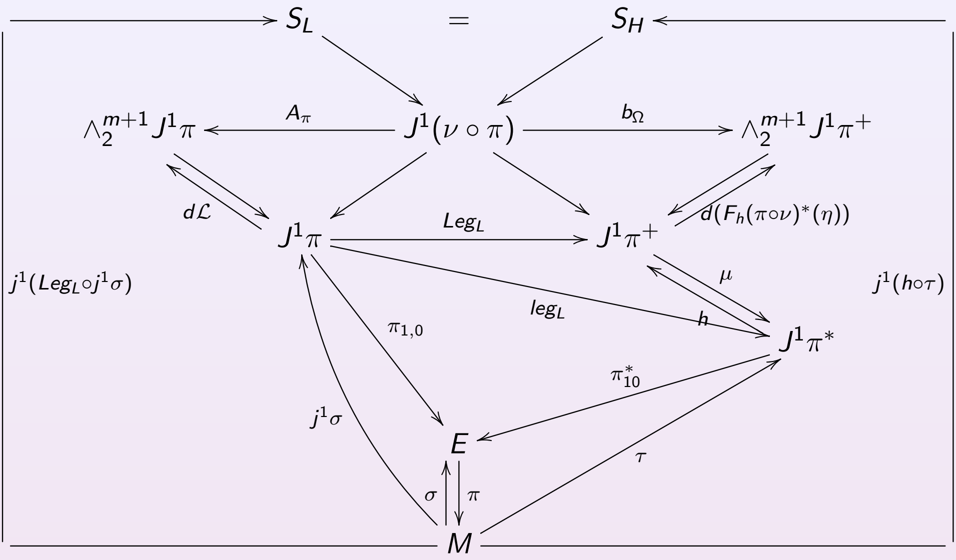
$$\beta_{\widetilde{\Omega}}(j_x^1 \lambda) = i_{h^*}\widetilde{\Omega} - (m+1)\widetilde{\Omega}$$

where $\widetilde{\Omega}$ is the canonical multisymplectic structure on $J^1\pi^+$ and

$$J^1(\pi \circ \nu) \longleftrightarrow h^* : T_\lambda J^1\pi^+ \longrightarrow T_\lambda J^1\pi^+$$

$$\beta_{\widetilde{\Omega}}(x^i, u^\alpha, p, p^i_\alpha, u^\alpha_j, p_j, p^i_{\alpha j}) = (p^j_{\alpha j} du^\alpha - 1 dp - u^\alpha_j dp^j_\alpha) \wedge \eta$$

Tulczyjew's triple for Classical Field Theory



Affine degenerate Lagrangians

Let $\gamma \colon E \to J^1\pi^+$ be a local section of the fibration $\nu \colon J^1\pi^+ \to E$.

$$\hat{\gamma} \colon J^1\pi \to \mathbb{R}$$

for all $j^1_x\phi \in J^1\pi$, $\hat{\gamma}(j^1_x\phi) = (\gamma(y))(j^1_x\phi)$, where $y = \pi_{10}(j^1_x\phi)$.

Then, we can consider $S_L = A_\pi^{-1}(d\mathcal{L}(J^1\pi))$ where the Lagrangian density $\mathcal{L} = \hat{\gamma}\eta$.

$$\gamma = \gamma_\circ(x, u)d^m x + \gamma^i_\alpha du^\alpha \wedge d^{m-1}x_i,$$

$$\hat{\gamma}(x^i, u^\alpha, u^\alpha_i) = \gamma_\circ(x, u) + \gamma^i_\alpha(x, u)u^\alpha_i.$$

$$S_L = \left\{(x^i, u^\alpha, p, p^i_\alpha, u^\alpha_j, p_j, p^i_{\alpha j}) : \; p^i_\alpha = \gamma^\alpha_i ; \; p^i_{\alpha i} = \frac{\partial \gamma_\circ}{\partial u^\alpha} + \frac{\partial \gamma^i_\beta}{\partial u^\alpha} u^\beta_i \right\}.$$

Examples: Metric-affine gravity, Dirac fermion fields.

Quadratic degenerate Lagrangians

Let $b\colon J^1\pi \to J^1\pi^+$ be a morphism of affine bundles over the identity of E such that $\mu \circ b\colon J^1\pi \to J^1\pi^*$, is an affine bundle isomorphim.
We define the quadratic Lagrangian $L(z) = \frac{1}{2} b(z)(z)$.

$$b(x^i, u^\alpha, u_i^\alpha) = (x^i, u^\alpha, \quad b_\circ(x,u) + b_\alpha^i(x,u)u_i^\alpha, \quad \tilde{b}_\alpha^i(x,u) + b_{\alpha\beta}^{ij}(x,u)u_j^\beta),$$

$$(\mu \circ b)(x^i, u^\alpha, u_i^\alpha) = (x^i, u^\alpha, \quad \tilde{b}_\alpha^i(x,u) + b_{\alpha\beta}^{ij}(x,u)u_j^\beta).$$

The condition that $\mu \circ b$ be an affine isomorphism is equivalent to $(b_{\alpha\beta}^{ij})$ is regular.

$$L(x^i, u^\alpha, u_i^\alpha) = \frac{1}{2}(b_\circ + (b_\alpha^i + \tilde{b}_\alpha^i)u_i^\alpha + b_{\alpha\beta}^{ij} u_i^\alpha u_j^\beta).$$

Then, let $S_L = A_\pi^{-1}(d\mathcal{L}(J^1\pi))$.

Example: Electromagnetic fields.

References:

- M de León, D Martín de Diego and A Santamaría-Merino, in Applied Differential Geometry and Mechanics, Editors W Sarlet and F Cantrijn, Univ. of Gent, Academia Press, (2003), 2147.
- F Cantrijn, A Ibort and M de León, J. Austral. Math. Soc. (Series A) **66** (1999), 303–330.
- E Echeverría-Enríquez, M de León, MC. Muñóz-Lecanda and N Román-Roy, Journal of Mathematical Physics **48**(2007) 112901.
- I Kolár and M Modugno, Annalti di Matematica pura ed applicata (IV). **CLVIII**(1991), 151-165.
- DJ Saunders, London Math. Soc., Lecture Note Series, **142** Cambridge Univ. Press, (1989).
- W Tulczyjew, C.R. Acad. Sci. Paris **283** (1976), 15–18.
- W Tulczyjew, C.R. Acad. Sci. Paris **283** (1976),

THANK YOU!