Some aspects of stochastic differential equations driven by fractional Brownian motions

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Based on joint works with L. Coutin, M. Hairer, C. Ouyang and S. Tindel

Motivation

The motivation of the talk is to present some results concerning solutions of stochastic differential equations on $\mathbb{R}^n$

$$X_t^x = x + \int_0^t V_0(X_s^x)ds + \sum_{i=1}^{d} \int_0^t V_i(X_s^x)dB_s^i \qquad (1)$$

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- Functional inequalities satisfied by the law of the solution and upper Gaussian bounds for the density.

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If $I = (i_1, \dots, i_k) \in \{0, \dots, d\}^k$, we denote by V_I the Lie commutator defined by

$$V_I = [V_{i_1}, [V_{i_2}, \dots, [V_{i_{k-1}}, V_{i_k}] \dots]$$

and

$$d(I) = k + n(I),$$

where $n(I)$ is the number of 0 in the word I.

Theorem (Baudoin-Hairer, PTRF'07)

Assume that, at some $x_0 \in \mathbb{R}^n$, there exists N such that:

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$$\mathbf{span}\{V_I(x_0), d(I) \le N\} = \mathbb{R}^n. \tag{2}$$

Then, for any $t > 0$, the law of the random variable $X_t^{x_0}$ has a smooth density with respect to the Lebesgue measure on $\mathbb{R}^n$.

Scheme of the proof

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- Smoothness with $H > 1/3$ in some particular cases by Hu-Tindel (2011)
- Smoothness with $H > 1/3$ in the general case by Hairer-Pillai (2011) + Cass-Litterer-Lyons (2011).

Operators associated with SDEs driven by fBms

Again, let us consider the stochastic differential equations on $\mathbb{R}^n$

$$X_t^{x_0} = x_0 + \sum_{i=1}^{d} \int_0^t V_i(X_s^{x_0})dB_s^i \tag{3}$$

where the V_i's are C^∞-bounded vector fields on $\mathbb{R}^n$ and B is a d dimensional fractional Brownian motion with Hurst parameter $H > \frac{1}{3}$.

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where the V_i's are C^∞-bounded vector fields on $\mathbb{R}^n$ and B is a d dimensional fractional Brownian motion with Hurst parameter $H > \frac{1}{3}$.
We denote by $\mathcal{C}_b^\infty(\mathbb{R}^n, \mathbb{R})$ the set of compactly supported smooth functions $\mathbb{R}^n \to \mathbb{R}$. If $f \in \mathcal{C}_b^\infty(\mathbb{R}^n, \mathbb{R})$, let us denote

$$\mathbf{P}_t f(x_0) = \mathbb{E}\left(f(X_t^{x_0})\right), \ \ t \geq 0,$$

where $X_t^{x_0}$ is the solution of (3) at time t.

Development in small times

Theorem (Baudoin-Coutin, SPA'07)

Assume $H > \frac{1}{3}$. *There exists a family* $\left(\Gamma_k^H\right)_{k \geq 0}$ *of differential operators such that:*
If $f \in \mathcal{C}_b^\infty(\mathbb{R}^n, \mathbb{R})$ *and* $x \in \mathbb{R}^n$, *then for every* $N \geq 0$, *when* $t \to 0$

$$\mathbf{P}_t f(x) = \sum_{k=0}^{N} t^{2kH} (\Gamma_k^H f)(x) + o(t^{(2N+1)H}).$$

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$$+\left(\frac{H}{4(4H-1)} - \frac{H}{4}\beta(2H,2H)\right)\sum_{i,j=1}^{d}(V_i V_j)^2,$$
where $\beta(a,b) = \int_0^1 x^{a-1}(1-x)^{b-1}dx$;

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$$+ \left(\frac{H}{4(4H-1)} - \frac{H}{4}\beta(2H,2H)\right)\sum_{i,j=1}^{d}(V_i V_j)^2,$$
where $\beta(a,b) = \int_0^1 x^{a-1}(1-x)^{b-1}dx$;

3. More generally, Γ_k^H is a homogeneous polynomial in the $V_i's$ of degree $2k$:
$$\Gamma_k^H = \sum_{I=(i_1,\dots i_{2k})} a_I V_{i_1}\dots V_{i_{2k}},$$

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- We conjecture that Γ_k^H, $k \geq 2$, has coefficients that are meromorphic functions of H with poles in the set $\{\frac{1}{2j}, 2 \leq j \leq k\}$.

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- It would be interesting to determine what is the smallest algebra of vector fields that contains the family $(\Gamma_k^H)_{k \geq 1}$ (In the case of Brownian motion, this algebra is the algebra generated by the operator $\sum_{i=1}^d V_i^2$).

Smoothing property

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has a smooth integral kernel. We prove here the following regularisation bounds, for $q > 1$,

$$|V_{i_1} \cdots V_{i_k} P_t f(x)| \le \frac{C_{k,q}(x)}{t^{kH}} \left(P_t |f|^q\right)^{1/q}(x), \quad 0 < t < 1.$$

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which implies

$$|V_{i_1} \cdots V_{i_k} P_t f(x)| \leq \frac{C_k(x)}{t^{kH}} \|f\|_\infty, \quad 0 < t < 1.$$

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We first have the following commutation

Lemma

$$V_i P_t f(x) = \mathbb{E}\left(\sum_{k=1}^{n} \alpha_i^k(t,x) V_k f(X_t^x)\right),$$

where α solves the following system of SDEs:

$$d\alpha_i^j(t,x) = \sum_{k,l=1}^{n} \alpha_i^k(t,x)\omega_{kl}^j(X_t^x)dB_t^l, \quad \alpha_i^j(0,x) = \delta_i^j.$$

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Then by ellipticity, we can find $\alpha_i^j(t,x)$ such that

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The change of variable formula shows that α solves the above system of SDEs. □

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Theorem

If f is C^∞-bounded and $\Phi \in \mathcal{K}_r$,

$$\mathbb{E}\left(\Phi(t,x)V_i f(X_t^x)\right) = \mathbb{E}\left((T_{V_i}\Phi)(t,x) f(X_t^x)\right)$$

for some $T_{V_i}\Phi \in \mathcal{K}_{r-1}$.

Integration by parts on the path space

$$\begin{aligned} \mathbf{D}_s^j f(X_t) &= \langle \nabla f(X_t), \mathbf{D}_s^j X_t^x \rangle \\ &= \langle \nabla f(X_t), \mathbf{J}_t \mathbf{J}_s^{-1} V_j(X_s^x) \rangle \\ &= \sum_{k,l=1}^{n} h_k^j(s,t,x) \alpha_l^k(t,x)(V_l f)(X_t^x). \end{aligned}$$

where

$$h_i(s,t,x) = (\beta_i^k(s,x)\mathbb{I}_{[0,t]}(s))_{k=1,\dots,n}, \qquad i = 1, \dots, n.$$

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Introduce $M_{i,j}(t,x)$ given by

$$M_{i,j}(t,x) = \frac{1}{t^{2H}} \langle h_i(\cdot,t,x), h_j(\cdot,t,x) \rangle_{\mathcal{H}}.$$

Hence

$$V_i f(X_t^x) = \frac{1}{t^{2H}} \sum_{j,l=1}^{n} \beta_j^i(t,x) M_{jl}^{-1}(t) \langle \mathbf{D} f(X_t^x), h_l(\cdot,t) \rangle_{\mathcal{H}}.$$

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$$T^*_{V_i}\Phi(t,x)=\sum_{k,l=1}^{n}\Bigg[\frac{1}{t^{2H}}\Phi(t,x)\beta^i_k(t,x)M^{-1}_{kl}(t)\delta h_l(\cdot,t)$$
$$-\frac{1}{t^{2H}}\langle \mathbf{D}(\Phi(t,x)\beta^i_k(t,x)M^{-1}_{kl}(t)),h_l(\cdot,t)\rangle_{\mathcal{H}}\Bigg]$$

Iterating the previous formulas and using Hölder's inequality, we finally conclude:

Regularizing bounds

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Theorem (Baudoin-Ouyang, 2011)

For $q > 1$,

$$|V_{i_1} \cdots V_{i_k} P_t f(x)| \le \frac{C_{k,q}(x)}{t^{kH}} \left(P_t |f|^q\right)^{1/q}(x), \quad 0 < t < 1.$$

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then we have the global bound

$$\sqrt{\sum_{i=1}^n (V_i P_t f)^2(x)} \le P_t \left(\sqrt{\sum_{i=1}^n (V_i f)^2} \right)(x).$$

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There exists M such that for every $T \geq 0$ and $\lambda \geq 0$,

$$\mathbb{P}\left(\sup_{0 \leq t \leq T} \|X_t^x\| - \mathbb{E}\left(\sup_{0 \leq t \leq T} \|X_t^x\| \right) \geq \lambda \right) \leq \exp\left(-\frac{\lambda^2}{2MT^{2H}} \right).$$

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and a corresponding Gaussian upper bound.

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For any $t \in \mathbb{R}_+^*$, the random variable X_t^x admits a smooth density $p_X(t,\cdot)$. Furthermore, there exist 3 positive constants $c_t^{(1)}, c_t^{(2)}, c_{t,x}^{(3)}$ such that

$$p_X(t,y) \leq c_t^{(1)} \exp\left(-c_t^{(2)}\left(\|y\| - c_{t,x}^{(3)}\right)^2\right),$$

for any $y \in \mathbb{R}^d$.

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a log-Sobolev inequality:

Theorem (Baudoin-Ouyang-Tindel, 2011)

$$P_t(f \ln f) - (P_t f)(\ln P_t f) \leq 2C t^{2H} P_t \left(\sum_{i=1}^{n} \frac{(V_i \ln f)^2}{f} \right)$$