A class of aggregation-diffusion equations: concentration and small-scale behaviour.

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Outline

What do we mean by small-scale behaviour?

Basic ideas: For example (case when the mass (L_1-norm) of $u \geq 0$ is conserved) most of the mass is concentrated on a small ball $B(\varepsilon)$ of radius ε.

More involved: for $p \geq 1$ the L_p norms behave as $\varepsilon^{-c(p,N)}$.

Indeed, if $\int_{B(\varepsilon)} u \geq C$, then by Hölder's inequality,

$$\left(\int_{B(\varepsilon)} u^p \right)^{1/p} \geq C|B(\varepsilon)|^{-(p-1)/p} = C\varepsilon^{-N(1-1/p)}.$$

For a reverse inequality, we would need for example an upper estimate for $|u|_\infty$.
If we have weaker concentration in the limit $\varepsilon \to 0$ (on a surface of dimension k rather than a point), we obtain a different exponent for ε (equal to $-(N-k)(1-1/p)$).

What do we mean by small-scale behaviour? (2)

Oscillations beyond concentration: we study the small-scale behaviour of the Sobolev semi-norms

$$|u|_{m,p} := \left(\int_{\mathbb{R}^n} \left| \frac{\partial^m u}{\partial x^m} \right|^p dx \right)^{1/p}.$$

In the language of hydrodynamics/turbulence theory (Kolmogorov, Kraichnan, Frisch...): typical small-scale quantities used to detect oscillations:

-$\hat{\mathbf{u}}(s)$ for large $|s|$.
-$\mathbf{u}(\mathbf{x}+\mathbf{r}) - \mathbf{u}(\mathbf{x})$ for small $\mathbf{r}$.

Small-scale quantities are related to Sobolev norms:

-$H^m = W^{m,2}$ Sobolev norms defined through spectrum.

-Hölder, Sobolev-Slobodeckij... defined through increments; then Sobolev injections.

What do we NOT mean by small-scale behaviour?

In this talk, we only consider space scales, not time scales. However, our lower estimates almost always involve time averaging due to the energy/moments method we use.

We also do not touch semiclassical, stationary phase... type phenomena which also involve PDEs with a small parameter.

What type of results?

Small-scale behaviour of solutions is studied for PDEs from:

- Hydrodynamics: Navier-Stokes, Burgers, Korteweg-De Vries,
- Quantum physics: nonlinear Schrödinger,
- Biology: aggregation-diffusion: Keller-Segel,
- Astrophysics: Burgers; aggregation-diffusion...

For Sobolev norms, estimates have been found for
2D Navier-Stokes, non-linear Schrödinger, Korteweg-de Vries...
with and without random forcing (typically on the torus).
See Kuksin '97-'98, the book of Kuksin-Shirikyan ('12) and the book of B.-Kuksin ('21).

However, these estimates are not sharp for $\varepsilon \to 0$ (different powers of ε for upper/lower bounds). Possible with simpler models? Yes!

1D Periodic Generalised Burgers Equation

$$v_t + (f(v))_x = \varepsilon v_{xx},\ t \geq 0,\ x \in S^1 = \mathbb{R}/\mathbb{Z}. \tag{1DB}$$

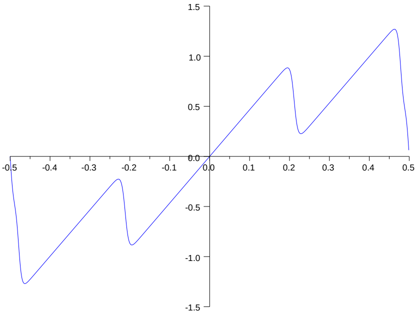
We assume that f is smooth, strongly convex ($f(v) = v^2/2$: usual Burgers).
So we never use the Cole-Hopf transformation.

"Pressureless turbulence" considered by many physicists, for instance Polyakov '95 (and Zeldovich in the multi-d case '89).

We assume that $\varepsilon > 0,\ \varepsilon \ll 1$. Again, only ε varies.

For simplicity, we assume that the integral $\int_{S^1} v(t,\cdot)$ vanishes for $t = 0$, and therefore for all t.
We may study the unforced problem or add random (smooth in space) forcing.

Typical Profile of a Burgers Solution



Amplitude of solution ~ 1. Cliffs (quasi-shocks): number of cliffs ~ 1, jump ~ -1, width $\sim \varepsilon$ (scaling argument).
Burgers turbulence or "Burgulence": see [Bec-Khanin 2007].
Ramp-cliff structure $\Rightarrow$ intermittency.

Estimates for the Sobolev Norms of the Solution

In [B. '14], I obtain sharp estimates for the (averaged) (1DB) solution.

$$\{|v|_{m,p}\} \overset{m,p}{\sim} \varepsilon^{-\gamma}, \quad \forall m \geq 1,\ 1 < p \leq \infty.$$

Here $\gamma(m,p) = m - 1/p$, and $\{\dots\}$ stands for averaging over a v_0-dependent time period $[T_1, T_2]$.

Upper and lower estimates the same up to a ε-independent constant, and necessarily depend on v_0.
These results can be adapted for a subcritical fractional damping $-\varepsilon(\partial_{xx})^{\alpha} u,\ 1/2 < \alpha < 1$ with a different scaling argument.
For more details, especially in a random setting, see the book [B.-Kuksin].

Estimates for the Sobolev Norms of the Solution: Ideas of Proofs

Precise upper estimates are obtained by using Oleinik's estimate $u_x \leq t^{-1}$.

Precise lower estimates follow from the energy balance:

$$\frac{d}{dt}|u|_2^2 = -2\varepsilon|u|_{1,2}^2.$$

combined with the 'inviscid energy dissipation'.

Propagation to higher order Sobolev norms follows from the Gagliardo-Nirenberg inequality and higher-order energy estimates.

Upper Bounds: Oleinik's Estimate

Consider unforced (1DB) on $S = (t, x) \in [0, T] \times S^1$:

$$u_t + uu_x = \varepsilon u_{xx}.$$

Consider $v = tu_x$. The function v can only reach a str. positive maximum for $t > 0$. Then we would have:

$$\underbrace{v_t}_{\geq 0} + u \underbrace{v_x}_{0} + t^{-1}(-v + v^2) = \underbrace{\varepsilon v_{xx}}_{\leq 0}.$$

Thus $v \leq 1$ on S. In other words, $u_x \leq t^{-1} \Rightarrow$ "damping".

Obtaining lower bounds

We have:

$$\frac{d}{dt}\int_{S^1} u^2 = \underbrace{-2\int_{S^1} uf'(u)u_x}_{0} + 2\varepsilon \int_{S^1} uu_{xx} = -2\varepsilon \int_{S^1} u_x^2.$$

Integrating in time, we get:

$$|u(T)|_2^2 - |u(0)|_2^2 = -2\varepsilon T\{|u|_{1,2}^2\}.$$

Using the upper estimates, for $T \geq 1$ we have that:

$$|u(T)|_2^2 \leq (\max_x u_x(0,x))^2 \leq CT^{-2}.$$

Consequently, for T large enough:

$$\{|u|_{1,2}^2\} \geq CT^{-1}\varepsilon^{-1}.$$

Chemotaxis to (ADE)

Chemotaxis=aggregation of bacteria, pollen, spermatozoids... through chemical signals.
Parabolic-parabolic Keller-Segel model:

$$u_t - \varepsilon\Delta u + \nabla\cdot(u\nabla c) = 0;$$
$$\delta c_t - \Delta c + \alpha c = u.$$

The quantities $u, c \geq 0$ stand for cell density and concentration of a chemical signal, respectively.
In the limit $\delta \to 0$ (instantaneously propagating information) we get the parabolic-elliptic aggregation-diffusion equation:

$$u_t - \varepsilon\Delta u + \nabla\cdot(u\nabla K * u) = 0, \ (ADE)$$

with K the kernel of the elliptic operator $-\Delta + \alpha Id$.

Our setting: pointy potentials

$$u_t - \varepsilon \Delta u + \nabla \cdot (u \nabla K * u) = 0.$$

Radial kernel $K = k(|\cdot|)$ satisfying $k' \in L^\infty \cap C^0([0,\infty))$ (like 1D chemiotaxis).

Properties: Preservation of positivity; conservation of mass $M = \int u$; global well-posedness (in $L_1 \cap L_p,\ p < \infty,\ L_1 \cap W^{m,1}$).

We assume that $k'(0) \neq 0$; therefore there is a mild singularity (pointy potential).

Typical examples $K(x) = -|x|;\ e^{-|x|}$.

Inviscid explosion (I)

For $\varepsilon = 0$, i.e. for the aggregation equation

$$u_t + \nabla \cdot (u \nabla K * u) = 0,$$

short-time well-posedness and long-time explosion if the kernel is attractive. This is proved by the (generalised) characteristics method or using gradient flow tools:
Bertozzi, Laurent, Rosado; Carrillo, James, Lagoutière, Vauchelet...
Carrillo, DiFrancesco, Figalli, Laurent, Slepčev...

Inviscid explosion (II)

Explosion in the radial attractive case: the quantity

$$D(u(t)) := u(0,t) \text{ if } \quad N = 1, \quad \int_{\mathbb{R}^n} \frac{u(x,t)}{|x|}\, dx \text{ if } \quad N \geq 2$$

explodes in finite time (Biler-Karch-Laurençot '09).

More precisely, my collaborators argue by contradiction, obtaining $D(u(T)) < 0$ for some $T(u_0)$.

Small-scale behaviour

[Biler-B.-Karch-Laurençot 1] Assume that u_0 is radially symmetric, concentrated near 0 and K is attractive near 0. Then the solution u of (ADE) satisfies

$$\int_0^{T_*} \int_{B(\lambda_* \varepsilon)} u(x,t)\, dx\, dt \geq C_* \ \Rightarrow \text{(Hölder)}$$

$$\int_0^{T_*} \left(\int_{B(\lambda_* \varepsilon)} u(x,t)^p\, dx \right)^{1/p} \geq C(p) \varepsilon^{-N(1-\frac{1}{p})},\ 1 \leq p < \infty,$$

for all $\varepsilon \in (0, \varepsilon_*)$. The constants with the $*$ only depend on u_0, K through a finite number of parameters.

These L^p estimates are sharp; the corresponding upper estimates hold on the whole space $\mathbb{R}^n$ and without time averaging.

Proof of small-scale concentration

The upper estimates hold under very general conditions: radial symmetry is not needed. We use an energy method.

To prove lower estimates, we consider again the quantity:

$$D(u(t)) := u(0,t) \text{ if } \quad N = 1, \quad \int_{\mathbb{R}^n} \frac{u(x,t)}{|x|}\, dx \text{ if } \quad N \geq 2$$

If $\varepsilon > 0$, no explosion. However, integrating by parts and using a symmetrization trick we obtain that for some $T_* > 0$:

$$\int_0^{T_*} D(u(t)) \geq C\varepsilon^{-1}.$$

Combining this lower estimate with the upper ones in Lebesgue spaces (and in H^1 if $N = 1$) and using Hölder's inequality, we obtain the lower bounds for L_p norms.

Sobolev norms

In [Biler-B.-Karch-Laurençot 2], we obtain ε-optimal Sobolev norms for u localised on a small ball as above.

Lower estimates: They follow from lower estimates for L_p norms and the GN (Gagliardo-Nirenberg) inequality.
For example, since (after averaging in time) we have (by conservation of the mass M):

$$C\varepsilon^{-N/2} \leq |u|_2 \leq CM^{2/(N+2)}|u|_{1,2}^{N/(N+2)},$$

we obtain that

$$|u|_{1,2} \geq CM^{-2/N}\varepsilon^{-(N/2+1)}.$$

Upper estimates: Energy method. Inequalities of Hölder, GN and HLS (Hardy-Littlewood-Sobolev), taking derivatives of K (convolution with $|x|^{-k}$ for $k < N$).

Analogy between Burgers and (ADE)

Formally, if v solves Burgers, $-v_x$ satisfies (ADE) with $K(x) = -|x|$. However:

1. Periodic setting so $-v_x = u \geq 0$ is impossible.
2. This is a purely 1D analogy.

However, this suggests $|u|_p \sim \varepsilon^{-(1-1/p)}$.

And this is indeed true only in 1D.

So dependence on N for (ADE). The explanation is that when $\varepsilon \to 0$, for Burgers (resp. (ADE)) we concentrate to a singularity of codimension 1 (resp. dimension 0).

Concluding Remarks

Our results give precise and rigorously proved small-scale estimates for a broad class of (deterministic and random) models.

Until recently, such results were only available for Burgers-type equations, relying heavily on versions of Oleinik's estimate $u_x \leq t^{-1}$.

Many perspectives on aggregation-diffusion equations (which do not have inviscid upper estimates like Oleinik's, but have obvious inviscid lower ones since solutions are positive!)

Natural question: what if the initial condition is not radially symmetric?

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