A Formal Power Series Approach to Nonlinear System Interconnections*

W. Steven Gray

Old Dominion University
Norfolk, Virginia USA

RPCCT 2011
San Diego, California

*This research was supported in part by NSF grant DMS 0960589.

Overview*

*See `www.ece.odu.edu/~sgray/RPCCT2011/grayslides.pdf`

1. Introduction

1.1 Fliess Operators

- Most complex systems encountered in engineering applications can be viewed in terms of interconnections of more elementary subsystems.
- A natural class of nonlinear systems to consider is the set of analytic input-output systems known as Fliess operators (Fliess, 1983).
- Such operators are described by convergent functional series which are indexed by words over a noncommutative alphabet. Their generating series are therefore specified in terms of noncommutative formal power series.
- Fliess operators constitute a special case of Volterra operators having analytic kernel functions.

- Let $X = \{x_0, x_1, \ldots, x_m\}$ be an alphabet and X^* the free monoid comprised of all words over X (including the empty word $\emptyset$) under the catenation product.
- A formal power series in X is any mapping of the form $c : X^* \to \mathbb{R}^\ell$, typically written as the formal sum $c = \sum_{\eta \in X^*} (c, \eta)\eta$. The set of all such mappings is denoted by $\mathbb{R}^\ell\langle\langle X \rangle\rangle$.
- For each $c \in \mathbb{R}^\ell\langle\langle X \rangle\rangle$, one can associate an m-input, ℓ-output operator F_c in the following manner:
  - With $t_0, T \in \mathbb{R}$ fixed and $T > 0$, define recursively for each $\eta \in X^*$ the mapping $E_\eta : L_1^m[t_0, t_0 + T] \to \mathcal{C}[t_0, t_0 + T]$ by

$$E_{x_i \bar{\eta}}[u](t, t_0) = \int_{t_0}^{t} u_i(\tau) E_{\bar{\eta}}[u](\tau, t_0)\, d\tau,$$

    where $E_\emptyset = 1$, $x_i \in X$, $\bar{\eta} \in X^*$ and $u_0 := 1$.

- The input-output operator corresponding to c is the Fliess operator

$$y = F_c[u](t) = \sum_{\eta \in X^*} (c, \eta)\, E_\eta[u](t, t_0).$$

- If there exist real numbers $K_c, M_c > 0$ such that

$$|(c, \eta)| \leq K_c M_c^{|\eta|} |\eta|!, \quad \forall\, \eta \in X^*,$$

where $|\eta|$ denotes the number of symbols in η, then c is said to be locally convergent. The set of all such series is denoted by $\mathbb{R}^{\ell}_{LC}\langle\langle X \rangle\rangle$.

- If there exist real numbers $K_c, M_c > 0$ such that

$$|(c, \eta)| \leq K_c M_c^{|\eta|}, \quad \forall\, \eta \in X^*,$$

where $|\eta|$ denotes the number of symbols in η, then c is said to be globally convergent. The set of all such series is denoted by $\mathbb{R}^{\ell}_{GC}\langle\langle X \rangle\rangle$.

- If $c \in \mathbb{R}^{\ell}_{LC}\langle\langle X \rangle\rangle$ then

$$F_c : B^m_{\mathfrak{p}}(R)[t_0, t_0 + T] \to B^{\ell}_{\mathfrak{q}}(S)[t_0, t_0 + T]$$

for sufficiently small $R, S, T > 0$, where the numbers $\mathfrak{p}, \mathfrak{q} \in [1, \infty]$ are conjugate exponents, i.e., $1/\mathfrak{p} + 1/\mathfrak{q} = 1$ (Gray & Wang, 2002).

- In particular, when $\mathfrak{p} = 1$, the series defining $y = F_c[u]$ converges provided

$$\max\{R, T\} < \frac{1}{M_c(m + 1)}$$

(Duffaut-Espinosa et al., 2009).

- The number $1/M_c(m + 1)$ will be referred to as the radius of convergence for c when M_c is the infimum of all possible geometric growth constants for which $|(c, \eta)| \leq K_c M_c^{|\eta|} |\eta|!$, $\forall\, \eta \in X^*$.

- If $c \in \mathbb{R}_{GC}\langle\langle X \rangle\rangle$ then $F_c : L^m_{\mathfrak{p},e}(t_0) \to C[t_0, t_0 + T]$ for any $T > 0$.

Definition 1.1: For any $u \in L_1^m[0,T]$ and $0 \leq t_0 \leq t_1 \leq T$, the corresponding Chen series in $\mathbb{R}\langle\langle X \rangle\rangle$ is

$$P[u](t_1,t_0) = \sum_{\eta \in X^*} \eta \, E_\eta[u](t_1,t_0).$$

Remarks:

- Every Chen series $P[u](t,0)$ is an exponential Lie series satisfying
$$\frac{d}{dt}P[u] = \left[x_0 + \sum_{i=1}^{m} x_i u_i\right] P[u], \quad P[u](0) = 1.$$
- Chen's Theorem: $P[u](t_2,t_1)P[u](t_1,t_0) = P[u](t_2,t_0)$.
- The set of driftless Chen series is a group under the Cauchy product.
- Every Fliess operator can be written in terms of a Chen series:
$$\begin{aligned} F_c[u] &= \sum_{\eta \in X^*} (c,\eta) E_\eta[u](t,0) = \sum_{\eta \in X^*} (c,\eta)(P[u](t,0),\eta) \\ &=: (c, P[u](t,0)). \end{aligned}$$

Definition 1.2: For any $c \in \mathbb{R}^{\ell}\langle\langle X\rangle\rangle$, its Hankel map is the $\mathbb{R}$-linear mapping $\mathcal{H}_c : \mathbb{R}\langle X\rangle \to \mathbb{R}^{\ell}\langle\langle X\rangle\rangle$ uniquely specified by

$$(\mathcal{H}_c(\eta), \xi) = (c, \xi\eta), \ \ \forall \xi, \eta \in X^*.$$

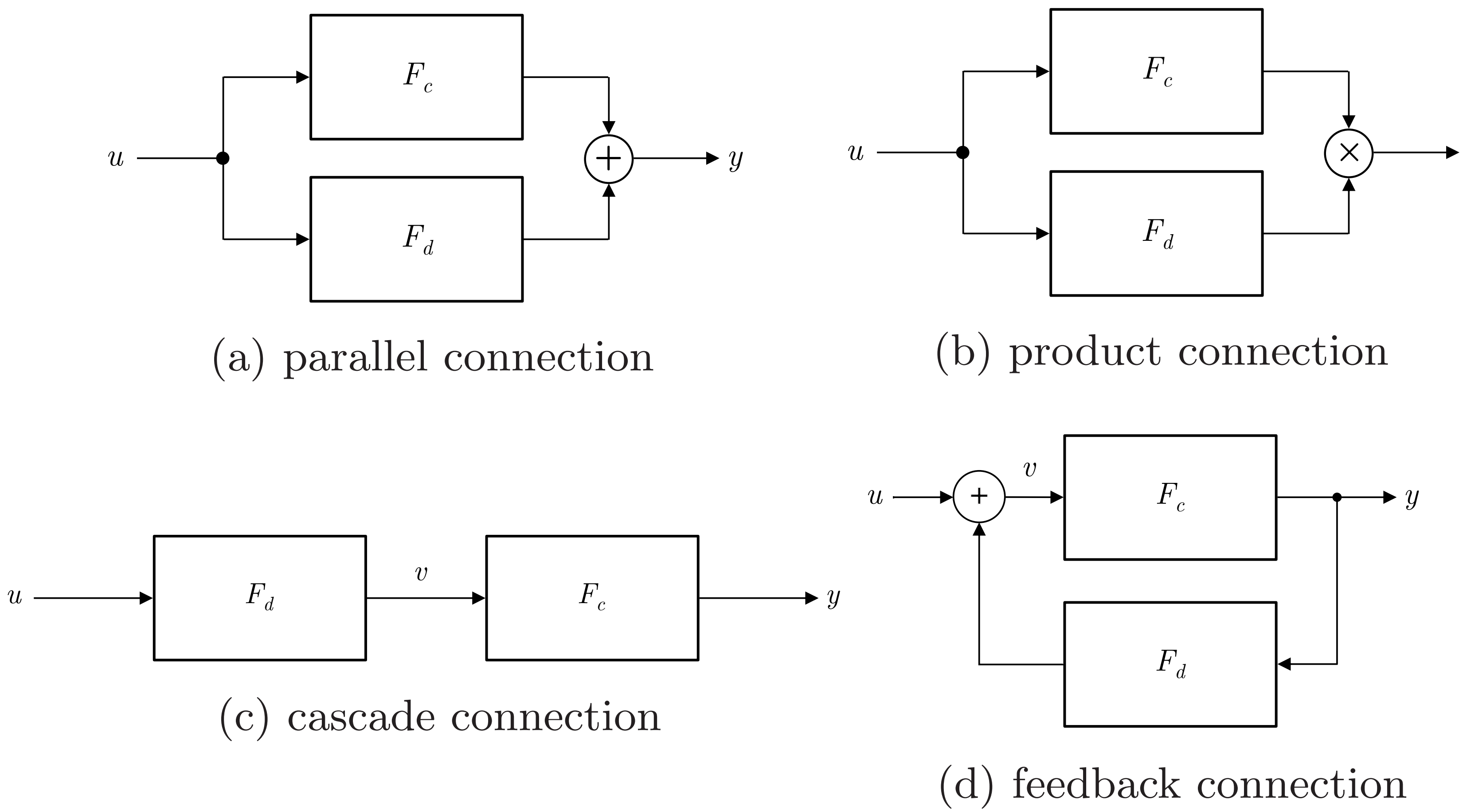
Definition 1.3: The Lie rank of $c \in \mathbb{R}^{\ell}\langle\langle X\rangle\rangle$ is $\rho_L(c) = \dim(\mathcal{H}_c(\mathcal{L}(X)))$, where $\mathcal{L}(X) \subset \mathbb{R}\langle X\rangle$ denotes the free Lie algebra generated by X.

Theorem 1.1: (Fliess, 1983) A series $c \in \mathbb{R}^{\ell}_{LC}\langle\langle X\rangle\rangle$ has finite Lie rank if and only if it is differentially generated, i.e., there exists a set of analytic vectors fields $g_0, g_1, \ldots, g_m$ and an analytic function h such that $(c, x_{i_k} \cdots x_{i_1}) = L_{g_{i_1}} \cdots L_{g_{i_k}} h(z_0)$ for some $z_0 \in \mathbb{R}^n$.

Remark: In which case, $F_c : u \mapsto y$ has the state space realization

$$\begin{aligned} \dot{z} &= g_0(z) + \sum_{i=1}^{m} g_i(z)\, u_i, \ \ z(t_0) = z_0 \\ y &= h(z). \end{aligned}$$

1.2 Interconnections of Fliess Operators



(a) parallel connection

(b) product connection

(c) cascade connection

(d) feedback connection

Fig. 1.1 Four elementary system interconnections

Basic questions to consider for each interconnection type:

1. Does the composite input-output system $u \mapsto y$ have a Fliess operator representation?
2. If so, what is its generating series?
3. On what class of inputs is the interconnection well-posed?
4. If all the subsystems are locally convergent, is the composite system locally convergent?
5. If all the subsystems are globally convergent, is the composite system globally convergent?
6. What is the corresponding radius of convergence for the interconnections?

What is known about these questions:

1. All four interconnections have a Fliess operator representation (Ferfera, 1980).

2. The generating series for parallel, product and cascade connections can be computed using Cauchy and shuffle products (Ferfera, 1980). The generating series for feedback can be written using the antipode of a Faà di Bruno Hopf algebra (Gray & Duffaut Espinosa, 2011).

3. All four connections are well-posed at least on $L_1^m[t_0, t_0 + T]$ (Gray & Wang, 2002; Gray & Li, 2005).

4. Local convergence is preserved: parallel (trivial), product (Wang, 1990), cascade (Gray & Li, 2005), feedback (Thitsa & Gray, 2011).

5. Global convergence is preserved by the parallel and product connections but not by the cascade or feedback connection (Gray et al. 2009).

6. The radius of convergence has been computed for all four interconnection types (Thitsa & Gray, 2011).

2. Nonrecursive System Interconnections

2.1 Parallel and Product Interconnections

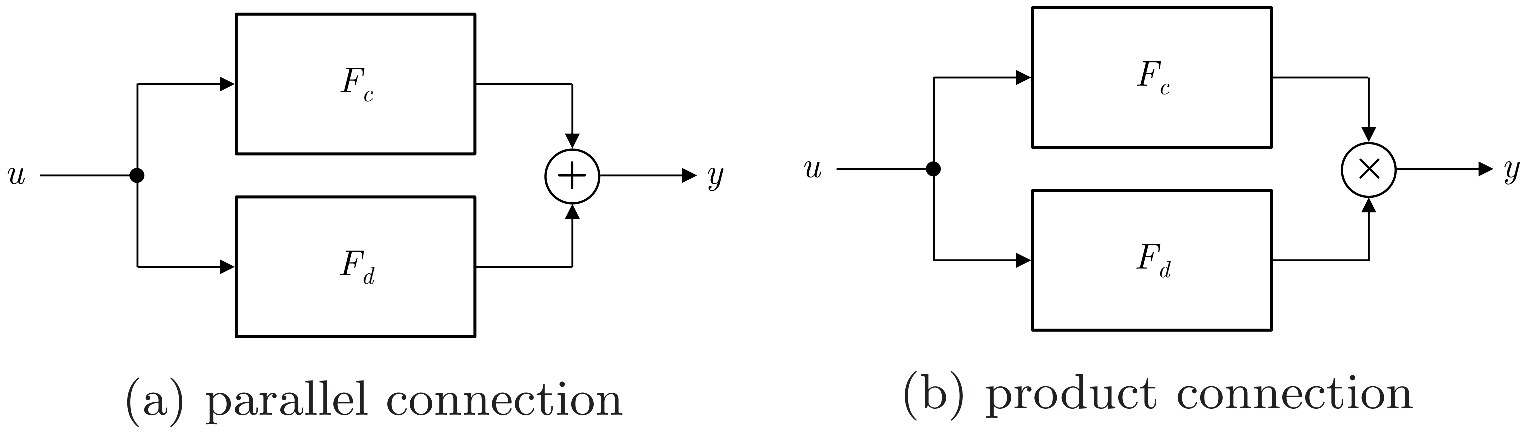


Fig. 2.1 Parallel and product system interconnections

Theorem 2.1: Let $X = \{x_0, x_1, \ldots, x_m\}$. If $c, d \in \mathbb{R}^\ell\langle\langle X\rangle\rangle$ then:

1. $F_c + F_d = F_{c+d}$

2. $F_c \cdot F_d = F_{c \,\sqcup\!\sqcup\, d}$.

Remark: It can be shown by integration by parts that the shuffle product, denoted by $\sqcup\!\sqcup$, satisfies $E_\eta E_\xi = E_{\eta \,\sqcup\!\sqcup\, \xi}$ (Fliess, 1981).

2.2 Cascade Interconnection

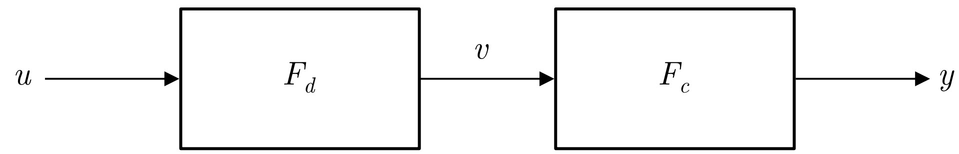


Fig. 2.2 Cascade interconnection

For any $x_i \in X$, $\eta \in X^*$ and $d \in \mathbb{R}^m\langle\langle X \rangle\rangle$ observe

$$E_{x_i\eta}[F_d[u]](t,t_0) = \int_{t_0}^{t} F_{d_i}[u](\tau)\, \underbrace{E_\eta[F_d[u]](\tau,t_0)}_{:=F_{\eta\circ d}[u](\tau)}\, d\tau = F_{x_0(d_i \sqcup\!\sqcup\, \eta\circ d)}[u](t).$$

Define the corresponding family of mappings

$$D_{x_i} : \mathbb{R}\langle\langle X \rangle\rangle \to \mathbb{R}\langle\langle X \rangle\rangle : e \mapsto x_0(d_i \sqcup\!\sqcup\, e),$$

where $i = 0, 1, \ldots, m$ and $d_0 := 1$. $D_\emptyset$ is the identity map on $\mathbb{R}\langle\langle X \rangle\rangle$. Such maps can be composed in an obvious way, $D_{x_i x_j} := D_{x_i} D_{x_j}$, to produce an $\mathbb{R}$-algebra.

Definition 2.1: The composition product of a word $\eta \in X^*$ and a series $d \in \mathbb{R}^m\langle\langle X\rangle\rangle$ is defined as

$$\underbrace{(x_{i_k} x_{i_{k-1}} \cdots x_{i_1})}_{\eta} \circ d = D_{x_{i_k}} D_{x_{i_{k-1}}} \cdots D_{x_{i_1}}(1) = D_\eta(1).$$

For any $c \in \mathbb{R}^\ell\langle\langle X\rangle\rangle$ define

$$c \circ d = \sum_{\eta \in X^*} (c, \eta)\, D_\eta(1).$$

Remarks:

- This product is always well defined (locally finite) and associative.
- The distributive property $(c \sqcup\!\sqcup d) \circ e = (c \circ e) \sqcup\!\sqcup (d \circ e)$ holds.
- This product is $\mathbb{R}$-linear in the left argument by not the right.

Theorem 2.2: Let $X = \{x_0, x_1, \ldots, x_m\}$. If $c \in \mathbb{R}^\ell\langle\langle X\rangle\rangle$ and $d \in \mathbb{R}^m\langle\langle X\rangle\rangle$ then $F_c \circ F_d = F_{c \circ d}$.

Lemma 2.1: (Berstel & Reutenauer, 1984) For any fixed real number σ such that $0 < \sigma < 1$, the $\mathbb{R}$-vector space $\mathbb{R}^{\ell}\langle\langle X\rangle\rangle$ with mapping

$$\begin{aligned} \mathrm{dist} \quad &: \quad \mathbb{R}^{\ell}\langle\langle X\rangle\rangle \times \mathbb{R}^{\ell}\langle\langle X\rangle\rangle \rightarrow \mathbb{R} \\ &: \quad (c,d) \mapsto \sigma^{\mathrm{ord}(c-d)}, \end{aligned}$$

where $\mathrm{ord}(c) := \min\{|\eta| \in X^* : \eta \in \mathrm{supp}(c)\}$, is a complete ultrametric space.

Remarks:

- The composition product is continuous in both arguments over the ultrametric topology.
- For any $c \in \mathbb{R}^m\langle\langle X\rangle\rangle$, the mapping $d \mapsto c \circ d$ is a contraction on $\mathbb{R}^m\langle\langle X\rangle\rangle$ with the ultrametric.

3.0 The Feedback Interconnection

3.1 Feedback Product Defined

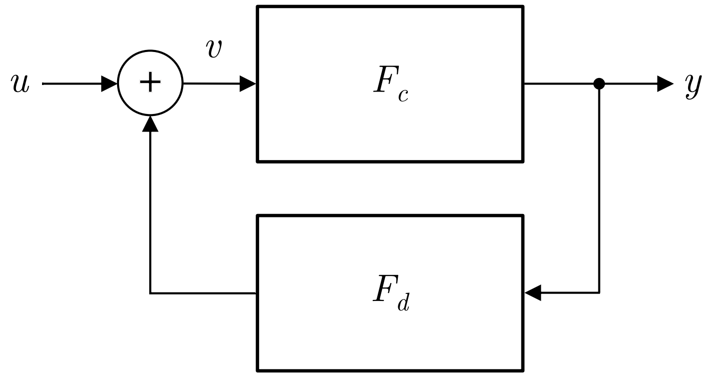


Fig. 3.1 Feedback connection of F_c and F_d.

Given $c, d \in \mathbb{R}^m\langle\langle X \rangle\rangle$, the output y satisfies the feedback equation

$$y = F_c[v] = F_c[u + F_d[y]].$$

If there exists a generating series e so that $y = F_e[u]$, then

$$F_e[u] = F_c[u + F_{d \circ e}[u]] = F_{c \tilde{\circ} (d \circ e)}[u],$$

where $\tilde{\circ}$ denotes the modified composition product.

Specifically, the modified composition product is defined as

$$c\tilde{\circ}d = \sum_{\eta \in X^*} (c,\eta)\, \tilde{D}_\eta(1),$$

where

$$\tilde{D}_{x_i} : \mathbb{R}\langle\langle X\rangle\rangle \to \mathbb{R}\langle\langle X\rangle\rangle : e \mapsto x_i e + x_0(d_i \sqcup\!\sqcup\, e)$$

with $d_0 := 0$.

Definition 3.1: The feedback product of c and d, namely $c@d$, is the unique fixed point of the contractive iterated map

$$\tilde{S} : e_i \mapsto e_{i+1} = c\tilde{\circ}(d \circ e_i).$$

Remarks:

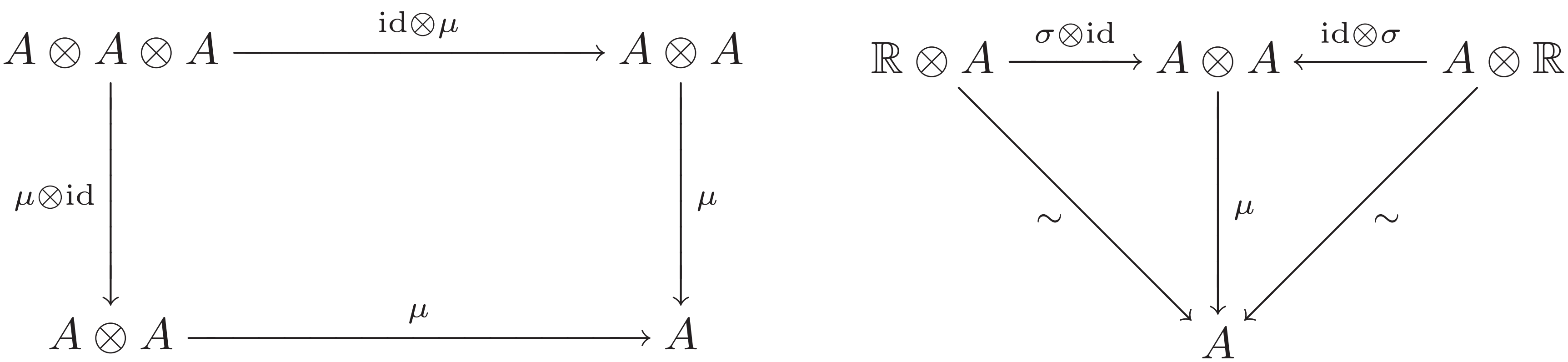
- For a unity feedback system (i.e., $F_d = I$), $\tilde{S}$ reduces to $e_{i+1} = c\tilde{\circ}e_i$.
- This approach yields no explicit way to compute $c@d$.

3.2 Hopf Algebra Fundamentals

Let A be an $\mathbb{R}$-vector space.

The triple (A, μ, σ) denotes an associative $\mathbb{R}$-algebra with the $\mathbb{R}$-bilinear multiplication map and an $\mathbb{R}$-linear unit map, respectively,

$$\mu : A \otimes A \to A, \quad \sigma : \mathbb{R} \to A.$$



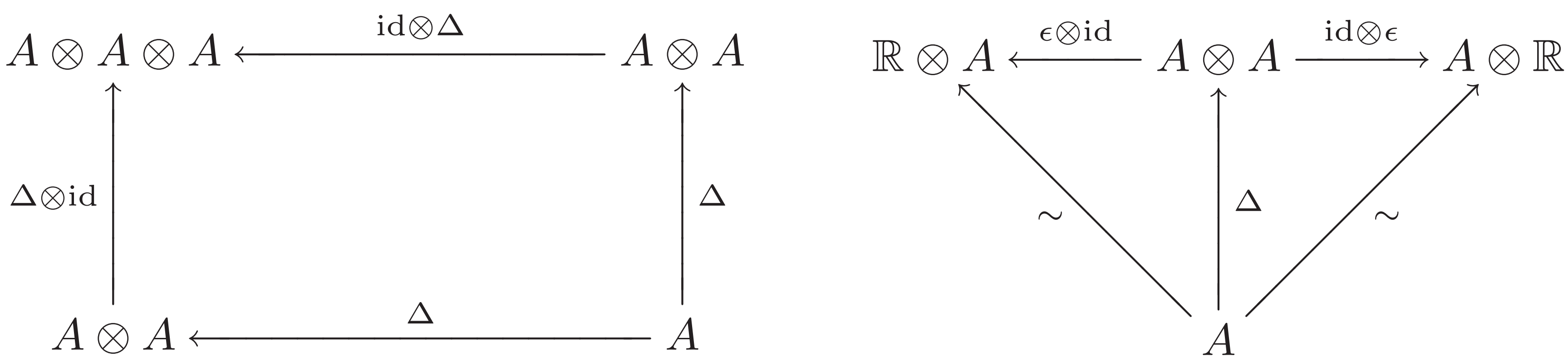
(a) The associative property

(b) The unitary property

Fig. 3.2 Defining properties of an $\mathbb{R}$-algebra (A, μ, σ).

The triple (A, Δ, ϵ) is an $\mathbb{R}$-coalgebra with the $\mathbb{R}$-linear comultiplication map and counit map, respectively,

$$\Delta : A \to A \otimes A, \quad \epsilon : A \to \mathbb{R}.$$



(a) The coassociative property

(b) The counitary property

Fig. 3.3 Defining properties of an $\mathbb{R}$-coalgebra (A, Δ, ϵ).

Definition 3.2: A morphism between two $\mathbb{R}$-algebras (A_1, μ_1, σ_1) and (A_2, μ_2, σ_2) is any $\mathbb{R}$-linear map $\psi : A_1 \longrightarrow A_2$ such that

$$\psi \circ \mu_1 = \mu_2 \circ (\psi \otimes \psi)$$
$$\psi \circ \sigma_1 = \sigma_2.$$

Remark: An analogous definition exists for a $\mathbb{R}$-coalgebra morphism.

Definition 3.3: The five-tuple $(A, \mu, \sigma, \Delta, \epsilon)$ is called an $\mathbb{R}$-bialgebra when Δ and ϵ are both $\mathbb{R}$-algebra morphisms.

Thus, $\Delta : A \longrightarrow A \otimes A$ must be an $\mathbb{R}$-algebra morphism between the $\mathbb{R}$-algebras (A, μ, σ) and $(A \otimes A, \mu_{A\otimes A}, \sigma_{A\otimes A})$, where

$$\begin{aligned} \mu_{A\otimes A} &: (A \otimes A) \otimes (A \otimes A) \longrightarrow A \otimes A \\ &: (a_1 \otimes a_2) \otimes (a_3 \otimes a_4) \longmapsto \mu(a_1 \otimes a_3) \otimes \mu(a_2 \otimes a_4) \\ \sigma_{A\otimes A} &: \mathbb{R} \longrightarrow A \otimes A \\ &: k \longmapsto \sigma(k) \otimes 1_A. \end{aligned}$$

In which case, it follows directly that

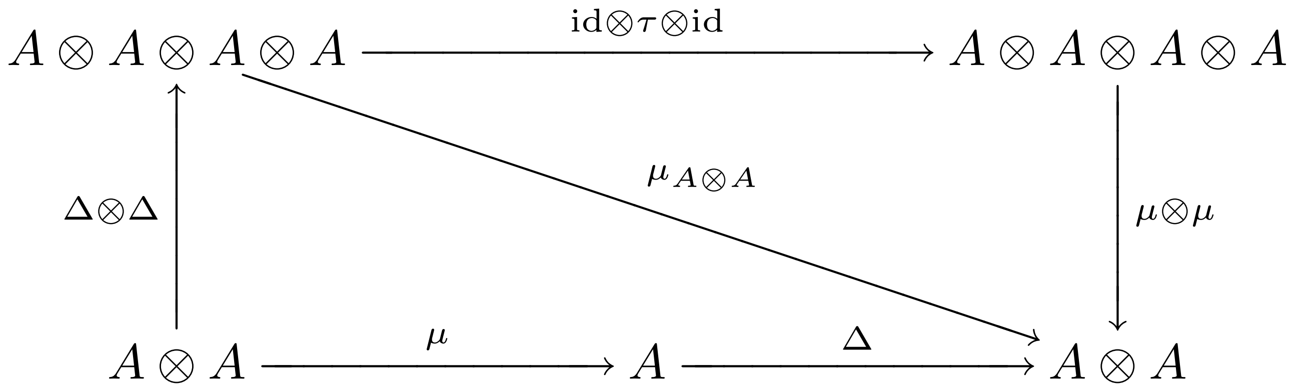
1. $\Delta \circ \mu = \mu_{A \otimes A} \circ (\Delta \otimes \Delta) = (\mu \otimes \mu) \circ (\mathrm{id} \otimes \tau \otimes \mathrm{id}) \circ (\Delta \otimes \Delta)$
2. $\Delta \circ \sigma = \sigma_{A \otimes A} = \sigma \otimes \sigma$,

where $\tau : A \otimes A \to A \otimes A : a \otimes a' \mapsto a' \otimes a$.

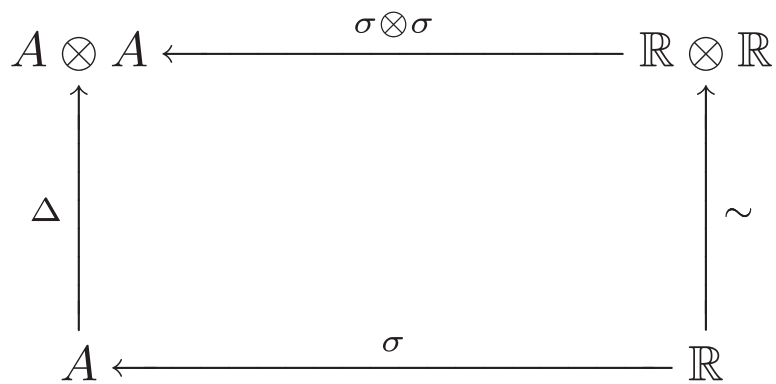
Similarly, $\epsilon : A \to \mathbb{R}$ must be an $\mathbb{R}$-algebra morphism between the $\mathbb{R}$-algebras (A, μ, σ) and $(\mathbb{R}, \mu_{\mathbb{R}}, \sigma_{\mathbb{R}})$. Therefore,

3. $\epsilon \circ \mu = \mu_{\mathbb{R}} \circ (\epsilon \otimes \epsilon) = \epsilon^2$
4. $\epsilon \circ \sigma = \sigma_{\mathbb{R}} = 1$.

Remark: An equivalent characterization of a bialgebra is one where μ and σ are both $\mathbb{R}$-coalgebra morphisms, yielding properties 1 and 3, and properties 2 and 4, respectively.

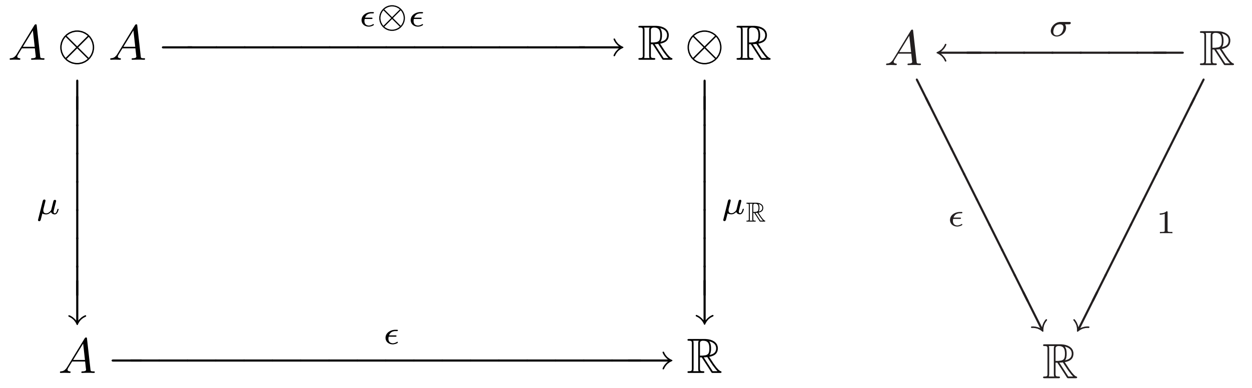


(a) Property 1



(b) Property 2

Fig. 3.4 Commutative diagrams describing Δ as an $\mathbb{R}$-algebra morphism



(c) Property 3 (d) Property 4

Fig. 3.5 Commutative diagrams describing ϵ as an $\mathbb{R}$-algebra morphism.

Consider the set of all $\mathbb{R}$-endomorphisms on A, End(A).

Given two arbitrary $f, g \in \text{End(A)}$, the Hopf convolution product is

$$f * g := \mu \circ (f \otimes g) \circ \Delta.$$

The triple $(\text{End(A)}, *, \vartheta)$ forms an associative $\mathbb{R}$-algebra with unit $\vartheta = \sigma \circ \epsilon$.

An element $\alpha \in \mathrm{End(A)}$ is an antipode of the bialgebra $(A, \mu, \sigma, \Delta, \epsilon)$ if

$$\mathrm{id} * \alpha = \alpha * \mathrm{id} = \vartheta,$$

where id is the identity map on A.

When it exists, the antipode is unique and and described by the series

$$\alpha = \mathrm{id}^{*-1} = (\vartheta - (\vartheta - \mathrm{id}))^{*-1} = \sum_{k=0}^{\infty} (\vartheta - \mathrm{id})^{*k}.$$

Definition 3.4: The six-tuple $(A, \mu, \sigma, \Delta, \epsilon, \alpha)$ is called an $\mathbb{R}$-Hopf algebra.

The set of $\mathbb{R}$-vector subspaces of A, $\{A_n\}_{n \geq 0}$, denotes a filtration of A.

The collection of $\mathbb{R}$-vector subspaces $\{A_{(n)}\}_{n \geq 0}$ is a grading of A.

3.3 A Faà di Bruno Hopf Algebra for Fliess Operators

For brevity, the treatment is restricted to the single-input, single-output case, i.e., $X = \{x_0, x_1\}$.

Define the set of operators

$$\mathscr{F}_\delta = \{I + F_c \ : \ c \in \mathbb{R}\langle\langle X \rangle\rangle\}.$$

It is convenient to introduce the Dirac symbol δ and the definition $F_\delta = I$ such that $I + F_c = F_{\delta+c} = F_{c_\delta}$ with $c_\delta := \delta + c$.

In which case,

$$c \,\tilde{\circ}\, d = c \circ (\delta + d).$$

The set of all such generating series for $\mathscr{F}_\delta$ will be denoted by $\mathbb{R}\langle\langle X_\delta \rangle\rangle$.

Consider the composition of two elements in $\mathscr{F}_\delta$:

$$\begin{aligned} F_{c_\delta} \circ F_{d_\delta} &= (I + F_c) \circ (I + F_d) \\ &= I + F_d + F_{c \,\tilde{\circ}\, d} \\ &= F_{\delta + d + c \,\tilde{\circ}\, d} \\ &=: F_{c_\delta \circ d_\delta}. \end{aligned}$$

Remarks:

- The composition products on $\mathscr{F}_\delta$ and $\mathbb{R}\langle\langle X_\delta \rangle\rangle$ are associative.
- The formal Laplace transform $\mathscr{L} : F_{c_\delta} \mapsto c_\delta$ is a semigroup isomorphism, i.e.,

$$\mathscr{L}_f(F_{c_\delta} \circ F_{d_\delta}) = \mathscr{L}_f(F_{c_\delta}) \circ \mathscr{L}_f(F_{d_\delta})$$

  and $\mathscr{L}_f(I) = \delta$.

Theorem 3.1: The triple $(\mathscr{F}_\delta, \circ, I)$, or equivalently $(\mathbb{R}\langle\langle X_\delta \rangle\rangle, \circ, \delta)$, forms a group.

Example 3.1: A linear series $c \in \mathbb{R}\langle\langle X \rangle\rangle$ has support in $L := \{x_0^{n_1} x_1 x_0^{n_0} : n_i \geq 0\}$.

When c is linear, the product $c \circ d$ is both left and right $\mathbb{R}$-linear.

It follows directly in this case that

$$c_\delta^{-1} = (\delta + c)^{-1} := \delta + c^{-1} = \delta - c + c^{\circ 2} - c^{\circ 3} + \cdots .$$

For example,

$$(\delta + x_1)^{-1} = \delta - x_1 + x_0 x_1 - x_0^2 x_1 + \cdots = \delta - (-x_0)^* x_1 .$$

In contrast, the series x_0 is not linear, and in this case

$$\delta - x_0 + x_0^{\circ 2} - x_0^{\circ 3} + \cdots = \delta - x_0 + x_0 - x_0 + \cdots ,$$

which is neither locally finite nor summable.

Nevertheless it can be verified directly that $(\delta + x_0)^{-1} = \delta - x_0$.

Example 3.2: Let $c \in \mathbb{R}\langle\langle X \rangle\rangle$ have finite Lie rank and thus realizable by (g_0, g_1, h, z_0).

If $F_c : u_1 \mapsto y_1$ then $I + F_c$ has a realization of the form

$$\begin{aligned} \dot{z} &= g_0(z) + g_1(z)u_1, \quad z(0) = z_0 \\ \tilde{y}_1 &= h(z) + u_1. \end{aligned}$$

The inverse $(I + F_c)^{-1} = I + F_{c^{-1}} : u_2 \mapsto y_2$ is described by Fig. 3.6

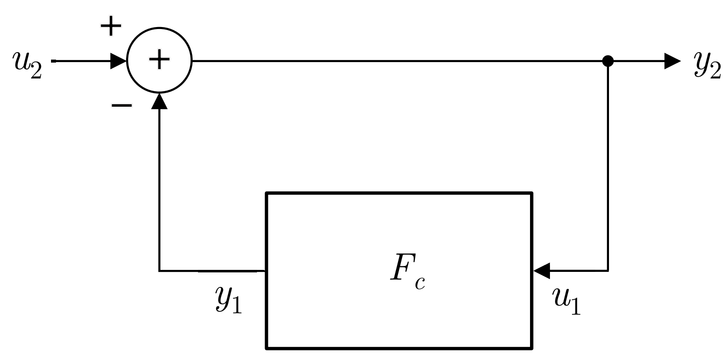


Fig. 3.6 Compositional inverse of $I + F_c$.

A simple calculation shows $F_{c^{-1}}$ is realizable by $(g_0 - g_1 h, g_1, -h, z_0)$.

Using this realization, one can compute as many coefficients of c^{-1} as desired by iterated Lie derivatives:

$$
\begin{aligned}
(c^{-1}, \emptyset) &= -(c, \emptyset) \\
(c^{-1}, x_0) &= -(c, x_0) + (c, \emptyset)(c, x_1) \\
(c^{-1}, x_1) &= -(c, x_1) \\
(c^{-1}, x_0^2) &= -(c, x_0^2) + (c, \emptyset)(c, x_0 x_1) + (c, x_0)(c, x_1) + (c, \emptyset)(c, x_1 x_0) - \\
&\qquad (c, \emptyset)(c, x_1)^2 - (c, \emptyset)^2 (c, x_1^2) \\
(c^{-1}, x_0 x_1) &= -(c, x_0 x_1) + (c, x_1)^2 + (c, \emptyset)(c, x_1^2) \\
(c^{-1}, x_1 x_0) &= -(c, x_1 x_0) + (c, \emptyset)(c, x_1^2) \\
(c^{-1}, x_1^2) &= -(c, x_1^2) \\
&\ \vdots
\end{aligned}
$$

The goal is to describe a Faà di Bruno Hopf algebra associated with the group $(\mathbb{R}\langle\langle X_\delta\rangle\rangle, \circ, \delta)$, where the antipode, α, satisfies the identity

$$c_\delta^{-1} = \delta + c^{-1} = \delta + \sum_{\eta \in X^*} (\alpha\, a_\eta)(c)\, \eta,$$

with

$$a_\eta : \mathbb{R}\langle\langle X\rangle\rangle \to \mathbb{R} : c \mapsto (c, \eta)$$

and $a_\delta(c_\delta) = 1$.

Define the commutative $\mathbb{R}$-algebra of polynomials denoted by

$$A = \mathbb{R}[a_\eta : \ \eta \in X^* \cup \delta],$$

where the product $\mu : A \otimes A \to A$ is defined so that

$$a_\eta a_\xi(c_\delta) = a_\eta(c_\delta) a_\xi(c_\delta).$$

We seek a bialgebra with a coproduct $\Delta : A \to A \otimes A$ such that

$$\begin{aligned} \mu(\Delta a_\nu(d_\delta \otimes c_\delta)) &= a_\nu(c_\delta \circ d_\delta) = (c_\delta \circ d_\delta, \nu) \\ &= (\delta + d + c \,\tilde{\circ}\, d, \nu) \\ &= (\delta, \nu) + (d, \nu) + \sum_{\eta \in X^*} (\tilde{D}_\eta(1), \nu)(c, \eta). \end{aligned}$$

Observe that any word $\eta \in X^*$ can be uniquely factored as

$$\eta = \underbrace{x_0^{n_k} x_1}_{\xi_k} \underbrace{x_0^{n_{k-1}} x_1}_{\xi_{k-1}} \cdots \underbrace{x_0^{n_1} x_1}_{\xi_1} \underbrace{x_0^{n_0}}_{\xi_0}.$$

For each factor ξ_j, $j \geq 1$ define two operators

$$\begin{aligned} I_{\xi_j} &: \mathbb{R}\langle\langle X \rangle\rangle \to \mathbb{R}\langle\langle X \rangle\rangle : e \mapsto \xi_j e \\ D_{\xi_j} &: X^* \times \mathbb{R}\langle\langle X \rangle\rangle \to \mathbb{R}\langle\langle X \rangle\rangle : (\theta, e) \mapsto x_0^{n_j+1}(\theta \sqcup\!\sqcup e). \end{aligned}$$

It then follows for any $\nu \neq \delta$ that

$$a_\nu(c_\delta \circ d_\delta) = a_\nu(d) + \sum_{k=1}^{|\nu|} \sum_{\eta=\xi_k\cdots\xi_0} \left(\left[\prod_{j=1}^{k} I_{\xi_j} + \sum_{\theta_j \in X^*} a_{\theta_j}(d) D_{\xi_j,\theta_j} \right] (\xi_0), \nu \right) a_\eta(c),$$

where $D_{\xi_k,\theta_j}(\cdot) := D_{\xi_k}(\theta_j, \cdot)$.

Expanding the product in j and suppressing the summations in θ_j gives

$$\begin{aligned}
\Delta a_\nu(d_\delta \otimes c_\delta) =& a_\nu \otimes 1 + 1 \otimes a_\nu + \sum_{k=1}^{|\nu|} \sum_{\eta=\xi_k\cdots\xi_0 \neq \nu} \\
& (D_{\xi_k,\theta_k} I_{\xi_{k-1}} \cdots I_{\xi_1}(\xi_0), \nu)\, a_{\theta_k} \otimes a_\eta + \\
& \cdots + (I_{\xi_k} \cdots I_{\xi_2} D_{\xi_1,\theta_1}(\xi_0), \nu)\, a_{\theta_1} \otimes a_\eta + \\
& (D_{\xi_k,\theta_k} D_{\xi_{k-1},\theta_{k-1}} I_{\xi_{k-2}} \cdots I_{\xi_1}(\xi_0), \nu)\, a_{\theta_k} a_{\theta_{k-1}} \otimes a_\eta + \\
& \cdots + (I_{\xi_k} \cdots I_{\xi_3} D_{\xi_2,\theta_2} D_{\xi_1,\theta_1}(\xi_0), \nu)\, a_{\theta_2} a_{\theta_1} \otimes a_\eta + \\
& \cdots + (D_{\xi_k,\theta_k} \cdots D_{\xi_1,\theta_1}(\xi_0), \nu)\, a_{\theta_k} \cdots a_{\theta_1} \otimes a_\eta .
\end{aligned}$$

With the aid of this last expression and using the equivalence $a_\delta \sim 1$, the first eight coproducts are found to be:

$$\begin{aligned}
\Delta 1 &= 1 \otimes 1 \\
\Delta a_\emptyset &= a_\emptyset \otimes 1 + 1 \otimes a_\emptyset \\
\Delta a_{x_0} &= a_{x_0} \otimes 1 + 1 \otimes a_{x_0} + a_\emptyset \otimes a_{x_1} \\
\Delta a_{x_1} &= a_{x_1} \otimes 1 + 1 \otimes a_{x_1} \\
\Delta a_{x_0^2} &= a_{x_0^2} \otimes 1 + 1 \otimes a_{x_0^2} + a_\emptyset \otimes a_{x_0 x_1} + a_{x_0} \otimes a_{x_1} + \\
&\quad\; a_\emptyset \otimes a_{x_1 x_0} + a_\emptyset^2 \otimes a_{x_1^2} \\
\Delta a_{x_0 x_1} &= a_{x_0 x_1} \otimes 1 + 1 \otimes a_{x_0 x_1} + a_{x_1} \otimes a_{x_1} + a_\emptyset \otimes a_{x_1^2} \\
\Delta a_{x_1 x_0} &= a_{x_1 x_0} \otimes 1 + 1 \otimes a_{x_1 x_0} + a_\emptyset \otimes a_{x_1^2} \\
\Delta a_{x_1^2} &= a_{x_1^2} \otimes 1 + 1 \otimes a_{x_1^2} \\
&\;\;\vdots
\end{aligned}$$

Define the unit and counit, respectively, as

$$\sigma : \mathbb{R} \longrightarrow A : \lambda \mapsto \lambda\, a_\delta$$

$$\epsilon : A \longrightarrow \mathbb{R} : a_{\eta_1} a_{\eta_2} \cdots a_{\eta_\ell} \mapsto a_{\eta_1}(\delta) a_{\eta_2}(\delta) \cdots a_{\eta_\ell}(\delta).$$

Theorem 3.2: The six-tuple $(A, \mu, \sigma, \Delta, \epsilon, \alpha)$ with

$$\alpha\, a_\nu = -a_\nu + \sum_{k=1}^{n} (-1)^{k+1} \mu^k \Delta'^{\,k} a_\nu, \quad \nu \in X^n, \quad \nu \neq \delta$$

and $\alpha\, 1 = 1$ is a graded $\mathbb{R}$-Hopf algebra with a grading given by

$$A_{(n)} = \mathrm{span}_{\mathbb{R}} \left\{ a_{\eta_1} a_{\eta_2} \cdots a_{\eta_l} \in A \;:\; \sum_{i=1}^{l} |\eta_i| = n \right\}, \quad n \geq 0,$$

where the degree is given by $\#(a_\eta) = |\eta|$ when $\eta \in X^*$ and $\#(\delta) := 0$.

Remarks:

- The bialgebra is not connected unless restricted to proper series, i.e., series where $a_\emptyset(c) = 0$.
- The first few antipode terms are:

$$
\begin{aligned}
\alpha\, 1 &= 1 \\
\alpha\, a_\emptyset &= -a_\emptyset \\
\alpha\, a_{x_0} &= -a_{x_0} + a_\emptyset a_{x_1} \\
\alpha\, a_{x_1} &= -a_{x_1} \\
\alpha\, a_{x_0^2} &= -a_{x_0^2} + a_\emptyset a_{x_0 x_1} + a_{x_0} a_{x_1} + a_\emptyset a_{x_1 x_0} - a_\emptyset a_{x_1}^2 - a_\emptyset^2 a_{x_1^2} \\
\alpha\, a_{x_0 x_1} &= -a_{x_0 x_1} + a_{x_1}^2 + a_\emptyset a_{x_1^2} \\
\alpha\, a_{x_1 x_0} &= -a_{x_1 x_0} + a_\emptyset a_{x_1^2} \\
\alpha\, a_{x_1^2} &= -a_{x_1^2} \\
&\vdots
\end{aligned}
$$

3.4 An Explicit Formula for the Feedback Product

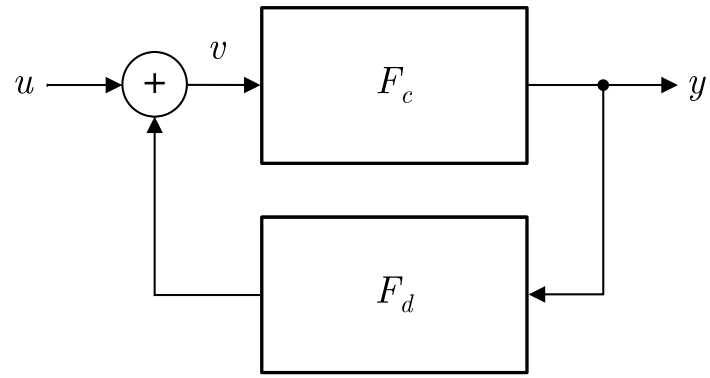


Fig. 3.7: Feedback connection of F_c and F_d.

Observe that

$$v = u + F_{d \circ c}[v],$$

and therefore,

$$\left(I + F_{-d \circ c}\right)[v] = u.$$

Simply apply the compositional inverse to both sides of this equation:

$$v = (I + F_{(-d)\circ c})^{-1}[u]$$
$$= \left(I + F_{(-d\circ c)^{-1}}\right)[u].$$

In which case,

$$F_{c@d}[u] = F_c[v] = F_{c\,\tilde{\circ}\,(-d\circ c)^{-1}}[u].$$

The new idea is that

$$c@d = c\,\tilde{\circ}\,(-d\circ c)^{-1} = c\circ(\delta - d\circ c)^{-1},$$

where

$$(\delta - d\circ c)^{-1} = \delta + \sum_{\eta\in X^*} (\alpha a_\eta)(-d\circ c)\,\eta.$$

Example 3.3: Consider the feedback connection of F_c and F_d, where $c = \sum_{\eta \in X^*} |\eta|!\, \eta$ and $d = \delta$.

In which case, by definition $c@\delta = c \,\tilde{\circ}\, (-c)^{-1}$.

It is easy to show that in general, $c \,\tilde{\circ}\, (-c)^{-1} = (-c)^{-1}$.

The series $(-c)^{-1}$ can be computed directly from the antipode formulas.

Table 3.1: Coefficients of the sequences in Example 3.2.

η	$\emptyset$	x_0	x_1	x_0^2	x_0x_1	x_1x_0	x_1^2
(c,η)	1	1	1	2	2	2	2
$((-c)^{-1},\eta)$	1	2	1	10	5	4	2
$(c@\delta,\eta)$	1	2	1	10	5	4	2
$L_{g_\eta}h(1)$	1	2	1	10	5	4	2

For this particular example, the Lie rank of c is finite. Hence, the state space method is also available to compute $c@\delta$

Recall that any Lie polynomial can be written in terms of a finite sum of its homogenous components, namely, the polynomials

$$\begin{aligned}
p_1 &= x_0 \\
p_2 &= x_1 \\
p_3 &= [x_0, x_1] = x_0 x_1 - x_1 x_0 \\
p_4 &= [x_1, x_0] = -p_3 \\
p_5 &= [x_0, [x_0, x_1]] = x_0^2 x_1 - 2x_0 x_1 x_0 + x_1 x_0^2 \\
p_6 &= [x_0, [x_1, x_0]] = -p_5 \\
p_7 &= [x_1, [x_1, x_0]] = x_1^2 x_0 - 2x_1 x_0 x_1 + x_0 x_1^2 \\
p_8 &= [x_1, [x_0, x_1]] = -p_7 \\
&\vdots
\end{aligned}$$

Observe that $\mathcal{H}_c(p_1) = \mathcal{H}_c(p_2) = \sum_{\eta \in X^*} (|\eta| + 1)!\, \eta$. The specific claim is that $\mathcal{H}_c(p_i) = 0$ for all $i \geq 3$, and thus, the Lie rank of c is one.

Since each polynomial p_i is homogeneous, it follows that

$$(\mathcal{H}_c(p_i), \eta) = (c, \eta p_i) = \sum_{\xi \in X^{\deg(p_i)}} (p_i, \xi)(c, \eta\xi) = \sum_{\xi \in X^{\deg(p_i)}} (p_i, \xi)\, |\eta\xi|!$$

$$= (|\eta| + \deg(p_i))! \sum_{\xi \in X^{\deg(p_i)}} (p_i, \xi).$$

But the latter summation is always zero for any p_i with $i \geq 3$. Why?

Every Lie polynomial p satisfies the identity (Ree, 1958)

$$(p, \eta \sqcup\!\sqcup\, \xi) = 0, \quad \eta, \xi \in X^+$$

and

$$\mathrm{char}(X^k) = \sum_{\substack{r_0, r_1 \geq 0 \\ r_0 + r_1 = k}} x_0^{r_0} \sqcup\!\sqcup\, x_1^{r_1}.$$

Applying these facts in the current context gives

$$\begin{aligned}\sum_{\xi \in X^{\deg(p_i)}} (p_i, \xi) &= \left(p_i, \operatorname{char}\left(X^{\deg(p_i)}\right)\right) \\ &= \sum_{\substack{r_0, r_1 \geq 0 \\ r_0 + r_1 = \deg(p_i)}} (p_i, x_0^{r_0} \sqcup\!\sqcup\, x_1^{r_1}) \\ &= 0\end{aligned}$$

when $i \geq 3$, and the claim is established.

To construct a one dimensional state space realization, the identity $\operatorname{char}(X^k) = \operatorname{char}(X)^{\sqcup\!\sqcup\, k}/k!$ is employed so that

$$c = \sum_{k=0}^{\infty} k! \operatorname{char}(X^k) = \sum_{k=0}^{\infty} \operatorname{char}(X)^{\sqcup\!\sqcup\, k}.$$

In which case,

$$F_c = \sum_{k=0}^{\infty} k!\, E_{\mathrm{char}(X^k)} = \sum_{k=0}^{\infty} E_{\mathrm{char}(X)\,{\sqcup\!\sqcup}\, k}$$
$$= \sum_{k=0}^{\infty} E^k_{\mathrm{char}(X)} = \frac{1}{1 - E_{\mathrm{char}(X)}}.$$

Defining $z = F_c$, it follows that

$$\dot{z} = z^2(1+u), \;\; z(0) = 1, \;\; y = z$$

realizes $y = F_c[u]$, and with unity feedback $u = z + v$

$$\dot{z} = z^2 + z^3 + z^2 v, \;\; z(0) = 1, \;\; y = z$$

realizes $y = F_{c@\delta}[v]$.

So $c@\delta$ can also be computed by the iterated Lie derivatives $L_{g_\eta} h(z_0)$, where $(g_0, g_1, h, z_0) = (z^2 + z^3, z^2, z, 1)$ as shown in Table 4.1.

Since in general $(-c)^{-1} = c@\delta$, the following theorem is proved using the work of Thitsa (2011) on the radius of convergence of feedback systems.

Theorem 3.2: The triple $(\mathbb{R}_{LC}\langle\langle X_\delta\rangle\rangle, \circ, \delta)$ is a subgroup of $(\mathbb{R}\langle\langle X_\delta\rangle\rangle, \circ, \delta)$. In particular,

$$|(c^{-1}, \eta)| \leq K\left(\mathcal{A}(K_c)M_c\right)^{|\eta|}|\eta|!, \quad \eta \in X^*,$$

for some $K > 0$, where

$$\mathcal{A}(K_c) = \frac{1}{1 - K_c \ln\left(1 + 1/K_c\right)}.$$

Theorem 3.3: If $c, d \in \mathbb{R}_{LC}\langle\langle X\rangle\rangle$ then $c@d \in \mathbb{R}_{LC}\langle\langle X\rangle\rangle$.

Proof. The composition product, the modified composition product, and the compositional inverse all preserve local convergence. Hence, the claim follows directly from the expression

$$c@d = c\,\tilde{\circ}\,(-d \circ c)^{-1}.$$

4. Conclusions and Future Work

- The four basic interconnections of Fliess operators can be described entirely in a formal power series setting.
- State space realizations (i.e., and therefore the need for local coordinates) can be entirely avoided.
- The basic issues concerning convergence of composite systems have been resolved for deterministic inputs. (More about this in Khin's talk today.)
- Open problems:
  - Noisy inputs (Luis's talk tomorrow), this is where rough path theory may play a useful role.
  - Computational issues
  - Deeper combinatoric interpretations