On the well-posedness of cascades of analytic nonlinear input-output systems driven by noise*

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Overview*

* See `www.ece.odu.edu/~sgray/RPCCT2011/duffautespinosaslides.pdf`

1. Fliess Operators

- Functional series expansions of nonlinear input-output operators have been utilized since the early 1900's in engineering, mathematics and physics (*V. Volterra, N. Wiener, etc*).
- A broad class of deterministic nonlinear systems can be described by Fliess operators, which are input-output maps constructed using the Chen-Fliess formalism (*Fliess* (1981)).
- Such operators are described by a summation of Lebesgue iterated integrals codified using the theory of noncommutative formal power series.

1.1 Formal Power Series

- Let $X = \{x_0, x_1, \ldots, x_m\}$ be an alphabet and X^* the set of all words over X (including the empty word $\emptyset$).
- A formal power series is any mapping $c : X^* \to \mathbb{R}^\ell$. Typically, c is written as a formal sum

$$c = \sum_{\eta \in X^*} (c, \eta)\eta.$$

- The set of all such series is denoted by $\mathbb{R}^\ell\langle\langle X\rangle\rangle$, and the subset denoted by $\mathbb{R}^\ell\langle X\rangle$ is the set of polynomials.
- A series c is rational if it belongs to the rational closure of $\mathbb{R}^\ell\langle X\rangle$.
- A series c is rational if and only if $(c, \eta) = \lambda\mu(\eta)\gamma, \;\; \forall \eta \in X^*$, where $\mu : X^* \to \mathbb{R}^{n\times n}$ is a monoid morphism, and $\gamma,\lambda^T \in \mathbb{R}^{n\times 1}$.
- c is called globally convergent when $|(c, \eta)| \leq KM^{|\eta|}$, $\forall\, \eta \in X^*$.
- c is called locally convergent when $|(c, \eta)| \leq KM^{|\eta|}\, |\eta|!$, $\forall\, \eta \in X^*$.

- For a measurable function $u : [a, b] \to \mathbb{R}^m$ with finite L_1-norm, define $E_\eta : L_1^m[t_0, t_0 + T] \to \mathcal{C}[t_0, t_0 + T]$ by $E_\emptyset[u] = 1$, and

$$E_{x_i\eta'}[u](t, t_0) = \int_{t_0}^{t} u_i(\tau)E_{\eta'}[u](\tau, t_0)\, d\tau, \quad (1)$$

  where $x_i \in X$, $\eta' \in X^*$ and $u_0 = 1$.

- Note that to each letter x_i is assigned a function u_i.

- Each $c \in \mathbb{R}^\ell\langle\langle X\rangle\rangle$ is associated with an m-input, ℓ-output system,

$$F_c[u](t) = \sum_{\eta\in X^*} (c, \eta)\, E_\eta[u](t, t_0),$$

  called a Fliess operator (*Fliess* (1981)).

Example 1 A linear input-output system $F : u \to y$ with $u(t) \in \mathbb{R}^m$ and $y(t) \in \mathbb{R}^\ell$ can be described by a convolution integral involving its impulse response $H(t,\tau) = (H_1(t,\tau), \ldots, H_m(t,\tau))'$ and the system input

$$y(t) = F[u](t) = \int_{t_0}^{t} H(t,\tau)u(\tau)\, d\tau,\ t \geq t_0. \tag{2}$$

If each H_i is real analytic on $D = \{(t,\tau) \in \mathbb{R}^2 : t_0 \leq \tau \leq t \leq t_0 + T\}$, then its Taylor series at (τ, t_0) is

$$H_i(t,\tau) = \sum_{n_1,n_2=0}^{\infty} c(n_2,i,n_1) \frac{(t-\tau)^{n_2}}{n_2!} \frac{(\tau-t_0)^{n_1}}{n_1!}, \tag{3}$$

where $c(n_2,i,n_1) \in \mathbb{R}^\ell$.

Substituting (3) into (2) and using the uniform convergence of the series on D, it follows that

$$y(t) = \sum_{n_1,n_2=0,i=1}^{\infty,m} c(n_2,i,n_1) \underbrace{\int_{t_0}^{t} \frac{(t-\tau)^{n_2}}{n_2!} u_i(\tau) \frac{(\tau-t_0)^{n_1}}{n_1!} d\tau}_{E_{x_0^{n_2} x_i x_0^{n_1}}[u](t,t_0)} . \qquad (4)$$

Thus, (4) can be written as

$$\boxed{y(t) = \sum_{n_1,n_2=0,i=1}^{\infty,m} c(n_2,i,n_1) E_{x_0^{n_2} x_i x_0^{n_1}}[u](t,t_0).}$$

Observe that the formal power series associated with system (2) is

$$\boxed{(c,\eta) = \begin{cases} c(n_2,i,n_1) & : \eta = x_0^{n_2} x_i x_0^{n_1},\ n_1,n_2 \geq 0,\ i \neq 0 \\ 0 & : \text{otherwise.} \end{cases}}$$

1.2 Fliess Operators with Stochastic Inputs

- System inputs in applications usually have noise.

- Several authors have formulated approaches where Wiener processes are admissible inputs to a Fliess operators (*G. B. Arous* (1989), *Fliess* (1977, 1981), *Fliess and Lamnabhi* (1981), *Sussmann* (1988)).

- A suitable mathematical formulation will use Stratonovich integrals:

  i. They obey the rules of ordinary differential calculus.

  ii. When schemes for solving stochastic differential equations use smooth functions to approximate white Gaussian noise, the appropriate model will use Stratonovich integrals.

Example 2 Let W be a Wiener process. Consider a system modeled by the stochastic differential equation (SDE) in Stratonovich form

$$z_t = z_0 + \int_0^t f(z_s)\, ds + \oint_0^t g(z_s)\, dW(s), \tag{5}$$

where $f(z)$ and $g(z)$ are suitably defined functions. For a $\mathcal{C}^2$ function F, the Stratonovich differential chain rule gives

$$F(z_t) \;=\; F(z_t) + \int_0^t f(z_s)\frac{\partial}{\partial z}F(z_s)\, ds + \oint_0^t g(z_s)\frac{\partial}{\partial z}F(z_s)\, dW(s). \tag{6}$$

Identifying operators $L_f = f(z)\frac{\partial}{\partial z}$ and $L_g = g(z)\frac{\partial}{\partial z}$, (6) becomes

$$F(z_t) = F(z_0) + \int_0^t L_f F(z_s)\, ds + \oint_0^t L_g F(z_s)\, dW(s).$$

Now let $F(z)$ in (6) be replaced by either f or g from (5) and substitute $f(z_t)$ and $g(z_t)$ back into (5). This yields

$$\begin{aligned} z_t \;=\; & z_0 + f(z_0)\int_0^t ds + g(z_0)\oint_0^t dW(s) \\ & + \int_0^t \int_0^s L_f f(z_r)\, dr ds + \int_0^t \oint_0^s L_g f(z_r)\, dW(r) ds \\ & + \oint_0^t \int_0^s L_f g(z_r)\, dr dW(s) + \oint_0^t \oint_0^s L_g g(z_r)\, dW(r) dW(s) \end{aligned}$$

$$\boxed{z_t = z_0 + f(z_0) \underbrace{\int_0^t ds}_{E_{x_0}[0](t,0)} + g(z_0) \underbrace{\oint_0^t dW(s)}_{E_{y_0}[0](t,0)} + R_1(z(t)).}$$

Continuing this way produces the usual Peano-Baker formula.

Let I be the identity map and define $X = \{x_0\}$, $Y = \{y_0\}$, $XY = X \cup Y$, $L_{g_{x_0\eta}} = L_{g_\eta} L_{g_{x_0}}$ and $L_{g_{y_0\eta}} = L_{g_\eta} L_{g_{y_0}}$, where $g_{x_0} = f$, $g_{y_0} = g$ and $\eta \in XY^*$. Thus, the solution of the SDE (5) in series form is

$$\boxed{y(t) \triangleq z(t) = \sum_{\eta \in XY^*} L_{g_\eta} I(z(0))\, E_\eta[0](t)} \tag{7}$$

Here, $(f, g, I, z(0))$ realizes F_c when $(c, \eta) = L_{g_\eta} I(z(0))$, $\forall \eta \in XY^*$.

Remarks:

- The output of this nonlinear input-output system is in general not a Wiener process. For example, equation (7) can be written as

$$\begin{aligned} y(t) &= (c, \emptyset) + \int_0^t \sum_{\eta \in XY^*} L_{g_{x_0\eta}} I(z(0))\, E_{x_0\eta}[0](s,0)\, ds \\ &\quad + \oint_0^t \sum_{\eta \in XY^*} L_{g_{y_0\eta}} I(z(0))\, E_{y_0\eta}[0](s,0)\, dW(s). \end{aligned}$$

- Note that $y(t)$ is not well-defined unless the integrands converge.

- Consider a Wiener process, denoted by $W(t)$, defined over $(\Omega, \mathcal{F}, P)$.
- Let $u : \Omega \times [t_0, t_0 + T] \to \mathbb{R}^m$ be a predictable function, and $\|u\|_p = \max\{\|u_i\|_{L_p} : 1 \leq i \leq m\}$.

Definition 1 (*Duffaut et al.* 2009) Consider the set of all m-dimensional stochastic processes over $[t_0, t_0 + T]$, denoted by $\widetilde{\mathcal{UV}}^m[t_0, t_0 + T]$, which can be written as

$$w(t) = \int_{t_0}^{t} u(s)\, ds + \oint\!\!\!\!\!\$_{t_0}^{t} v(s)\, dW(s).$$

The set $\mathcal{UV}^m[t_0, t_0 + T] \subset \widetilde{\mathcal{UV}}^m[t_0, t_0 + T]$ will refer to processes satisfying:

i. Each m-dimensional integrand has $\mathbf{E}[u_i(t)] < \infty$, $\mathbf{E}[v_i(t)] < \infty$, $t \in [t_0, t_0 + T]$ and are mutually independents.

ii. Also, $\|u\|_{L_2}, \|v\|_{L_2}, \|v\|_{L_4} \leq R \in \mathbb{R}^+$.

Definition 2 (*Duffaut et al.* 2010) Let $X(t) = \oint_0^t v(s)\,dW(s)$, where v is an m-dimensional L_2-Itô process. The set $\mathcal{UV}^m[0,\tau_R]$ is defined as the set of processes $w \in \mathcal{UV}^m[0,T]$ stopped at

$$\tau_R \triangleq \min_{i\in\{0,1,\cdots,m\}} \inf\left\{t \in \mathcal{T} \,:\, \left|\oint_0^t v_i(s)\,dW(s)\right| = R\right\}.$$

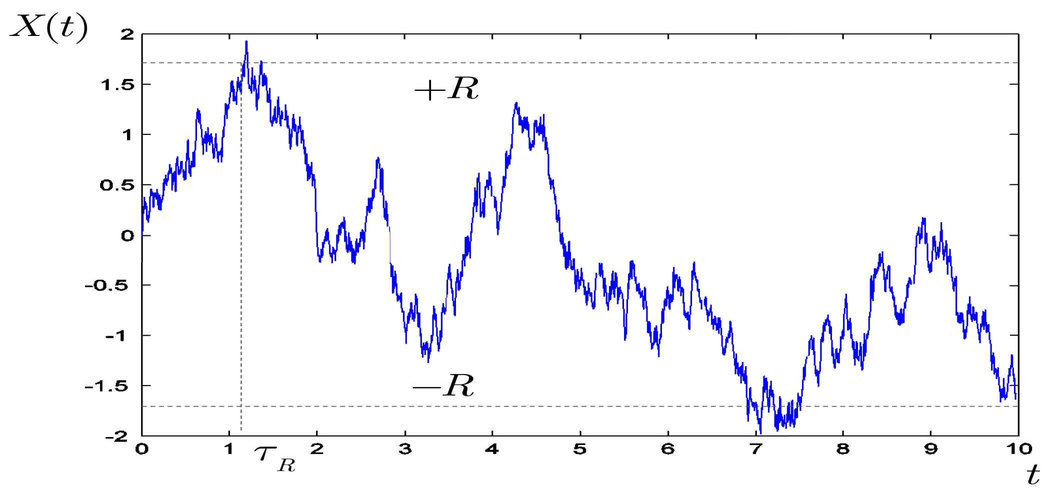

Figure 1: First time process $X(t)$ hits the barrier R.

Remark: τ_R is a strictly positive stopping time for any real $R > 0$.

- Let $X = \{x_0, x_1, \ldots, x_m\}$, $Y = \{y_0, y_1, \ldots, y_m\}$ and $XY = X \cup Y$.
- An iterated integral over $\mathcal{UV}^m[t_0, t_0 + T]$ is defined recursively by

$$E_{x_i\eta'}[w](t, t_0) = \int_{t_0}^{t-} u_i(s) E_{\eta'}[w](s)\, ds, \;\; x_i \in X,$$

$$E_{y_i\eta'}[w](t, t_0) = \oint_{t_0}^{t-} v_i(s) E_{\eta'}[w](s)\, dW(s), \;\; y_i \in Y,$$

where $\eta' \in XY^*$, $E_\emptyset = 1$ and $u_0 = v_0 = 1$.

Definition 3 (*Duffaut et al.* 2009) An m-input, ℓ-output Fliess operator F_c, $c \in \mathbb{R}^\ell\langle\langle XY \rangle\rangle$, driven by $w \in \mathcal{UV}^m[0, T]$ is formally defined as

$$F_c[w](t) = \sum_{\eta \in XY^*} (c, \eta)\, E_\eta[w](t, t_0). \tag{8}$$

Definition 4 For any $T > 0$, $w \in \mathcal{UV}^m[0,T]$ and $t \in [0,T]$, the Chen series associated with a formal power series in $\mathbb{R}^\ell\langle\langle XY \rangle\rangle$ is defined as

$$P[w](t,t_0) = \sum_{\eta \in XY^*} \eta\, E_\eta[w](t,t_0).$$

- The Chen series satisfies the stochastic differential equation

$$dP[w](t,t_0) = \left(\sum_{i=0}^{m} x_i u_i(t)\, dt + y_i v_i(t)\, dW(t) \right) P[w](t,t_0).$$

- For any t, $(P[u], \xi \sqcup\!\sqcup\, \nu) = (P[u], \xi)\,(P[u], \nu)\,, \ \ \forall \xi, \nu \in XY^*$. Therefore, from Ree's theorem $P[u]$, is an exponential Lie series.

- The Fliess operator (8) can be written as

$$F_c[w](t) = (c, P[w](t,0))$$

- $P[w]$ satisfies $P[w](t,t_0) = P[w](t,t')P[w](t',t_0)$ (Chen's identity).

2. Convergence of Fliess Operators with Stochastic Inputs

- It was shown by *Gray and Wang* (2002) that for $u \in L_1[t_0, t_0 + T]$ and any $\eta \in X^*$

$$\boxed{|E_\eta[u](t,t_0)| \leq \prod_{i=0}^{m} \frac{\bar{U}_i^{\alpha_i}(t)}{\alpha_i!},} \tag{9}$$

where $\bar{U}_i(t) = \int_{t_0}^{t} |u_i(\tau)|\, d\tau$, and $\alpha_i = |\eta|_{x_i}$ is the number of x_i in η.

- If $|(c,\eta)| \leq KM^{|\eta|}$, $\forall\, \eta \in X^*$, then $F_c[u]$ converges absolutely on $[t_0, \infty)$ for $u \in L_{p,e}(t_0)$.

- If $|(c,\eta)| \leq KM^{|\eta|}\, |\eta|!$, $\forall\, \eta \in X^*$, then

$$\boxed{F_c : B_p^m(R)[t_0,\, t_0 + T] \rightarrow B_q^\ell(S)[t_0,\, t_0 + T],}$$

for sufficiently small $R, S, T > 0$ and $1/p + 1/q = 1$.

Notation:

- Define the language $X^kY^n = \{\eta \in XY^*; |\eta|_X = k, |\eta|_Y = n\}$.
- For a fixed word $\eta \in X^kY^n$, define the vectors $\boldsymbol{\alpha} = (\alpha_m, \cdots, \alpha_0) \in \mathbb{N}^{m+1}$ and $\boldsymbol{\beta} = (\beta_m, \cdots, \beta_0) \in \mathbb{N}^{m+1}$, where $\alpha_i = |\eta|_{x_i}$, $\beta_i = |\eta|_{y_i}$, $k = \sum_{i=0}^m \alpha_i$ and $n = \sum_{i=0}^m \beta_i$.

Remark: Convergence is not easy to characterize using Stratonovich integrals. So a formula for E_η in terms of Itô integrals is needed.

Theorem 1 (*Duffaut et al.* 2009) Let $\eta \in X^kY^n$ and $w \in \mathcal{UV}^m[0,T]$. Then

$$E_\eta[w](t) = \sum_{r_1=0, r_2=0}^{n, \lfloor \frac{n}{2} \rfloor} \frac{1}{2^{r_1} 2^{r_2}} \sum_{\substack{\boldsymbol{s_{r_1}} \in A_{nr_1}^{\boldsymbol{\bar{s}_{r_2}}} \\ \boldsymbol{\bar{s}_{r_2}} \in \bar{A}_{nr_2}}} \mathbf{I}_\eta^{\substack{\boldsymbol{s_{r_1}} \\ \boldsymbol{\bar{s}_{r_2}}}}[w](t) \, ,$$

where $\bar{A}_{nr_2}$ and $A_{nr_1}^{\boldsymbol{\bar{s}_{r_2}}}$ are subsets of indexes in η, and $\mathbf{I}_\eta^{\substack{\boldsymbol{s_{r_1}} \\ \boldsymbol{\bar{s}_{r_2}}}}[w](t)$ is an Itô iterated integral.

Remarks:

- Recall *Gray and Wang* (2002) showed that for $u \in L_1[t_0, t_0+T]$ and any $\eta \in X^*$
$$|E_\eta[u](t)| \leq \prod_{i=0}^{m} \frac{U_i^{\alpha_i}(t)}{\alpha_i!},$$
where $U_i(t) = \int_0^t |u_i(\tau)|\, d\tau$, and $\alpha_i = |\eta|_{x_i}$ is the number of x_i in η.
- For the stochastic case, analogous bounds for Itô iterated integrals have been developed.

Theorem 2 (*Duffaut et al.* 2009) Let $\eta \in X^k Y^n$ and $w \in \mathcal{U}\mathcal{V}^m$ be arbitrary. Then for a fixed $t \in [0, T]$

$$\|E_\eta[w](t)\|_2 < \frac{(R\sqrt{t})^k(\sqrt{2R}(\sqrt{t}+2))^{2n}}{(\alpha!)^{\frac{1}{2}}(\beta!)^{\frac{1}{4}}}, \tag{10}$$

where $\max\{\|u\|_{L_2}, \|v\|_{L_2}, \|v_0\|_{L_2}, \|v\|_{L_4}\} \leq R$, $\alpha! \triangleq \alpha_0! \cdots \alpha_m!$ and $\beta! \triangleq \beta_0! \cdots \beta_m!$.

2.1 Global convergence

Example 3 Consider the following system driven by a Wiener process

$$\begin{aligned} dz_1(t) &= M_1 z_1(t)\, dW(t),\ z_1(0) = 1 \\ y_1(t) &= K_1 z_1(t). \end{aligned} \qquad (11)$$

The generating series of (11) is $(c_1, x_1^k) = K_1 M_1^k$, $k \geq 0$. Thus,

$$y_1(t) = F_{c_1}[0](t) = \sum_{k=0}^{\infty} \underbrace{K_1 M_1^k}_{(c_1, x_1^k)} \int_0^t \cdots \int_0^{t_2} dW(t_1) \cdots dW(t_k).$$

Since $\int_0^t \frac{W^k(s)}{k!}\, dW(s) = \frac{W^{k+1}(t)}{(k+1)!}$, $k \geq 0$. Then

$$\boxed{y_1(t) = F_{c_1}[0](t) = \sum_{k=0}^{\infty} K_1 M_1^k \frac{W^k(t)}{k!} = K_1 e^{M_1 W(t)},\ t \in [0, \infty).}$$

Theorem 3 (*Duffaut et al.* 2009) Suppose for a series $c \in \mathbb{R}^{\ell}\langle\langle XY \rangle\rangle$ there exists real numbers $K > 0$ and $M > 0$ such that

$$|(c, \eta)| \leq KM^{|\eta|}, \quad \forall \eta \in XY^*.$$

Then for any random process $w \in \mathcal{UV}^m[0, T]$, $T > 0$, the Fliess operator defined by series (8) converges absolutely in the mean square sense to a well defined random vector $y(t) = F_c[w](t)$, $t \in [0, T]$.

Remark: Recall that for any $w \in \mathcal{UV}^m[0, T]$, R is a bound for $\|u\|_1$, $\|v\|_2$ and $\|v\|_4$. This theorem is valid for all $t \in [0, T]$, where $T, R \geq 0$ are arbitrarily large but finite. Therefore, this theorem is viewed as a global convergence result.

2.2. Local convergence

Example 4 Consider the system

$$dz_2(t) = M_2 z_2^2(t)\, dW(t), \quad z_2(0) = 1, \quad y_2(t) = K_2 z_2(t). \tag{12}$$

The generating series of (12) is $(c_2, x_1^k) = K_2 M_2^k k!$, $k \geq 0$. Thus,

$$y_2(t) = F_{c_2}[0](t) = \sum_{k=0}^{\infty} K_2 M_2^k k! \oint_0^t \cdots \oint_0^{t_2} dW(t_1) \cdots dW(t_k).$$

Then the output is written by the divergent series

$$\boxed{y_2(t) = F_{c_2}[0](t) = \sum_{k=0}^{\infty} K_2 M_2^k W^k(t).}$$

But if $\tau = \inf\{t : |M_2 W(t)| = R\}$, $R < 1$, then $y_2(t) = \frac{K_2}{1 - M_2 W(t)}$, $t < \tau$.

Remarks:

- $[0, \tau]$ is random, i.e., $[0, \tau] = \{0 \leq t \leq \tau(\omega) : (\tau, \omega) \in \mathbb{R}^+ \times \Omega\}$.
- The solution by variable separation of (12) is $z_2(t) = \frac{K_2}{1 - M_2 W(t)}$.

Some Results and Notation: (*Duffaut et al.* 2009, 2010)

- Let $\eta, \xi \in XY^*$ and $q_i, q_j \in XY$. The shuffle product is
$$q_i\eta \sqcup\!\sqcup\, q_j\xi = q_i[\eta \sqcup\!\sqcup\, q_j\xi] + q_j[q_i\eta \sqcup\!\sqcup\, \xi],$$
where $\emptyset \sqcup\!\sqcup\, \emptyset = \emptyset$ and $\xi \sqcup\!\sqcup\, \emptyset = \emptyset \sqcup\!\sqcup\, \xi = \xi$.
- $\mathbb{R}^\ell\langle\langle XY\rangle\rangle$ with the shuffle product forms an $\mathbb{R}$-algebra.
- For any $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{N}^{m+1}$ define the polynomials $p_{\boldsymbol{\alpha}} = x_0^{\alpha_0} \sqcup\!\sqcup \cdots \sqcup\!\sqcup\, x_m^{\alpha_m}$ and $p_{\boldsymbol{\beta}} = y_0^{\beta_0} \sqcup\!\sqcup \cdots \sqcup\!\sqcup\, y_m^{\beta_m}$, respectively.
- Observe that $\boldsymbol{X^k Y^n} \triangleq \sum_{\eta \in X^k Y^n} \eta = \sum_{\|\alpha\|=k, \|\beta\|=n} p_{\boldsymbol{\alpha}} \sqcup\!\sqcup\, p_{\boldsymbol{\beta}}$.
- Define $\mathbf{S}_{\boldsymbol{\alpha},\boldsymbol{\beta}}[w](t) \triangleq F_{p_{\boldsymbol{\alpha}} \sqcup\!\sqcup\, p_{\boldsymbol{\beta}}}[w](t) = F_{p_{\boldsymbol{\alpha}}}[w](t) F_{p_{\boldsymbol{\beta}}}[w](t)$.
- Independence of the inputs gives
$$\|\mathbf{S}_{\boldsymbol{\alpha},\boldsymbol{\beta}}[w](t)\|_2^2 = \|F_{p_{\boldsymbol{\alpha}}}[w](t)\|_2^2 \left\|F_{p_{\boldsymbol{\beta}}}[w](t)\right\|_2^2.$$
- The L_2-norm of $F_{p_{\boldsymbol{\alpha}}}[w](t)$ is $\|F_{p_{\boldsymbol{\alpha}}}[w](t)\|_2^2 \leq \prod_{i=0}^{m} \frac{\bar{U}_i^{2\alpha_i}(t)}{(\alpha_i!)^2} \leq \frac{R^{2k}}{(\alpha!)^2}$.

Theorem 4 (*Duffaut et al.* 2010) Suppose that for a series $c \in \mathbb{R}^{\ell}\langle\langle XY \rangle\rangle$, there exists real numbers $K > 0$ and $M > 0$ such that

$$|(c,\eta)| \leq KM^{|\eta|}\,|\eta|!, \quad \forall \eta \in XY^*.$$

Then for any random process $w \in \mathcal{UV}^m[0,T]$, $T > 0$, the series

$$\boxed{F_c[w](t) = \sum_{j=0}^{\infty}\sum_{k=0}^{j}\sum_{\eta \in X^kY^{j-k}} (c,\eta)E_\eta[w](t)} \tag{13}$$

converges in the mean square sense to a random vector $y(t)$, $t \in [0, \tau_R]$, where

$$\tau_R \triangleq \min_{i \in \{0,\ldots,m\}} \inf \left\{ t \in [0,T] : \left| \int_0^t v_i(s)dW(s) \right| = R \right\}.$$

Remark: Note in (13) that there is an implied order to the summation over XY^*. Thus, the current result is strictly speaking for conditional convergence.

Definition 5 (*Fliess* (1981)) Let $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{N}^{m+1}$ and define the language

$$\boldsymbol{L}_{\boldsymbol{\alpha},\boldsymbol{\beta}} = \left\{ \eta \in XY^*, |\eta|_{x_i} = \alpha_i, |\eta|_{y_i} = \beta_i, i = 0, 1, \ldots, m \right\}.$$

A series $c \in \mathbb{R}^\ell\langle\langle XY \rangle\rangle$ is called exchangeable if all the words in $\boldsymbol{L}_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ have the same image under c for any given $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{N}^{m+1}$.

Corollary 1 (*Duffaut et al.* 2010) Let $c \in \mathbb{R}^\ell\langle\langle XY \rangle\rangle$ be exchangeable and locally convergent. Then, for an arbitrary $w \in \mathcal{UV}^m[0, T]$, there exist an $R > 0$ and a stopping time $\tau_R > 0$ such that $F_c[w]$ converges absolutely over $[0, \tau_R]$.

Remarks:

- Every process $y = F_c[w]$ is a well-defined L_2-Itô process. But the independence of the inputs is not preserved at the output.
- It is conjectured that there may exist a maximal exchangeable series which ensures absolute convergence for any locally convergent series.

2.3 Solving a type of polynomial differential equations

Consider an analytic input u and $X = \{x_0\}$,

$$u(t) = \sum_{n=0}^{\infty} c_u(n) \frac{(t-t_0)^n}{n!} \qquad \Rightarrow \qquad c = \sum_{n \geq 0} (c, x_0^n) x_0^n$$

Note $c_u(n) = (c, x_0^n)$. Thus, a transform $\mathfrak{L}_f : u \mapsto c$ can be defined.

Remark: For $t_0 = 0$, the one-sided Laplace transform of u will be

$$\begin{aligned} \mathfrak{L}[u](s) &= \int_0^{\infty} u(t) e^{st} dt = \int_0^{\infty} \sum_{n \geq 0} (c, x_0^n) \frac{t^n}{n!} e^{st} dt \\ &= \sum_{n \geq 0} (c, x_0^n) \int_0^{\infty} \frac{t^n}{n!} e^{st} dt = s^{-1} \sum_{n \geq 0} (c, x_0^n)(s^{-1})^n. \end{aligned}$$

Then

$$\boxed{\mathfrak{L}[u](s) = x_0 \mathfrak{L}_f[u] \Big|_{x_0 \to s^{-1}}.}$$

Definition 6 (*Gray & Li* 2006) Let $\mathfrak{F} \triangleq \{F_c : c \in \mathbb{R}^\ell\langle\langle X\rangle\rangle\}$. The formal Laplace and Borel transforms are, respectively,

$$
\begin{aligned}
\mathfrak{L}_f &: \mathfrak{F} \to \mathbb{R}^\ell\langle\langle X\rangle\rangle \\
&: F_c \mapsto c \\
\mathfrak{B}_f &: \mathbb{R}^\ell\langle\langle X\rangle\rangle \to \mathfrak{F} \\
&: c \mapsto F_c.
\end{aligned}
$$

Table 1: Some formal Laplace transforms.

F_c	$\mathfrak{L}_f[F_c] = c$
$u \to 1$	1
$u \to t^n$	$n! x_0^n$
$u \to \left(\sum_{i=0}^{n-1} \frac{\binom{n-1}{i}}{i!} a^i t^i \right) e^{at}$	$(1 - a x_0)^{-n}$
$u \to \exp\left(\int_0^t \sum_{j=1}^{k} u_{i_j}(\tau)\, d\tau \right)$	$(x_{i_1} + \cdots + x_{i_k})^*$

Remarks:

- The star operation is related to the inverse of a series with respect to the Cauchy product. If c is not proper then $c = (c, \emptyset)(1 - c')$ with c' proper, and

$$c^{-1} = \frac{1}{(c,\emptyset)}(1-c')^{-1} = \frac{1}{(c,\emptyset)}\,(c')^{*}.$$

Observe $cc^{-1} = (c,\emptyset)(1-c')\dfrac{1}{(c,\emptyset)}(1+c'+c'^{2}+\cdots) = 1.$

- Some properties of these transforms are:

i Linearity

$$\begin{aligned} \mathfrak{L}_f[\alpha F_c + \beta F_d] &= \alpha \mathfrak{L}_f[F_c] + \beta \mathfrak{L}_f[F_d] \\ \mathfrak{B}_f[\alpha c + \beta d] &= \alpha \mathfrak{B}_f[c] + \beta \mathfrak{B}_f[d] \end{aligned}$$

ii Integration:

$$\mathfrak{L}_f[I^n F_c] = x_0^n c, \quad \mathfrak{B}_f[x_0^n c] = I^n F_c$$

iii Multiplication:

$$\mathfrak{L}_f[F_c \cdot F_d] = \mathfrak{L}_f[F_c] \sqcup\!\sqcup\, \mathfrak{L}_f[F_d], \quad \mathfrak{B}_f[c \sqcup\!\sqcup\, d] = \mathfrak{B}_f[c] \cdot \mathfrak{B}_f[d]$$

- If u admits an expansion in terms of x_0 and y_0, then

$$u \xrightarrow{\mathfrak{L}_f} c_u = \sum_{\eta \in X_0 Y_0{}^*} (c_u, \eta)\eta.$$

Moreover, if c is globally convergent then u is a well-defined L_2-Itô process. This allows one to apply the formal-Laplace transform to stochastic processes (*Duffaut* 2009).

- By integration by parts for Stratonovich integrals,

$$\begin{aligned} F_c \cdot F_d &= \sum_{\eta, \in XY^*} (c,\eta)E_\eta \sum_{\xi \in X^*} (d,\xi)E_\xi \\ &= \sum_{\eta,\xi \in XY^*} (c,\eta)(d,\xi)E_\eta E_\xi \\ &= \sum_{\eta,\xi \in XY^*} (c,\eta)(d,\xi)E_{\eta \,\sqcup\!\sqcup\, \xi} \end{aligned}$$

Consider the differential equation

$$\sum_{i=0}^{n} \ell_i \frac{d^i}{dt^i} y(t) + \beta \sum_{i=2}^{m} a_i y^i(t) u(t) = \sum_{i=0}^{n-1} q_i \frac{d^i}{dt^i} u(t), \tag{14}$$

with initial conditions $y^{(i)}(0) = y_{i0}, \;\; u^{(i)}(0) = u_{i0}$.

Applying the formal Laplace Transform: $(y(t) = F_c[u](t) \;\; \Rightarrow \;\; \mathfrak{L}_f[y] = c)$

$$\sum_{i=0}^{n} \ell_i x_0^{n-i} c + \beta \sum_{i=2}^{m} a_i x_0^{n-i} x_1 c^{\sqcup\!\sqcup\, i} = \sum_{i=0}^{n-1} q_i x_0^{n-i-1} x_1 + \sum_{i=0}^{n-1} \bar{\ell}_i x_0^i + \sum_{i=1}^{n-1} \bar{q}_i x_0^i,$$

Example 5 Consider $y^{(i)}(0) = 0$, $u^{(i)}(0) = 0$, $\ell = 1$ and $\beta = 0$

$$c = \left(1 + \sum_{i=0}^{n} \ell_i x_0^{n-i}\right)^{-1} \sum_{i=0}^{n-1} q_i x_0^{n-i-1} x_1.$$

Use $\mathfrak{L}[u](s) = x_0 \mathfrak{L}_f[u] \Big|_{x_0 \to s^{-1}} \quad \Rightarrow \quad \frac{Y(s)}{U(s)} = \frac{\sum_{i=0}^{n-1} q_i s^i}{s^n + \sum_{i=0}^{n} \ell_i s^i}.$

Remarks:

- If u is stochastic then $u = \frac{d}{dt}w = \dot{w} = \bar{u} + v\bar{w}$ (abusing notation), where $\bar{w}$ is white Gaussian noise (*Duffaut* 2009). Therefore,

$$\int u(s)\,ds \xrightarrow{\mathfrak{L}_f} x_1 + y_1$$

- The generating series c of (14) can be calculated as $c = \sum_{k=0}^{\infty} c_k$, where

$$c_0 = \left(1 + \sum_{i=0}^{n} \ell_i x_0^{n-i}\right)^{-1} \left(\sum_{i=0}^{n-1} q_i x_0^{n-i-1} x_1 + \sum_{i=0}^{n-1} \bar{\ell}_i x_0^i + \sum_{i=1}^{n-1} \bar{q}_i x_0^i\right),$$

$$c_k = \left(1 + \sum_{i=0}^{n} \ell_i x_0^{n-i}\right)^{-1} x_0^n x_1 \beta \sum_{i=2}^{m} a_i \sum_{\substack{j_1+j_2+\cdots+j_i=k-1 \\ j_1, j_2, \cdots, j_i < k}} c_{j_1} \sqcup\!\sqcup\, c_{j_2} \sqcup\!\sqcup \cdots \sqcup\!\sqcup\, c_{j_i}.$$

Example 6 Consider the following state space system

$$\frac{d}{dt}z(t) + k_1 z(t) = k_2 \bar{w}(t), \quad z(0) = z_0, \quad y(t) = z(t) \tag{15}$$

Applying the formal Laplace transform to (15) gives

$$c - 1 + k_1 x_0 c = k_2 y_1 \qquad \Rightarrow \qquad c = \frac{k_2 y_1}{(1 + k_1 x_0)} + \frac{z_0}{(1 + k_1 x_0)}$$

Remark: $$\int_0^t \frac{(t-s)^n}{n!}\, dW(s) = \int \cdots \int \int dW(s) \underbrace{dt \cdots dt}_{n \ \text{times}}$$

Applying the Borel transform gives

$$\begin{aligned} y(t) &= F_c[u](t) = k_2 \sum_{n \geq 0} (-k_1)^n E_{x_0^n y_1}[u](t) + z_0 e^{-k_1 t} \\ &= k_2 \sum_{n \geq 0} \int_0^t \frac{(-k_1(t-s))^n}{n!}\, dW(s) + z_0 e^{-k_1 t} \\ &= k_2 \int_0^t e^{-k_1(t-s)}\, dW(s) + z_0 e^{-k_1 t}. \end{aligned}$$

Example 7 Consider the following state space system

$$\frac{d}{dt}z(t) = z^3(t)\,\bar{w}, \quad y(t) = z(t), \quad z(0) = 1. \tag{16}$$

Applying the formal Laplace transform to (16) gives

$$c = 1 + (x_1 + y_1)\, c^{\sqcup\!\sqcup\, 3}.$$

Thus, c can be calculated as $c = \sum_{k=0}^{\infty} c_k$, where

$$c_0 = 1, \qquad c_k = (x_1 + y_1) \sum_{\substack{i_1+i_2+i_3=k-1 \\ i_1, i_2, i_3 < k}} c_{i_1} \sqcup\!\sqcup\, c_{i_2} \sqcup\!\sqcup\, c_{i_3}.$$

The next three c_k's are given below:

$$c_1 = (x_1 + y_1), \;\; c_2 = 3\,(x_1 + y_1)^2, \;\; c_3 = 15\,(x_1 + y_1)^3.$$

Note, $c_k = (2k-1)!!(x_1 + y_1)^k$, $k \geq 0$. Since $(2k-1)!! = \frac{C_k^{2k}}{2^k}\, k! \leq 2^k k!$.
Therefore, c is locally convergent and exchangeable.

Hence, c is the generating series of the input-output operator

$$\begin{aligned} y &= \sum_{k=0}^{\infty} \frac{C_k^{2k} k!}{2^k} E_{(x_1+y_1)^{\sqcup\!\sqcup\, k}}[w](t) \\ &= \sum_{k=0}^{\infty} \frac{C_k^{2k}}{2^k} \underbrace{E_{(x_1+y_1)}^k[w](t)}_{w^k(t)} = \frac{1}{\sqrt{1-2\,w(t)}}, \end{aligned}$$

where $t \in [0, \tau_{\bar{R}}]$ with $\tau_{\bar{R}} = \inf\{t > 0 : |2\,w(t)| \ = \ \bar{R}\}$ and $\bar{R} < 1$.

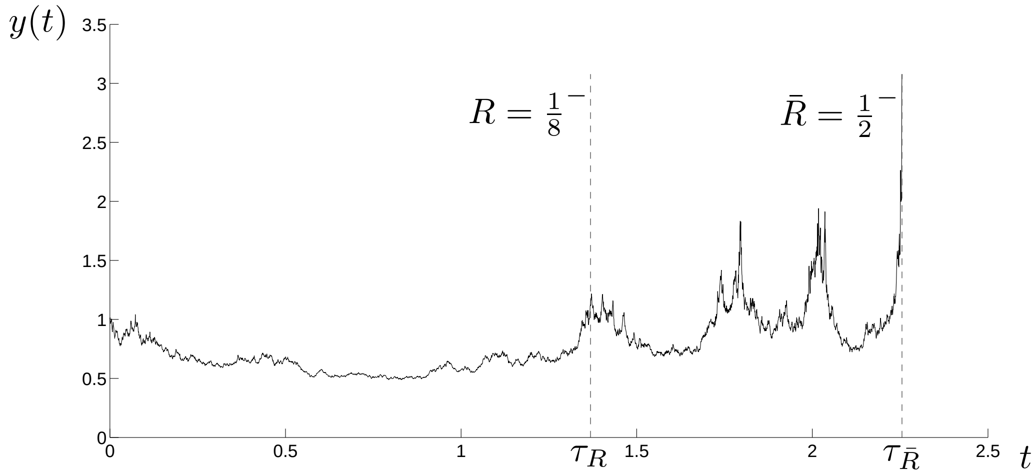

Figure 2: Sample path of $y(t)$.

3. System Interconnections with Stochastic Inputs

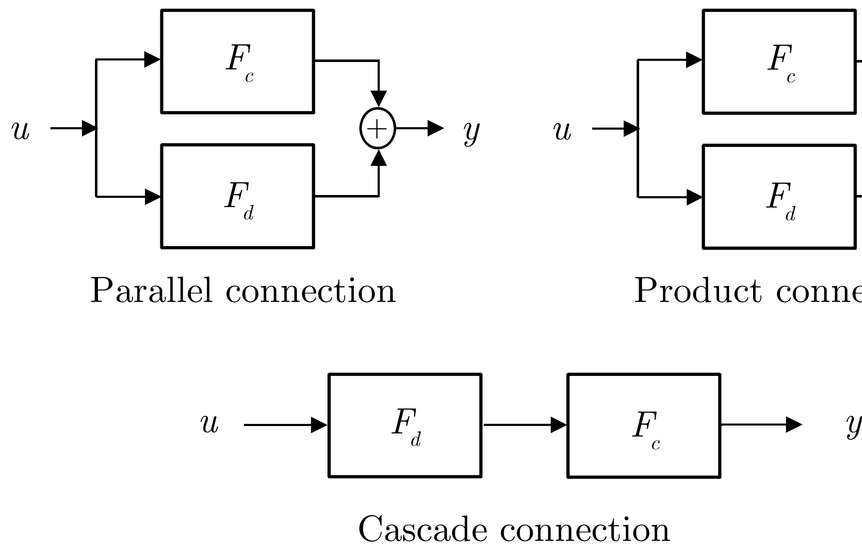

Figure 3: Elementary system interconnections.

$$
\begin{aligned}
F_c[u] + F_d[u] &= F_{c+d}[u] \\
F_c[u] \cdot F_d[u] &= F_{c \,\sqcup\!\sqcup\, d}[u] \\
F_c[F_d[u]] &= F_{c \circ d}[u].
\end{aligned}
$$

Let $X_0Y_0 = \{x_0, y_0\}$. Any $\eta \in XY^*$ can be written as

$$\eta = \eta_k q_{i_k}^{l_k} \eta_{k-1} q_{i_{k-1}}^{l_{k-1}} \ldots \eta_1 q_{i_1}^{l_1} \eta_0,$$

where $\eta_i \in X_0Y_0^*$ and $q_{i_j}^{l_j} = x_{i_j}$ when $l_j = 1$, $q_{i_j}^{l_j} = y_{i_j}$ when $l_j = 2$.

Definition 7 For $\eta \in XY^*$ and $d \in \mathbb{R}^m\langle\langle XY\rangle\rangle$ the composition product is

$$\eta \circ d = \begin{cases} \eta & : \quad |\eta|_{x_i, y_i} = 0, \ \forall\, i \neq 0 \\ \eta' q_0^l [d_i^j \sqcup\!\sqcup (\bar{\eta} \circ d)] & : \quad \eta = \eta' q_i^l \bar{\eta}, \ i \neq 0, \ l \in \{1, 2\}, \\ & \quad\;\; \eta' \in X_0Y_0^*, \ \bar{\eta} \in XY^*, \end{cases}$$

where $d_i^l : \xi \mapsto (d, \xi)_i^l$, and $(d, \xi)_i^l$ is the i-th component of $(d, \xi)^l$ with $l = 1$ representing drifts and $l = 2$ representing diffusions.

For $c \in \mathbb{R}^\ell\langle\langle XY\rangle\rangle$ and $d \in \mathbb{R}^m\langle\langle XY\rangle\rangle$,

$$c \circ d = \sum_{\eta \in XY^*} (c, \eta) \eta \circ d.$$

Remark: If $Y = \varnothing$, then $\circ$ reduces to the usual deterministic definition.

The following questions can be formulated:

- Can each interconnection of two Fliess operators with stochastic inputs be represented by another Fliess operator?
- What is the nature of the generating series of the composite Fliess operator given that the component generating series are either globally convergent or locally convergent?
- What conditions need to be imposed to obtain a well-defined stochastic process at the output of the interconnected system?
- Are all the signals in the interconnected system well-defined?

3.1 Formal Interconnections

Definition 8 Let $c_w \in \mathbb{R}^m\langle\langle X_0Y_0\rangle\rangle$. A formal stochastic process w is defined by

$$w(t) = \sum_{\eta \in X_0Y_0^*} (c_w, \eta) E_\eta[0](t). \tag{17}$$

The set of all formal stochastic processes is denoted by $\mathscr{W}$.

Remarks:

- For any $w \in \mathscr{W}$, there exist a corresponding generating series $c_w \in \mathbb{R}\langle\langle X_0Y_0\rangle\rangle$.
- Since c_w is arbitrary, w is simply a formal summation of iterated integrals.

Theorem 5 Let $c_w \in \mathbb{R}\langle\langle X_0 Y_0 \rangle\rangle$ be the generating series for a given $w \in \mathscr{W}$.

i. If c_w is a globally convergent series then $w \in \widetilde{\mathcal{UV}}^m[0, T]$.

ii. If w is ordered in the sense that

$$w(t) = \sum_{j=0}^{\infty} \sum_{k=0}^{j} \sum_{\eta \in X_0^k Y_0^{j-k}} (c_w, \eta) E_\eta[0](t), \tag{18}$$

and c_w is locally convergent then $w \in \widetilde{\mathcal{UV}}^m[0, \tau_R]$, where $\tau_R = \inf\{t \in [0, T] : |W(t)| = R\}$.

iii. If c_w is exchangeable and locally convergent then $w \in \widetilde{\mathcal{UV}}^m[0, \tau_R]$ regardless the order implied in (18).

Definition 9 The class of formal Fliess operators on $\mathbb{R}^m\langle\langle X_0 Y_0\rangle\rangle$ is the collection of mappings

$$\mathscr{F} \triangleq \{c\circ : \mathbb{R}^m\langle\langle X_0 Y_0\rangle\rangle \to \mathbb{R}^\ell\langle\langle X_0 Y_0\rangle\rangle : c_w \mapsto c_y = c \circ c_w, c \in \mathbb{R}^\ell\langle\langle XY\rangle\rangle\}.$$

Theorem 6 Let $c, d \in \mathbb{R}\langle\langle XY\rangle\rangle$ and $c_w \in \mathbb{R}\langle\langle X_0 Y_0\rangle\rangle$. The parallel, product and cascade connections of formal Fliess operators are characterized by the operations $+$, $\sqcup\!\sqcup$, and $\circ$ on $\mathbb{R}\langle\langle XY\rangle\rangle$ as

$$\begin{aligned} c \circ c_w + d \circ c_w &= (c + d) \circ c_w \\ (c \circ c_w) \sqcup\!\sqcup (d \circ c_w) &= (c \sqcup\!\sqcup d) \circ c_w \\ c \circ (d \circ c_w) &= (c \circ d) \circ c_w. \end{aligned}$$

Remark: The operator $c\circ$ is a formal operator in that it acts on a formal input, i.e., one that has a series representation.

3.2 Parallel and Product Interconnections

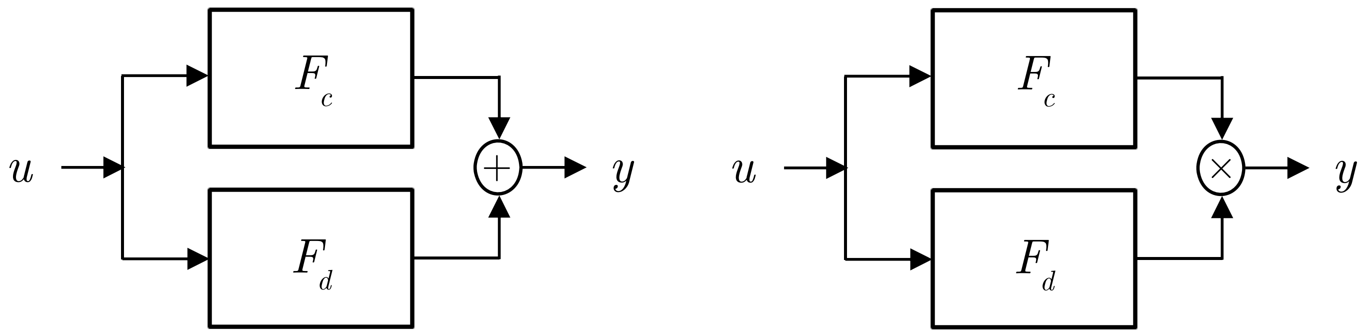

Figure 4: The parallel and product connections

What are the generating series corresponding to these interconnections?

$$F_c[w] + F_d[w] = F_{c+d}[w] \quad ?$$

$$F_c[w] \cdot F_d[w] = F_{c \sqcup\!\sqcup\, d}[w] \quad ?$$

Conditions for the series $c + d$ and $c \sqcup\!\sqcup\, d$ will establish the convergence of the product and parallel connections.

Theorem 7 If $c, d \in \mathbb{R}^\ell\langle\langle X Y\rangle\rangle$ are globally convergent then $c + d$ and $c \sqcup\!\sqcup\, d$ are globally convergent. Moreover, if $c, d \in \mathbb{R}^\ell\langle\langle X Y\rangle\rangle$ are locally convergent then $c + d$ and $c \sqcup\!\sqcup\, d$ are locally convergent.

Corollary 1 Let $c \in \mathbb{R}^\ell\langle\langle X Y\rangle\rangle$ and $d \in \mathbb{R}^\ell\langle\langle X Y\rangle\rangle$ be globally convergent series. For any $w \in \mathcal{UV}^m[0, T]$, $F_{c+d}[w]$ and $F_{c \sqcup\!\sqcup\, d}[w]$ produce well-defined L_2-Itô output processes over $[0, T]$ for any $T > 0$.

Corollary 2 Let $c, d \in \mathbb{R}^\ell\langle\langle X Y\rangle\rangle$ be locally convergent series. For any $w \in \mathcal{UV}^m[0, T]$, there exist an $R > 0$ and a stopping time τ_R such that $F_{c+d}[w]$ and $F_{c \sqcup\!\sqcup\, d}[w]$, respectively, produce L_2-Itô processes over $[0, \tau_R]$ assuming the order of summation defined as in (13).

Remark: If $c + d$ and $c \sqcup\!\sqcup\, d$ are exchangeable, then $F_{c+d}[w]$ and $F_{c \sqcup\!\sqcup\, d}[w]$ will be convergent unconditionally.

3.3 Cascade interconnections

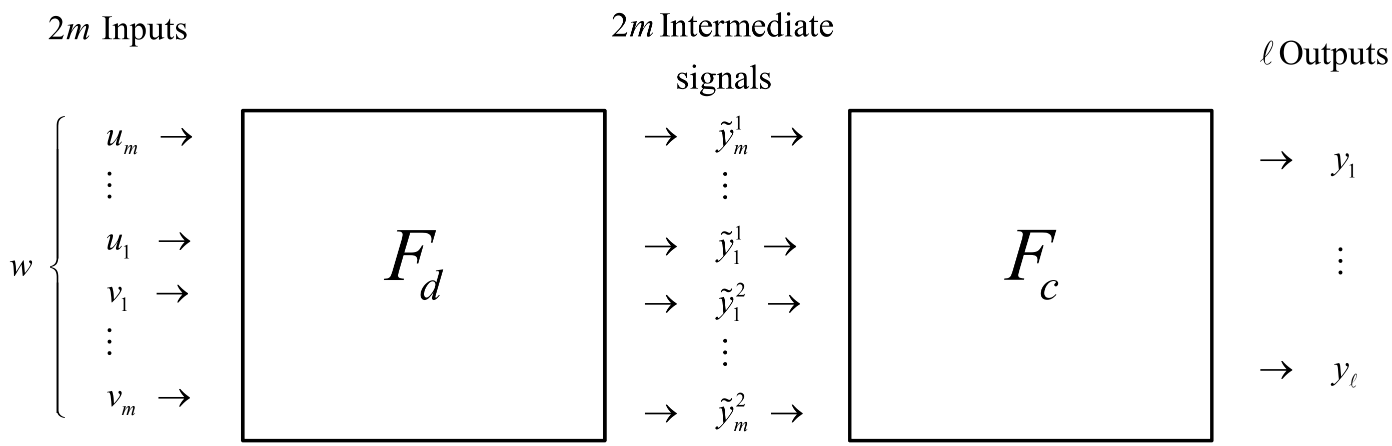

Figure 5: Cascade connection.

What is the generating series corresponding to the cascade connection?

$$F_c[F_d[w]] = F_{c\circ d}[w] \quad ?$$

Remark: For c and d in $\mathbb{R}\langle\langle XY\rangle\rangle$ locally convergent, the series $c \circ d$ is also locally convergent.

To illustrate the problems encountered in the cascade of systems driven by stochastic processes, consider

$$F_c[\tilde{y}] = F_c[f[u,v]](t),$$

where $c \in \mathbb{R}\langle\langle XY \rangle\rangle$, and $\tilde{y} = (\tilde{y}_1, \tilde{y}_2)^T$ is given by

$$\begin{aligned}
\tilde{y}_1(t) &= f_1[u,v](t) = \int_0^t u(s) \not\int_0^s v(r)\, dW(r)\, ds \\
\tilde{y}_2(t) &= f_2[u,v](t) = \not\int_0^t v(s) \int_0^s u(r)\, dr\, dW(s),
\end{aligned}$$

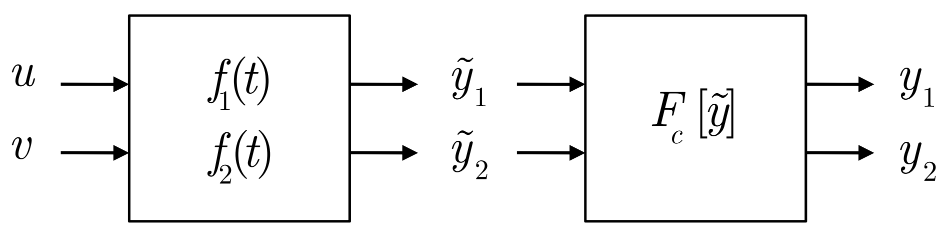

Figure 6: Cascade of input-output maps.

Even u and v are mutually independent, the intermediate signals $\tilde{y}_1$ and $\tilde{y}_2$ are correlated. Implying

$$\mathbf{E}[\tilde{y}_1\tilde{y}_2] \neq \mathbf{E}[\tilde{y}_1]\mathbf{E}[\tilde{y}_2].$$

Since F_c is only defined for independent inputs, it cannot be driven by $\tilde{y}$.

Thus, the cascade connection is at present not well-posed because the inputs and outputs are not compatible.

Remarks:

- The formulation of Fliess operators on Banach spaces (for rough paths) is very likely to solve this obstacle. In that context, no requirement for independence is needed.
- It is believed that seeing input paths as rough paths may give better estimates of the mapping E_η.
- However, the so-called control function must be better understood in the systems terminology.

4. Conclusions

- An extension of the notion of a Fliess operators for L_2-Itô process inputs was presented.
- To consider system interconnections, the notion of global and local stochastic convergence for these operators was considered.
- Local absolute convergence over random intervals of time was not achieved in general. The same limitation is expected for rough paths.
- The generating series of the cascade connection of formal Fliess operators was presented.
- The cascade connection was shown not to be well-posed under the current setting since the inputs and outputs are not compatible.
- It is expected that the limitations found in the cascade connection, because of the way stochastic inputs were characterized, can be overcome by using Lyon's rough path theory. However, many concepts have to be adapted to the systems terminology.