Uniqueness issues for the fast diffusion equation

GABRIELE GRILLO

Dipartimento di Matematica, Politecnico di Milano
Research group "Analysis@Polimi"

Joint works with M. Muratori, F. Punzo, K. Ishige

MAY 12, 2023,
Workshop "Diffusion in Warsaw"

The fast diffusion equation on manifolds

We study the following Cauchy problem, associated to the fast diffusion equation (FDE):

The fast diffusion equation on manifolds

We study the following Cauchy problem, associated to the fast diffusion equation (FDE):

$$\begin{cases} u_t = \Delta u^m & \text{in } M \times (0, +\infty), \\ u = u_0 & \text{on } M \times \{0\}, \end{cases} \tag{1}$$

The fast diffusion equation on manifolds

We study the following Cauchy problem, associated to the fast diffusion equation (FDE):

$$\begin{cases} u_t = \Delta u^m & \text{in } M \times (0, +\infty)\,, \\ u = u_0 & \text{on } M \times \{0\}\,, \end{cases} \tag{1}$$

where $m \in (0, 1)$, the initial datum u_0 belongs to a suitable class to be specified, M is a complete, connected, noncompact n-dimensional Riemannian manifold and Δ is the Laplace-Beltrami operator on M. When dealing with sign-changing solutions, we set $u^m := \mathrm{sign}(u)|u|^m$.

The fast diffusion equation on manifolds

We study the following Cauchy problem, associated to the fast diffusion equation (FDE):

$$\begin{cases} u_t = \Delta u^m & \text{in } M \times (0, +\infty)\,, \\ u = u_0 & \text{on } M \times \{0\}\,, \end{cases} \tag{1}$$

where $m \in (0, 1)$, the initial datum u_0 belongs to a suitable class to be specified, M is a complete, connected, noncompact n-dimensional Riemannian manifold and Δ is the Laplace-Beltrami operator on M. When dealing with sign-changing solutions, we set $u^m := \text{sign}(u)|u|^m$.

More general nonlinearities will be considered in the second part of the talk.

The Cauchy problem (1) in the case $M \equiv \mathbb{R}^n$ was thoroughly investigated in a seminal paper by Herrero and Pierre (TAMS 1985).

The Cauchy problem (1) in the case $M \equiv \mathbb{R}^n$ was thoroughly investigated in a seminal paper by Herrero and Pierre (TAMS 1985).

They show that, for any $L^1_{\mathrm{loc}}(\mathbb{R}^n)$ initial datum, there exists a global in time solution, which is also unique under the additional assumption that u_t is locally integrable, i.e. in the class of strong solutions.

The Cauchy problem (1) in the case $M \equiv \mathbb{R}^n$ was thoroughly investigated in a seminal paper by Herrero and Pierre (TAMS 1985).

They show that, for any $L^1_{\mathrm{loc}}(\mathbb{R}^n)$ initial datum, there exists a global in time solution, which is also unique under the additional assumption that u_t is locally integrable, i.e. in the class of strong solutions.

In particular, no requirement at all on the behavior at infinity of u_0 is necessary.

The Cauchy problem (1) in the case $M \equiv \mathbb{R}^n$ was thoroughly investigated in a seminal paper by Herrero and Pierre (TAMS 1985).

They show that, for any $L^1_{\mathrm{loc}}(\mathbb{R}^n)$ initial datum, there exists a global in time solution, which is also unique under the additional assumption that u_t is locally integrable, i.e. in the class of strong solutions.

In particular, no requirement at all on the behavior at infinity of u_0 is necessary.

A crucial tool in their proofs is the celebrated Herrero-Pierre estimate. It has an analogue in our context:

The following results are taken from G., Muratori, Punzo, TAMS 2021.

The following results are taken from G., Muratori, Punzo, TAMS 2021.

Proposition

Let $m \in (0,1)$. Let $u, v \in L^1_{\mathrm{loc}}(M \times (0,+\infty))$ with $u \geq v$. Suppose that

$$u_t = \Delta u^m \quad \text{and} \quad v_t = \Delta v^m \qquad \text{in } \mathcal{D}'(M \times (0,+\infty))\,. \tag{2}$$

The following results are taken from G., Muratori, Punzo, TAMS 2021.

Proposition

Let $m \in (0,1)$. Let $u, v \in L^1_{\mathrm{loc}}(M \times (0,+\infty))$ with $u \geq v$. Suppose that

$$u_t = \Delta u^m \quad \text{and} \quad v_t = \Delta v^m \qquad \text{in } \mathcal{D}'(M \times (0,+\infty)). \tag{2}$$

Let $R > 0$. Then the following estimate holds, for a suitable constant $\mathcal{H}_R > 0$:

The following results are taken from G., Muratori, Punzo, TAMS 2021.

Proposition

Let $m \in (0,1)$*. Let* $u, v \in L^1_{\mathrm{loc}}(M \times (0,+\infty))$ *with* $u \geq v$*. Suppose that*

$$u_t = \Delta u^m \quad \text{and} \quad v_t = \Delta v^m \qquad \text{in } \mathcal{D}'(M \times (0,+\infty)) \,. \tag{2}$$

Let $R > 0$*. Then the following estimate holds, for a suitable constant* $\mathcal{H}_R > 0$*:*

$$\left[\int_{\Omega_R} [u(t) - v(t)] \, d\mu \right]^{1-m} \leq \left[\int_{\Omega_{2R}} [u(s) - v(s)] \, d\mu \right]^{1-m} + \mathcal{H}_R \, |t - s| \tag{3}$$

for almost every $t, s \in (0,+\infty)$*.*

The following results are taken from G., Muratori, Punzo, TAMS 2021.

Proposition

Let $m \in (0,1)$*. Let* $u, v \in L^1_{\mathrm{loc}}(M \times (0,+\infty))$ *with* $u \geq v$*. Suppose that*

$$u_t = \Delta u^m \quad \text{and} \quad v_t = \Delta v^m \qquad \text{in } \mathcal{D}'(M \times (0,+\infty))\,. \tag{2}$$

Let $R > 0$*. Then the following estimate holds, for a suitable constant* $\mathcal{H}_R > 0$*:*

$$\left[\int_{\Omega_R} [u(t) - v(t)]\, d\mu\right]^{1-m} \leq \left[\int_{\Omega_{2R}} [u(s) - v(s)]\, d\mu\right]^{1-m} + \mathcal{H}_R\, |t - s| \tag{3}$$

for almost every $t, s \in (0,+\infty)$*.* (3) *still holds, with moduli of the differences under the integrals, when* u, v *are not necessarily ordered but* $u_t, v_t \in L^1_{\mathrm{loc}}(M \times (0,+\infty))$*.*

The following results are taken from G., Muratori, Punzo, TAMS 2021.

Proposition

Let $m \in (0,1)$. Let $u, v \in L^1_{\mathrm{loc}}(M \times (0,+\infty))$ with $u \geq v$. Suppose that

$$u_t = \Delta u^m \quad \text{and} \quad v_t = \Delta v^m \qquad \text{in } \mathcal{D}'(M \times (0,+\infty))\,. \tag{2}$$

Let $R > 0$. Then the following estimate holds, for a suitable constant $\mathcal{H}_R > 0$:

$$\left[\int_{\Omega_R} [u(t) - v(t)]\, d\mu\right]^{1-m} \leq \left[\int_{\Omega_{2R}} [u(s) - v(s)]\, d\mu\right]^{1-m} + \mathcal{H}_R\, |t - s| \tag{3}$$

for almost every $t, s \in (0,+\infty)$. (3) *still holds, with moduli of the differences under the integrals, when u, v are not necessarily ordered but $u_t, v_t \in L^1_{\mathrm{loc}}(M \times (0,+\infty))$.*

Here, μ is the Riemannian measure and Ω_R are "morally" Riemannian balls.

To introduce our main results, we first define our concept of distributional solutions.

To introduce our main results, we first define our concept of distributional solutions.

Definition 1 (Distributional solutions)

Let $u_0 \in L^1_{\mathrm{loc}}(M)$. *We say that a function* $u \in L^1_{\mathrm{loc}}(M \times [0, +\infty))$ *is a distributional solution of the Cauchy problem* (1) *if it satisfies*

To introduce our main results, we first define our concept of distributional solutions.

Definition 1 (Distributional solutions)

Let $u_0 \in L^1_{\mathrm{loc}}(M)$. We say that a function $u \in L^1_{\mathrm{loc}}(M \times [0, +\infty))$ is a distributional solution of the Cauchy problem (1) *if it satisfies*

$$u_t = \Delta u^m \qquad \text{in } \mathcal{D}'(M \times (0, +\infty)) \tag{4}$$

To introduce our main results, we first define our concept of distributional solutions.

Definition 1 (Distributional solutions)

Let $u_0 \in L^1_{\mathrm{loc}}(M)$*. We say that a function* $u \in L^1_{\mathrm{loc}}(M \times [0, +\infty))$ *is a distributional solution of the Cauchy problem* (1) *if it satisfies*

$$u_t = \Delta u^m \qquad \text{in } \mathcal{D}'(M \times (0, +\infty)) \tag{4}$$

and (in the sense of essential limits)

$$\lim_{t \to 0^+} \int_M u(x,t)\, \psi(x)\, d\mu = \int_M u_0\, \psi\, d\mu \qquad \forall \psi \in C_c^\infty(M)\,. \tag{5}$$

Our main existence result, without further assumptions on u_0, is the following.

Our main existence result, without further assumptions on u_0, is the following.

Theorem 1 (Existence of solutions)

Let $u_0 \in L^1_{\mathrm{loc}}(M)$. *There exists a solution* u *of problem* (1), *in the sense of Definition 1. In addition* $u \in C([0,+\infty); L^1_{\mathrm{loc}}(M))$.

Our main existence result, without further assumptions on u_0, is the following.

Theorem 1 (Existence of solutions)

Let $u_0 \in L^1_{\mathrm{loc}}(M)$. There exists a solution u of problem (1)*, in the sense of Definition 1. In addition $u \in C([0, +\infty); L^1_{\mathrm{loc}}(M))$.*

For nonnegative initial data that also belong to $L^2_{\mathrm{loc}}(M)$, we can establish existence of the minimal solution in the class of nonnegative distributional solutions.

Our main existence result, without further assumptions on u_0, is the following.

Theorem 1 (Existence of solutions)

Let $u_0 \in L^1_{\mathrm{loc}}(M)$. There exists a solution u of problem (1)*, in the sense of Definition 1. In addition $u \in C([0,+\infty); L^1_{\mathrm{loc}}(M))$.*

For nonnegative initial data that also belong to $L^2_{\mathrm{loc}}(M)$, we can establish existence of the minimal solution in the class of nonnegative distributional solutions.

Theorem 2 (Existence of the minimal solution)

Given $u_0 \in L^2_{\mathrm{loc}}(M)$, with $u_0 \geq 0$, there exists a nonnegative distributional solution $\underline{u} \in L^2_{\mathrm{loc}}(M \times [0,+\infty))$ of problem (1)*, in the sense of Definition 1, which is minimal in the class of nonnegative distributional solutions belonging to $L^2_{\mathrm{loc}}(M \times [0,+\infty))$.*

In our first result about uniqueness, a further property of solutions, which amounts to asking that the time derivative is a locally integrable function, has to be required.

In our first result about uniqueness, a further property of solutions, which amounts to asking that the time derivative is a locally integrable function, has to be required.

Definition 2 (Strong solutions)

We say that a function u is a strong solution of problem (1) *if it is a distributional solution in the sense of Definition 1 and in addition*

$$u_t \in L^1_{\mathrm{loc}}(M \times (0, +\infty)) .$$

In our first result about uniqueness, a further property of solutions, which amounts to asking that the time derivative is a locally integrable function, has to be required.

Definition 2 (Strong solutions)

We say that a function u is a strong solution of problem (1) *if it is a distributional solution in the sense of Definition 1 and in addition*

$$u_t \in L^1_{\mathrm{loc}}(M \times (0, +\infty))\,.$$

However, curvature conditions are necessary to prove uniqueness results.

Let $o \in M$ be a fixed reference point, and let $r(x)$ denote the geodesic distance between x and o.

Let $o \in M$ be a fixed reference point, and let $r(x)$ denote the geodesic distance between x and o. Assume for simplicity $\mathrm{cut}(o) = \emptyset$.

Let $o \in M$ be a fixed reference point, and let $r(x)$ denote the geodesic distance between x and o. Assume for simplicity $\mathtt{cut}(o) = \emptyset$.We assume that the *radial* Ricci curvature with respect to o satisfies the following bound from below:

$$\mathrm{Ric}_o(x) \geq -(n-1)\,\frac{\psi''(r(x))}{\psi(r(x))} \qquad \forall x \in M, \tag{6}$$

Let $o \in M$ be a fixed reference point, and let $r(x)$ denote the geodesic distance between x and o. Assume for simplicity $\mathtt{cut}(o) = \emptyset$.We assume that the *radial* Ricci curvature with respect to o satisfies the following bound from below:

$$\mathrm{Ric}_o(x) \geq -(n-1)\, \frac{\psi''(r(x))}{\psi(r(x))} \qquad \forall x \in M, \tag{6}$$

for some sufficiently regular function ψ such that
$\psi' \geq 0,\ \psi(0) = 0,\ \psi'(0) = 1$ and

Let $o \in M$ be a fixed reference point, and let $r(x)$ denote the geodesic distance between x and o. Assume for simplicity $\mathtt{cut}(o) = \emptyset$.We assume that the *radial* Ricci curvature with respect to o satisfies the following bound from below:

$$\mathrm{Ric}_o(x) \geq -(n-1)\,\frac{\psi''(r(x))}{\psi(r(x))} \qquad \forall x \in M, \tag{6}$$

for some sufficiently regular function ψ such that $\psi' \geq 0,\ \psi(0) = 0,\ \psi'(0) = 1$ and

$$\int_0^\infty \frac{\int_0^r \psi(\rho)^{n-1}\,d\rho}{\psi(r)^{n-1}}\,dr = \infty\,. \tag{7}$$

Let $o \in M$ be a fixed reference point, and let $r(x)$ denote the geodesic distance between x and o. Assume for simplicity $\mathrm{cut}(o) = \emptyset$.We assume that the *radial* Ricci curvature with respect to o satisfies the following bound from below:

$$\mathrm{Ric}_o(x) \geq -(n-1)\, \frac{\psi''(r(x))}{\psi(r(x))} \qquad \forall x \in M, \tag{6}$$

for some sufficiently regular function ψ such that $\psi' \geq 0,\ \psi(0) = 0,\ \psi'(0) = 1$ and

$$\int_0^{\infty} \frac{\int_0^r \psi(\rho)^{n-1}\, d\rho}{\psi(r)^{n-1}}\, dr = \infty\,. \tag{7}$$

In particular, we can consider the relevant, borderline case

$$\mathrm{Ric}_o(x) \geq -C\left[1 + r(x)^2\right] \qquad \forall x \in M \setminus (\{o\} \cup \mathrm{cut}(o))\,. \tag{8}$$

If we assume the Ricci curvature complies with conditions (7), we obtain a uniqueness result for *strong solutions*. We stress that, even in the Euclidean setting, uniqueness of possibly *sign-changing* solutions was proved within the class of strong solutions only.

If we assume the Ricci curvature complies with conditions (7), we obtain a uniqueness result for *strong solutions*. We stress that, even in the Euclidean setting, uniqueness of possibly *sign-changing* solutions was proved within the class of strong solutions only.

Theorem 3 (Uniqueness of strong solutions)

Let the curvature conditions (6)–(7) *be satisfied. Let u and v be any two strong solutions of problem* (1)*, in the sense of Definition 2, such that* $|u(\cdot,t) - v(\cdot,t)| \to 0$ *in* $L^1_{\mathrm{loc}}(M)$ *as* $t \to 0^+$*. Then* $u = v$ *almost everywhere in* $M \times (0, +\infty)$*.*

For nonnegative initial data belonging to $L^2_{\mathrm{loc}}(M)$, we do not need solutions to be strong in order to establish uniqueness. Indeed, we can show that any nonnegative distributional solution must coincide with the minimal one, constructed in Theorem 2.

For nonnegative initial data belonging to $L^2_{\mathrm{loc}}(M)$, we do not need solutions to be strong in order to establish uniqueness. Indeed, we can show that any nonnegative distributional solution must coincide with the minimal one, constructed in Theorem 2. To the best of our knowledge this is new also in the case $M \equiv \mathbb{R}^n$, unless one deals with bounded solutions, see Peletier and Zhao, Nonlin. Anal. 1991.

For nonnegative initial data belonging to $L^2_{\mathrm{loc}}(M)$, we do not need solutions to be strong in order to establish uniqueness. Indeed, we can show that any nonnegative distributional solution must coincide with the minimal one, constructed in Theorem 2. To the best of our knowledge this is new also in the case $M \equiv \mathbb{R}^n$, unless one deals with bounded solutions, see Peletier and Zhao, Nonlin. Anal. 1991.

Theorem 4 (Uniqueness of nonnegative distributional solutions)

Let the curvature conditions (6)–(7) *be satisfied and* $u_0 \in L^2_{\mathrm{loc}}(M)$, *with* $u_0 \geq 0$. *Let* $u \in L^2_{\mathrm{loc}}(M \times [0, +\infty))$ *be a nonnegative distributional solution of problem* (1), *in the sense of Definition 1. Then* $u = \underline{u}$ *almost everywhere in* $M \times (0, +\infty)$, *where* $\underline{u}$ *is the minimal solution provided by Theorem 2.*

For nonnegative initial data belonging to $L^2_{\mathrm{loc}}(M)$, we do not need solutions to be strong in order to establish uniqueness. Indeed, we can show that any nonnegative distributional solution must coincide with the minimal one, constructed in Theorem 2. To the best of our knowledge this is new also in the case $M \equiv \mathbb{R}^n$, unless one deals with bounded solutions, see Peletier and Zhao, Nonlin. Anal. 1991.

Theorem 4 (Uniqueness of nonnegative distributional solutions)

Let the curvature conditions (6)–(7) *be satisfied and* $u_0 \in L^2_{\mathrm{loc}}(M)$, *with* $u_0 \geq 0$. *Let* $u \in L^2_{\mathrm{loc}}(M \times [0, +\infty))$ *be a nonnegative distributional solution of problem* (1), *in the sense of Definition 1. Then* $u = \underline{u}$ *almost everywhere in* $M \times (0, +\infty)$, *where* $\underline{u}$ *is the minimal solution provided by Theorem 2.*

Is the L^2 condition really necessary? We don't know, but it is important to stress that this is anyway a local condition on data.

Both our uniqueness results rely on a crucial nonexistence theorem for the nonlinear elliptic equation

$$\Delta W = W\,|W|^{p-1} \qquad \text{in } M\,. \tag{9}$$

in the spirit of Keller and Osserman.

Both our uniqueness results rely on a crucial nonexistence theorem for the nonlinear elliptic equation

$$\Delta W = W\,|W|^{p-1} \qquad \text{in } M\,. \tag{9}$$

in the spirit of Keller and Osserman. Let us emphasize that the latter is of independent interest and the proof we will provide is self-contained and does not exploit methods of parabolic equations.

Both our uniqueness results rely on a crucial nonexistence theorem for the nonlinear elliptic equation

$$\Delta W = W |W|^{p-1} \qquad \text{in } M. \tag{9}$$

in the spirit of Keller and Osserman. Let us emphasize that the latter is of independent interest and the proof we will provide is self-contained and does not exploit methods of parabolic equations.

Theorem 5 (Nonexistence for the elliptic equation)

Let $p > 1$ and the curvature conditions (7) be satisfied. Then:
(i) there exists no nonnegative, nontrivial, distributional subsolution to (9)*;*

Both our uniqueness results rely on a crucial nonexistence theorem for the nonlinear elliptic equation

$$\Delta W = W\,|W|^{p-1} \qquad \text{in } M\,. \tag{9}$$

in the spirit of Keller and Osserman. Let us emphasize that the latter is of independent interest and the proof we will provide is self-contained and does not exploit methods of parabolic equations.

Theorem 5 (Nonexistence for the elliptic equation)

Let $p > 1$ and the curvature conditions (7) *be satisfied. Then:*
(i) there exists no nonnegative, nontrivial, distributional subsolution to (9)*;*
(ii) there exists no nontrivial distributional solution of (9)*.*

Stochastic completeness: an overview

We assume that M is a complete, connected, noncompact Riemannian manifold.

Stochastic completeness: an overview

We assume that M is a complete, connected, noncompact Riemannian manifold. We say that M is stochastically complete if the heat kernel $k(t,x,y)$ of M preserves the mass, namely

$$\int_M k(t,x,y)\, d\mu(y) = 1 \qquad \forall (x,t) \in M \times \mathbb{R}^+ \, . \tag{MC}$$

Stochastic completeness: an overview

We assume that M is a complete, connected, noncompact Riemannian manifold. We say that M is stochastically complete if the heat kernel $k(t, x, y)$ of M preserves the mass, namely

$$\int_M k(t,x,y)\, d\mu(y) = 1 \qquad \forall (x,t) \in M \times \mathbb{R}^+ . \tag{MC}$$

We recall that, for each $x \in M$, by definition $(y,t) \mapsto k(t,x,y)$ is the (minimal) solution to the Cauchy problem

$$\begin{cases} u_t = \Delta u & \text{in } M \times (0,T)\,, \\ u(\cdot,0) = u_0 & \text{in } M\,, \end{cases} \tag{HE}$$

with $u_0 = \delta_x$ and $T = +\infty$.

Stochastic completeness: an overview

We assume that M is a complete, connected, noncompact Riemannian manifold. We say that M is stochastically complete if the heat kernel $k(t,x,y)$ of M preserves the mass, namely

$$\int_M k(t,x,y)\,d\mu(y) = 1 \qquad \forall (x,t) \in M \times \mathbb{R}^+ . \tag{MC}$$

We recall that, for each $x \in M$, by definition $(y,t) \mapsto k(t,x,y)$ is the (minimal) solution to the Cauchy problem

$$\begin{cases} u_t = \Delta u & \text{in } M \times (0,T)\,, \\ u(\cdot,0) = u_0 & \text{in } M\,, \end{cases} \tag{HE}$$

with $u_0 = \delta_x$ and $T = +\infty$. Notice that (MC) holds $\iff u \equiv 1$ is the unique solution to (HE) with $u_0 \equiv 1$.

How is such a property related to stochastic processes? M is stochastically complete if and only if $\forall x \in M$ the lifetime of the Brownian Motion (BM) $\{X_t^x\}$ that starts from x at time $t = 0$ is a.s. $+\infty$.

How is such a property related to stochastic processes? M is stochastically complete if and only if $\forall x \in M$ the lifetime of the Brownian Motion (BM) $\{X_t^x\}$ that starts from x at time $t = 0$ is a.s. $+\infty$.

In other words, almost surely the trajectories of BM don't go to infinity in finite time, so that BM is "completely" confined in the manifold.

How is such a property related to stochastic processes? M is stochastically complete if and only if $\forall x \in M$ the lifetime of the Brownian Motion (BM) $\{X_t^x\}$ that starts from x at time $t = 0$ is a.s. $+\infty$.

In other words, almost surely the trajectories of BM don't go to infinity in finite time, so that BM is "completely" confined in the manifold.

More rigorously, $k(t, x, y)$ is the transition probability density of BM: hence the failure of (MC) means that

$$A \mapsto \int_A k(t, x, y)\, d\mu(y)$$

is actually not a probability measure for all $(x, t) \in M \times \mathbb{R}^+$. Indeed,

$$1 - \int_M k(t, x, y)\, d\mu(y) > 0$$

is the probability that $\{X_t^x\}$ has already escaped to infinity at time t.

There is a further analytic characterization of stochastic completeness, which is the one we shall work with.

There is a further analytic characterization of stochastic completeness, which is the one we shall work with.

Theorem (See e.g. Grigor'yan, '09)

Let $\alpha > 0$, $T \in (0, +\infty]$ and $u_0 \in L^\infty(M)$, with $u_0 \geq 0$. Then the following assertions are equivalent:

There is a further analytic characterization of stochastic completeness, which is the one we shall work with.

Theorem (See e.g. Grigor'yan, '09)

Let $\alpha > 0$, $T \in (0, +\infty]$ and $u_0 \in L^\infty(M)$, with $u_0 \geq 0$. Then the following assertions are equivalent:

- *M is stochastically complete.*

There is a further analytic characterization of stochastic completeness, which is the one we shall work with.

Theorem (See e.g. Grigor'yan, '09)

Let $\alpha > 0$, $T \in (0, +\infty]$ and $u_0 \in L^\infty(M)$, with $u_0 \geq 0$. Then the following assertions are equivalent:

- *M is stochastically complete.*
- *The Cauchy problem* (HE) *has a unique nonnegative solution in $L^\infty(M \times (0, T))$.*

There is a further analytic characterization of stochastic completeness, which is the one we shall work with.

Theorem (See e.g. Grigor'yan, '09)

Let $\alpha > 0$, $T \in (0, +\infty]$ and $u_0 \in L^\infty(M)$, with $u_0 \geq 0$. Then the following assertions are equivalent:

- *M is stochastically complete.*
- *The Cauchy problem* (HE) *has a unique nonnegative solution in $L^\infty(M \times (0, T))$.*
- *The elliptic equation*

$$\Delta V = \alpha V \qquad \text{in } M$$

  does not admit any nonnegative, nontrivial, bounded solution.

There is a further analytic characterization of stochastic completeness, which is the one we shall work with.

Theorem (See e.g. Grigor'yan, '09)

Let $\alpha > 0$, $T \in (0, +\infty]$ and $u_0 \in L^\infty(M)$, with $u_0 \geq 0$. Then the following assertions are equivalent:

- *M is stochastically complete.*
- *The Cauchy problem* (HE) *has a unique nonnegative solution in $L^\infty(M \times (0, T))$.*
- *The elliptic equation*

$$\Delta V = \alpha V \qquad \text{in } M$$

  does not admit any nonnegative, nontrivial, bounded solution.

Note: here positivity is not essential, but it will be from now on.

A class of nonlinear diffusion (and elliptic) equations

We denote by $\mathfrak{C}$ the set of all functions $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ s.t.

$$\phi \text{ is concave, strictly increasing with } \phi(0) = 0\,,$$

$$\phi \in C(\mathbb{R}^+) \cap C^1(\mathbb{R}^+ \setminus \{0\})\,.$$

A class of nonlinear diffusion (and elliptic) equations

We denote by $\mathfrak{C}$ the set of all functions $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ s.t.

$$\phi \text{ is concave, strictly increasing with } \phi(0) = 0\,,$$
$$\phi \in C(\mathbb{R}^+) \cap C^1(\mathbb{R}^+ \setminus \{0\})\,.$$

For any $\phi \in \mathfrak{C}$, any $T \in (0, +\infty]$ and any nonnegative $u_0 \in L^\infty(M)$, we consider the nonlinear parabolic problem

$$\begin{cases} u_t = \Delta\phi(u) & \text{in } M \times (0, T)\,, \\ u(\cdot, 0) = u_0 & \text{in } M\,, \end{cases} \qquad \text{(NPP)}$$

which always has an $L^\infty(M \times (0, T))$ nonnegative (minimal) solution.

A class of nonlinear diffusion (and elliptic) equations

We denote by $\mathfrak{C}$ the set of all functions $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ s.t.

$$\phi \text{ is concave, strictly increasing with } \phi(0) = 0\,,$$
$$\phi \in C(\mathbb{R}^+) \cap C^1(\mathbb{R}^+ \setminus \{0\})\,.$$

For any $\phi \in \mathfrak{C}$, any $T \in (0, +\infty]$ and any nonnegative $u_0 \in L^\infty(M)$, we consider the nonlinear parabolic problem

$$\begin{cases} u_t = \Delta\phi(u) & \text{in } M \times (0, T)\,, \\ u(\cdot, 0) = u_0 & \text{in } M\,, \end{cases} \tag{NPP}$$

which always has an $L^\infty(M \times (0, T))$ nonnegative (minimal) solution. We also consider nonnegative, nontrivial, bounded solutions to the semilinear elliptic equation

$$\Delta W = \phi^{-1}(W) \qquad \text{in } M\,. \tag{E}$$

A class of nonlinear diffusion (and elliptic) equations

We denote by $\mathfrak{C}$ the set of all functions $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ s.t.

$$\phi \text{ is concave, strictly increasing with } \phi(0) = 0\,,$$
$$\phi \in C(\mathbb{R}^+) \cap C^1(\mathbb{R}^+ \setminus \{0\})\,.$$

For any $\phi \in \mathfrak{C}$, any $T \in (0, +\infty]$ and any nonnegative $u_0 \in L^\infty(M)$, we consider the nonlinear parabolic problem

$$\begin{cases} u_t = \Delta\phi(u) & \text{in } M \times (0, T)\,, \\ u(\cdot, 0) = u_0 & \text{in } M\,, \end{cases} \tag{NPP}$$

which always has an $L^\infty(M \times (0, T))$ nonnegative (minimal) solution. We also consider nonnegative, nontrivial, bounded solutions to the semilinear elliptic equation

$$\Delta W = \phi^{-1}(W) \qquad \text{in } M\,. \tag{E}$$

Note: here “solutions” are always meant in the distributional sense.

Non-uniqueness [G., Ishige, Muratori, JMPA 2020]

Theorem 6 (Stochastic incompleteness)

Let $\phi \in \mathfrak{C}$, $T \in (0, +\infty]$ and $u_0 \in L^\infty(M)$, with $u_0 \geq 0$. Then the following assertions are equivalent:

Non-uniqueness [G., Ishige, Muratori, JMPA 2020]

Theorem 6 (Stochastic incompleteness)

Let $\phi \in \mathfrak{C}$, $T \in (0, +\infty]$ and $u_0 \in L^\infty(M)$, with $u_0 \geq 0$. Then the following assertions are equivalent:

(a) *M is stochastically incomplete.*

Non-uniqueness [G., Ishige, Muratori, JMPA 2020]

Theorem 6 (Stochastic incompleteness)

Let $\phi \in \mathfrak{C}$, $T \in (0, +\infty]$ *and* $u_0 \in L^\infty(M)$, *with* $u_0 \geq 0$. *Then the following assertions are equivalent:*

(a) *M is stochastically incomplete.*

(b) *The Cauchy problem* (NPP) *admits at least two nonnegative solutions in* $L^\infty(M \times (0, T))$.

Non-uniqueness [G., Ishige, Muratori, JMPA 2020]

Theorem 6 (Stochastic incompleteness)

Let $\phi \in \mathfrak{C}$, $T \in (0, +\infty]$ and $u_0 \in L^\infty(M)$, with $u_0 \geq 0$. Then the following assertions are equivalent:

(a) *M is stochastically incomplete.*

(b) *The Cauchy problem* (NPP) *admits at least two nonnegative solutions in $L^\infty(M \times (0, T))$.*

(c) *The semilinear elliptic equation* (E) *admits a nonnegative, nontrivial, bounded solution.*

Non-uniqueness [G., Ishige, Muratori, JMPA 2020]

Theorem 6 (Stochastic incompleteness)

Let $\phi \in \mathfrak{C}$, $T \in (0, +\infty]$ *and* $u_0 \in L^\infty(M)$, *with* $u_0 \geq 0$. *Then the following assertions are equivalent:*

(a) *M is stochastically incomplete.*

(b) *The Cauchy problem* (NPP) *admits at least two nonnegative solutions in* $L^\infty(M \times (0, T))$.

(c) *The semilinear elliptic equation* (E) *admits a nonnegative, nontrivial, bounded solution.*

Note 1: in the significant case of the FDE, it means in particular that there are manifolds where uniqueness fails even for bounded solutions!

Non-uniqueness [G., Ishige, Muratori, JMPA 2020]

Theorem 6 (Stochastic incompleteness)

Let $\phi \in \mathfrak{C}$, $T \in (0, +\infty]$ and $u_0 \in L^\infty(M)$, with $u_0 \geq 0$. Then the following assertions are equivalent:

(a) *M is stochastically incomplete.*

(b) *The Cauchy problem* (NPP) *admits at least two nonnegative solutions in $L^\infty(M \times (0, T))$.*

(c) *The semilinear elliptic equation* (E) *admits a nonnegative, nontrivial, bounded solution.*

Note 1: in the significant case of the FDE, it means in particular that there are manifolds where uniqueness fails even for bounded solutions!

Note 1: The curvature condition (7) is sharp for stochastic completeness in the significant class of spherically symmetric manifolds.

Theorem 6 can be stated equivalently in the following way.

Theorem 6 can be stated equivalently in the following way.

Theorem 7 (Stochastic completeness)

Let $\phi \in \mathfrak{C}$, $T \in (0, +\infty]$ and $u_0 \in L^\infty(M)$, with $u_0 \geq 0$. Then the following assertions are equivalent:

Theorem 6 can be stated equivalently in the following way.

Theorem 7 (Stochastic completeness)

Let $\phi \in \mathfrak{C}$, $T \in (0, +\infty]$ *and* $u_0 \in L^\infty(M)$, *with* $u_0 \geq 0$. *Then the following assertions are equivalent:*

(a') *M is stochastically complete.*

Theorem 6 can be stated equivalently in the following way.

Theorem 7 (Stochastic completeness)

Let $\phi \in \mathfrak{C}$, $T \in (0, +\infty]$ and $u_0 \in L^\infty(M)$, with $u_0 \geq 0$. Then the following assertions are equivalent:

(a') *M is stochastically complete.*

(b') *The Cauchy problem* (NPP) *has a unique nonnegative solution in $L^\infty(M \times (0, T))$.*

Theorem 6 can be stated equivalently in the following way.

Theorem 7 (Stochastic completeness)

Let $\phi \in \mathfrak{C}$, $T \in (0, +\infty]$ *and* $u_0 \in L^\infty(M)$, *with* $u_0 \geq 0$. *Then the following assertions are equivalent:*

(a') *M is stochastically complete.*

(b') *The Cauchy problem* (NPP) *has a unique nonnegative solution in* $L^\infty(M \times (0, T))$.

(c') *The semilinear elliptic equation* (E) *does not admit any nonnegative, nontrivial, bounded solution.*

Theorem 6 can be stated equivalently in the following way.

Theorem 7 (Stochastic completeness)

Let $\phi \in \mathfrak{C}$, $T \in (0, +\infty]$ *and* $u_0 \in L^\infty(M)$, *with* $u_0 \geq 0$. *Then the following assertions are equivalent:*

(a') *M is stochastically complete.*

(b') *The Cauchy problem* (NPP) *has a unique nonnegative solution in* $L^\infty(M \times (0, T))$.

(c') *The semilinear elliptic equation* (E) *does not admit any nonnegative, nontrivial, bounded solution.*

We can finally summarize the results entailed by Theorems 6 and 7.

Theorem 8 (Summary and main consequences)

The following statements are equivalent, and each of them is equivalent to stochastic incompleteness (resp. completeness).

Theorem 8 (Summary and main consequences)

The following statements are equivalent, and each of them is equivalent to stochastic incompleteness (resp. completeness).

- *For some function $\phi \in \mathfrak{C}$, some $T \in (0, +\infty]$ and some bdd initial datum $u_0 \geq 0$ the Cauchy problem* (NPP) *admits at least two nonnegative bdd solutions (resp. a unique nonnegative solution).*

Theorem 8 (Summary and main consequences)

The following statements are equivalent, and each of them is equivalent to stochastic incompleteness (resp. completeness).

- *For some function $\phi \in \mathfrak{C}$, some $T \in (0, +\infty]$ and some bdd initial datum $u_0 \geq 0$ the Cauchy problem* (NPP) *admits at least two nonnegative bdd solutions (resp. a unique nonnegative solution).*
- *For all functions $\phi \in \mathfrak{C}$, all $T \in (0, +\infty]$ and all bdd initial data $u_0 \geq 0$ the Cauchy problem* (NPP) *admits at least two nonnegative bdd solutions (resp. a unique nonnegative solution).*

Theorem 8 (Summary and main consequences)

The following statements are equivalent, and each of them is equivalent to stochastic incompleteness (resp. completeness).

- *For some function $\phi \in \mathfrak{C}$, some $T \in (0, +\infty]$ and some bdd initial datum $u_0 \geq 0$ the Cauchy problem* (NPP) *admits at least two nonnegative bdd solutions (resp. a unique nonnegative solution).*
- *For all functions $\phi \in \mathfrak{C}$, all $T \in (0, +\infty]$ and all bdd initial data $u_0 \geq 0$ the Cauchy problem* (NPP) *admits at least two nonnegative bdd solutions (resp. a unique nonnegative solution).*
- *For some function $\phi \in \mathfrak{C}$ the semilinear elliptic equation* (E) *admits (resp. does not admit) a nonnegative, nontrivial, bounded solution.*

Theorem 8 (Summary and main consequences)

The following statements are equivalent, and each of them is equivalent to stochastic incompleteness (resp. completeness).

- *For some function $\phi \in \mathfrak{C}$, some $T \in (0, +\infty]$ and some bdd initial datum $u_0 \geq 0$ the Cauchy problem* (NPP) *admits at least two nonnegative bdd solutions (resp. a unique nonnegative solution).*
- *For all functions $\phi \in \mathfrak{C}$, all $T \in (0, +\infty]$ and all bdd initial data $u_0 \geq 0$ the Cauchy problem* (NPP) *admits at least two nonnegative bdd solutions (resp. a unique nonnegative solution).*
- *For some function $\phi \in \mathfrak{C}$ the semilinear elliptic equation* (E) *admits (resp. does not admit) a nonnegative, nontrivial, bounded solution.*
- *For all functions $\phi \in \mathfrak{C}$ the semilinear elliptic equation* (E) *admits (resp. does not admit) a nonnegative, nontrivial, bounded solution.*

Sketch of proof: $(a) \Rightarrow (b)$

Strategy of the proof: we rewrite the nonlinear parabolic problem (NPP) as

$$\psi'(w)\, w_t = \Delta w\,, \qquad \text{where} \quad \psi(w) := \phi^{-1}(w)\,, \tag{10}$$

Stochastic incompleteness ensures the existence of a nonnegative, nontrivial solution V to

$$\Delta V = V \quad \text{in } M\,, \qquad \|V\|_\infty = 1\,.$$

Sketch of proof: $(a) \Rightarrow (b)$

Strategy of the proof: we rewrite the nonlinear parabolic problem (NPP) as

$$\psi'(w)\, w_t = \Delta w\,, \qquad \text{where} \quad \psi(w) := \phi^{-1}(w)\,, \tag{10}$$

Stochastic incompleteness ensures the existence of a nonnegative, nontrivial solution V to

$$\Delta V = V \quad \text{in } M\,, \qquad \|V\|_\infty = 1\,.$$

We shall use the solution to $\Delta V = V$ to construct a nontrivial bdd subsolution to (10) that starts from $u_0 \equiv 0$

Sketch of proof: $(a) \Rightarrow (b)$

Strategy of the proof: we rewrite the nonlinear parabolic problem (NPP) as

$$\psi'(w)\, w_t = \Delta w\,, \qquad \text{where} \quad \psi(w) := \phi^{-1}(w)\,, \tag{10}$$

Stochastic incompleteness ensures the existence of a nonnegative, nontrivial solution V to

$$\Delta V = V \quad \text{in } M\,, \qquad \|V\|_\infty = 1\,.$$

We shall use the solution to $\Delta V = V$ to construct a nontrivial bdd subsolution to (10) that starts from $u_0 \equiv 0$

Even if it is not necessary, let us assume for simplicity $\phi(+\infty) = +\infty$, so that ψ is defined on the whole $\mathbb{R}^+$, and $T = +\infty$. The other cases can be treated similarly up to small modifications.

Even if it is not necessary, let us assume for simplicity $\phi(+\infty) = +\infty$, so that ψ is defined on the whole $\mathbb{R}^+$, and $T = +\infty$. The other cases can be treated similarly up to small modifications.

First of all, we observe that

$$w(x,t) = g(t)V(x)\,, \qquad \text{where} \quad \begin{cases} g_t = \frac{g}{\psi'(g)}\,, \\ g(0) = 1\,, \end{cases}$$

is a subsolution to (10).

Even if it is not necessary, let us assume for simplicity $\phi(+\infty) = +\infty$, so that ψ is defined on the whole $\mathbb{R}^+$, and $T = +\infty$. The other cases can be treated similarly up to small modifications.

First of all, we observe that

$$w(x,t) = g(t)V(x)\,, \qquad \text{where} \quad \begin{cases} g_t = \frac{g}{\psi'(g)}\,, \\ g(0) = 1\,, \end{cases}$$

is a subsolution to (10).
Indeed, thanks to convexity of ψ, $0 \le V \le 1$ and $g_t \ge 0$, there holds

$$\psi'(w)\,w_t = \psi'(gV)\,g_t V \le \psi'(g)\,g_t V = gV = \Delta(gV) = \Delta w \quad \text{in } M \times \mathbb{R}^+\,.$$

We would like to modify w so that it is identically zero at $t = 0$.

We would like to modify w so that it is identically zero at $t = 0$.

To this end, it is enough to note that

$$\underline{w} := (w - 1) \vee 0$$

is also a nonnegative subsolution to (10), since

$$\psi'(w-1)\,\partial_t(w-1) \leq \psi'(w)\,w_t \qquad \text{and} \qquad \Delta w = \Delta(w-1)\,.$$

We would like to modify w so that it is identically zero at $t = 0$.

To this end, it is enough to note that

$$\underline{w} := (w - 1) \vee 0$$

is also a nonnegative subsolution to (10), since

$$\psi'(w-1)\,\partial_t(w-1) \le \psi'(w)\,w_t \qquad \text{and} \qquad \Delta w = \Delta(w-1)\,.$$

Moreover, it has the remarkable properties

$$\underline{w}(\cdot,0) \equiv 0\,, \qquad \lim_{t\to+\infty} \underline{w}(x,t) = +\infty \quad \forall x:\ V(x) > 0\,. \tag{11}$$

Equivalently, we have shown that the function $(x, t) \mapsto \psi(\underline{w}(x, t))$ is a subsolution to the original parabolic equation with $u_0 \equiv 0$.

Equivalently, we have shown that the function $(x,t) \mapsto \psi(\underline{w}(x,t))$ is a subsolution to the original parabolic equation with $u_0 \equiv 0$.

For each $R > 0$, let us solve the approximate Dirichlet problems

$$\begin{cases} (u_R)_t = \Delta\phi(u_R) & \text{in } B_R \times \mathbb{R}^+ \,, \\ u_R(\cdot,t) = \psi(g(t) \vee \phi(\|u_0\|_\infty)) & \text{on } \partial B_R \times \mathbb{R}^+ \,, \\ u_R(\cdot,0) = u_0 & \text{in } B_R \,. \end{cases}$$

Equivalently, we have shown that the function $(x,t) \mapsto \psi(\underline{w}(x,t))$ is a subsolution to the original parabolic equation with $u_0 \equiv 0$.

For each $R > 0$, let us solve the approximate Dirichlet problems

$$\begin{cases} (u_R)_t = \Delta\phi(u_R) & \text{in } B_R \times \mathbb{R}^+\,, \\ u_R(\cdot,t) = \psi(g(t) \vee \phi(\|u_0\|_\infty)) & \text{on } \partial B_R \times \mathbb{R}^+\,, \\ u_R(\cdot,0) = u_0 & \text{in } B_R\,. \end{cases}$$

Since $\psi(\underline{w}(x,t)) \le \psi(g(t))$, we infer that

$$u_R(x,t) \ge \psi(\underline{w}(x,t)) \qquad \text{in } B_R \times \mathbb{R}^+\,.$$

Equivalently, we have shown that the function $(x,t) \mapsto \psi(\underline{w}(x,t))$ is a subsolution to the original parabolic equation with $u_0 \equiv 0$.

For each $R > 0$, let us solve the approximate Dirichlet problems

$$\begin{cases} (u_R)_t = \Delta\phi(u_R) & \text{in } B_R \times \mathbb{R}^+ \,, \\ u_R(\cdot,t) = \psi(g(t) \vee \phi(\|u_0\|_\infty)) & \text{on } \partial B_R \times \mathbb{R}^+ \,, \\ u_R(\cdot,0) = u_0 & \text{in } B_R \,. \end{cases}$$

Since $\psi(\underline{w}(x,t)) \le \psi(g(t))$, we infer that

$$u_R(x,t) \ge \psi(\underline{w}(x,t)) \qquad \text{in } B_R \times \mathbb{R}^+ \,.$$

Moreover, the family $\{u_R\}_{R>0}$ is nonincreasing w.r.t. R, so that by letting $R \uparrow +\infty$ we obtain a solution u to (NPP) s.t.

$$u(x,t) \ge \psi(\underline{w}(x,t)) \qquad \text{in } M \times \mathbb{R}^+ \,. \tag{12}$$

Equivalently, we have shown that the function $(x, t) \mapsto \psi(\underline{w}(x, t))$ is a subsolution to the original parabolic equation with $u_0 \equiv 0$.

For each $R > 0$, let us solve the approximate Dirichlet problems

$$\begin{cases} (u_R)_t = \Delta\phi(u_R) & \text{in } B_R \times \mathbb{R}^+ \,, \\ u_R(\cdot, t) = \psi(g(t) \vee \phi(\|u_0\|_\infty)) & \text{on } \partial B_R \times \mathbb{R}^+ \,, \\ u_R(\cdot, 0) = u_0 & \text{in } B_R \,. \end{cases}$$

Since $\psi(\underline{w}(x, t)) \leq \psi(g(t))$, we infer that

$$u_R(x, t) \geq \psi(\underline{w}(x, t)) \qquad \text{in } B_R \times \mathbb{R}^+ \,.$$

Moreover, the family $\{u_R\}_{R>0}$ is nonincreasing w.r.t. R, so that by letting $R \uparrow +\infty$ we obtain a solution u to (NPP) s.t.

$$u(x, t) \geq \psi(\underline{w}(x, t)) \qquad \text{in } M \times \mathbb{R}^+ \,. \tag{12}$$

Uniqueness fails because (16) and (12) imply that $\|u(t)\|_\infty$ at some time exceeds $\|u_0\|_\infty$, so u cannot coincide with the minimal solution.

Sketch of proof: $(b) \Rightarrow (c)$

Strategy of the proof: *we multiply by* e^{-t} *the equation solved by* $u_* - u$*, where* u_* *is any solution and* u *is the minimal one, then we integrate in time to obtain a nontrivial bdd subsolution to*

$$\Delta W \geq \psi(W) \qquad \text{in } M \, .$$

Sketch of proof: $(b) \Rightarrow (c)$

Strategy of the proof: *we multiply by e^{-t} the equation solved by $u_* - u$, where u_* is any solution and u is the minimal one, then we integrate in time to obtain a nontrivial bdd subsolution to*

$$\Delta W \geq \psi(W) \qquad \text{in } M\,.$$

Let us denote by u the minimal solution, which always exists, and u_* any other solution: by assumption we know that $u_* \not\equiv u$ and $u_* \geq u$.

Sketch of proof: $(b) \Rightarrow (c)$

Strategy of the proof: *we multiply by* e^{-t} *the equation solved by* $u_* - u$*, where* u_* *is any solution and* u *is the minimal one, then we integrate in time to obtain a nontrivial bdd subsolution to*

$$\Delta W \geq \psi(W) \qquad \textit{in } M\,.$$

Let us denote by u the minimal solution, which always exists, and u_* any other solution: by assumption we know that $u_* \not\equiv u$ and $u_* \geq u$.

For simplicity we suppose again that $T = +\infty$.

Sketch of proof: $(b) \Rightarrow (c)$

Strategy of the proof: *we multiply by e^{-t} the equation solved by $u_* - u$, where u_* is any solution and u is the minimal one, then we integrate in time to obtain a nontrivial bdd subsolution to*

$$\Delta W \geq \psi(W) \qquad \text{in } M\,.$$

Let us denote by u the minimal solution, which always exists, and u_* any other solution: by assumption we know that $u_* \not\equiv u$ and $u_* \geq u$.

For simplicity we suppose again that $T = +\infty$.

Having in mind a smart trick due to [Murata, JFA '96], mostly used in the linear case, we multiply the two differential equations in (NPP) by e^{-t}, subtract and and integrate in time, so as to obtain

$$\Delta\left(\int_0^\infty \phi(u_*)\, e^{-s} ds - \int_0^\infty \phi(u)\, e^{-s} ds\right) = \int_0^\infty (u_* - u)\, e^{-s} ds\,.$$

Sketch of proof: $(b) \Rightarrow (c)$

Strategy of the proof: *we multiply by e^{-t} the equation solved by $u_* - u$, where u_* is any solution and u is the minimal one, then we integrate in time to obtain a nontrivial bdd subsolution to*

$$\Delta W \geq \psi(W) \qquad in\ M\,.$$

Let us denote by u the minimal solution, which always exists, and u_* any other solution: by assumption we know that $u_* \not\equiv u$ and $u_* \geq u$.

For simplicity we suppose again that $T = +\infty$.

Having in mind a smart trick due to [Murata, JFA '96], mostly used in the linear case, we multiply the two differential equations in (NPP) by e^{-t}, subtract and and integrate in time, so as to obtain

$$\Delta\left(\int_0^\infty \phi(u_*)\, e^{-s} ds - \int_0^\infty \phi(u)\, e^{-s} ds\right) = \int_0^\infty (u_* - u)\, e^{-s} ds\,.$$

Now we take advantage of concavity.

Indeed, since ϕ is concave and monotone increasing, thanks to Jensen's inequality we deduce that

$$\begin{aligned}\Delta\left(\int_0^\infty \phi(u_*)\, e^{-s} ds - \int_0^\infty \phi(u)\, e^{-s} ds\right) &= \int_0^\infty \phi^{-1}[\phi(u_* - u)]\, e^{-s} ds \\ &\geq \phi^{-1}\left[\int_0^\infty \phi(u_* - u)\, e^{-s} ds\right].\end{aligned}$$

Indeed, since ϕ is concave and monotone increasing, thanks to Jensen's inequality we deduce that

$$\Delta \left(\int_0^\infty \phi(u_*)\, e^{-s} ds - \int_0^\infty \phi(u)\, e^{-s} ds \right) = \int_0^\infty \phi^{-1}[\phi(u_* - u)]\, e^{-s} ds$$
$$\geq \phi^{-1} \left[\int_0^\infty \phi(u_* - u)\, e^{-s} ds \right].$$

On the other hand, still concavity yields

$$\phi(u_* - u) \geq \phi(u_*) - \phi(u)\,;$$

Indeed, since ϕ is concave and monotone increasing, thanks to Jensen's inequality we deduce that

$$\Delta\left(\int_0^\infty \phi(u_*)\, e^{-s} ds - \int_0^\infty \phi(u)\, e^{-s} ds\right) = \int_0^\infty \phi^{-1}[\phi(u_* - u)]\, e^{-s} ds$$
$$\geq \phi^{-1}\left[\int_0^\infty \phi(u_* - u)\, e^{-s} ds\right].$$

On the other hand, still concavity yields

$$\phi(u_* - u) \geq \phi(u_*) - \phi(u)\,;$$

we can thus assert that the (nonnegative, nontrivial, bounded) function

$$\underline{W} := \int_0^\infty [\phi(u_*) - \phi(u)]\, e^{-s} ds$$

is a subsolution to (E). By means of an approximation procedure on balls as above, existence of a solution with the same properties follows.

Sketch of proof: $(c) \Rightarrow (a)$

Strategy of the proof: *we combine ideas from the two previous implications to construct a non-constant solution starting from a constant $c > 0$, then we integrate in time to obtain a nontrivial bdd subsolution to*

$$\Delta V \geq \frac{1}{\phi'(c)} V \qquad \text{in } M\,.$$

Sketch of proof: $(c) \Rightarrow (a)$

Strategy of the proof: *we combine ideas from the two previous implications to construct a non-constant solution starting from a constant* $c > 0$*, then we integrate in time to obtain a nontrivial bdd subsolution to*

$$\Delta V \geq \frac{1}{\phi'(c)} V \qquad \text{in } M\,.$$

First of all we observe that convexity of $\psi = \phi^{-1}$ yields

$$\Delta W = \psi(W) \geq \frac{W}{2}\, \psi'\left(\frac{W}{2}\right) \qquad \text{in } M\,.$$

Sketch of proof: $(c) \Rightarrow (a)$

Strategy of the proof: *we combine ideas from the two previous implications to construct a non-constant solution starting from a constant $c > 0$, then we integrate in time to obtain a nontrivial bdd subsolution to*

$$\Delta V \geq \frac{1}{\phi'(c)} V \qquad \text{in } M\,.$$

First of all we observe that convexity of $\psi = \phi^{-1}$ yields

$$\Delta W = \psi(W) \geq \frac{W}{2}\, \psi'\left(\frac{W}{2}\right) \qquad \text{in } M\,.$$

Hence computations similar to those carried out in $(a) \Rightarrow (b)$ show that

$$\underline{w}(x,t) := \frac{e^{\frac{t}{2}}}{4}\, W(x)$$

is a subsolution to

$$\psi'(\underline{w})\, \underline{w}_t \leq \Delta \underline{w} \qquad \text{in } M \times [0, \log 4]\,.$$

So if we solve the approximate Dirichlet problems (let $R > 0$)

$$\begin{cases} (u_R)_t = \Delta\phi(u_R) & \text{in } B_R \times \mathbb{R}^+ \,, \\ u_R(\cdot,t) = \psi\left(\frac{e^{\frac{t}{2}}}{4}\,\|W\|_\infty\right) \wedge \psi(\|W\|_\infty) & \text{on } \partial B_R \times \mathbb{R}^+ \,, \\ u_R(\cdot,0) = c := \psi\left(\frac{\|W\|_\infty}{4}\right) > 0 & \text{in } B_R \,, \end{cases}$$

So if we solve the approximate Dirichlet problems (let $R > 0$)

$$\begin{cases} (u_R)_t = \Delta\phi(u_R) & \text{in } B_R \times \mathbb{R}^+ \,, \\ u_R(\cdot, t) = \psi\left(\frac{e^{\frac{t}{2}}}{4} \, \|W\|_\infty\right) \wedge \psi(\|W\|_\infty) & \text{on } \partial B_R \times \mathbb{R}^+ \,, \\ u_R(\cdot, 0) = c := \psi\left(\frac{\|W\|_\infty}{4}\right) > 0 & \text{in } B_R \,, \end{cases}$$

and let $R \uparrow +\infty$, we obtain a global solution u_* to (NPP) with $u_0 \equiv c$ s.t.

$$c \le u_* \le \psi(\|W\|_\infty) \quad \text{in } M \times \mathbb{R}^+ \quad \text{and} \quad u_* \ge \psi(\underline{w}) \quad \text{in } M \times [0, \log 4] \,.$$

So if we solve the approximate Dirichlet problems (let $R > 0$)

$$\begin{cases} (u_R)_t = \Delta\phi(u_R) & \text{in } B_R \times \mathbb{R}^+ \,, \\ u_R(\cdot, t) = \psi\left(\frac{e^{\frac{t}{2}}}{4} \|W\|_\infty\right) \wedge \psi(\|W\|_\infty) & \text{on } \partial B_R \times \mathbb{R}^+ \,, \\ u_R(\cdot, 0) = c := \psi\left(\frac{\|W\|_\infty}{4}\right) > 0 & \text{in } B_R \,, \end{cases}$$

and let $R \uparrow +\infty$, we obtain a global solution u_* to (NPP) with $u_0 \equiv c$ s.t.

$$c \le u_* \le \psi(\|W\|_\infty) \quad \text{in } M \times \mathbb{R}^+ \quad \text{and} \quad u_* \ge \psi(\underline{w}) \quad \text{in } M \times [0, \log 4] \,.$$

As a consequence,

$$\|u_*(t)\|_\infty \ge \psi\left(\frac{e^{\frac{t}{2}}}{4} \|W\|_\infty\right) \qquad \forall t \in [0, \log 4] \,,$$

so that u_* is larger than c, the latter being a constant solution.

So if we solve the approximate Dirichlet problems (let $R > 0$)

$$\begin{cases} (u_R)_t = \Delta\phi(u_R) & \text{in } B_R \times \mathbb{R}^+ \,, \\ u_R(\cdot, t) = \psi\left(\frac{e^{\frac{t}{2}}}{4} \|W\|_\infty\right) \wedge \psi(\|W\|_\infty) & \text{on } \partial B_R \times \mathbb{R}^+ \,, \\ u_R(\cdot, 0) = c := \psi\left(\frac{\|W\|_\infty}{4}\right) > 0 & \text{in } B_R \,, \end{cases}$$

and let $R \uparrow +\infty$, we obtain a global solution u_* to (NPP) with $u_0 \equiv c$ s.t.

$$c \le u_* \le \psi(\|W\|_\infty) \quad \text{in } M \times \mathbb{R}^+ \quad \text{and} \quad u_* \ge \psi(\underline{w}) \quad \text{in } M \times [0, \log 4] \,.$$

As a consequence,

$$\|u_*(t)\|_\infty \ge \psi\left(\frac{e^{\frac{t}{2}}}{4} \|W\|_\infty\right) \qquad \forall t \in [0, \log 4] \,,$$

so that u_* is larger than c, the latter being a constant solution.

Now note that the following trivial inequality holds:

$$u_* - c \geq \frac{\phi(u_*) - \phi(c)}{\phi'(c)}.$$

Now note that the following trivial inequality holds:

$$u_* - c \geq \frac{\phi(u_*) - \phi(c)}{\phi'(c)}.$$

Hence, by means of computations similar to those carried out in $(b) \Rightarrow (c)$, we end up with the different inequality

$$\Delta \left[\int_0^\infty \phi(u_*)\, e^{-s} ds \right] \geq \frac{1}{\phi'(c)} \int_0^\infty [\phi(u_*) - \phi(c)]\, e^{-s} ds\,.$$

Now note that the following trivial inequality holds:

$$u_* - c \geq \frac{\phi(u_*) - \phi(c)}{\phi'(c)}.$$

Hence, by means of computations similar to those carried out in $(b) \Rightarrow (c)$, we end up with the different inequality

$$\Delta \left[\int_0^\infty \phi(u_*)\, e^{-s} ds \right] \geq \frac{1}{\phi'(c)} \int_0^\infty [\phi(u_*) - \phi(c)]\, e^{-s} ds \, .$$

We have shown that the (nonnegative, nontrivial, bounded) function

$$\underline{V} = \int_0^\infty [\phi(u_*) - \phi(c)]\, e^{-s} ds$$

is in fact a subsolution to

$$\Delta V = \alpha V \qquad \text{in } M \, , \qquad \text{where } \alpha = \frac{1}{\phi'(c)} \, .$$

Now note that the following trivial inequality holds:

$$u_* - c \geq \frac{\phi(u_*) - \phi(c)}{\phi'(c)}.$$

Hence, by means of computations similar to those carried out in $(b) \Rightarrow (c)$, we end up with the different inequality

$$\Delta \left[\int_0^\infty \phi(u_*)\, e^{-s} ds\right] \geq \frac{1}{\phi'(c)} \int_0^\infty [\phi(u_*) - \phi(c)]\, e^{-s} ds\,.$$

We have shown that the (nonnegative, nontrivial, bounded) function

$$\underline{V} = \int_0^\infty [\phi(u_*) - \phi(c)]\, e^{-s} ds$$

is in fact a subsolution to

$$\Delta V = \alpha V \qquad \text{in } M\,, \qquad \text{where } \alpha = \frac{1}{\phi'(c)}\,.$$

Existence of a corresponding solution, thus stochastic incompleteness, then follows by the usual approximation scheme.

Stochastic completeness & curvature/volume bounds

Proposition (See e.g. Grigor'yan, Bull. AMS 99)

M is stochastically incomplete if one of the following conditions holds:

Stochastic completeness & curvature/volume bounds

Proposition (See e.g. Grigor'yan, Bull. AMS 99)

M is stochastically incomplete if one of the following conditions holds:

- *M is nonparabolic, i.e. it admits a minimal positive Green function G, and G satisfies $\int_{U^c} G(x,y)\,dy < \infty$ for some precompact open set $U \subset M$ and $x \in U$.*

Stochastic completeness & curvature/volume bounds

Proposition (See e.g. Grigor'yan, Bull. AMS 99)

M is stochastically incomplete if one of the following conditions holds:

- *M is nonparabolic, i.e. it admits a minimal positive Green function G, and G satisfies $\int_{U^c} G(x,y)\,dy < \infty$ for some precompact open set $U \subset M$ and $x \in U$.*
- *M has a pole o and the sectional curvatures Sec_ω w.r.t. planes containing the radial direction w.r.t. o satisfy for all $x \in M \setminus \{o\}$.*

$$\mathrm{Sec}_\omega(x) \le -\frac{\psi''(r)}{\psi(r)} \qquad \text{with} \quad \int_0^\infty \frac{1}{\psi^{N-1}(r)} \int_0^r \psi^{N-1}(s)\,ds\,dr < \infty\,,$$

where $r := d(x,o)$ and $\psi : [0,\infty) \to [0,\infty)$ is regular and strictly positive for $r > 0$, with $\psi(0) = 0$ and $\psi'(0) = 1$.

Stochastic completeness & curvature/volume bounds

Proposition (See e.g. Grigor'yan, Bull. AMS 99)

M is stochastically incomplete if one of the following conditions holds:

- *M is nonparabolic, i.e. it admits a minimal positive Green function G, and G satisfies $\int_{U^c} G(x,y)\,dy < \infty$ for some precompact open set $U \subset M$ and $x \in U$.*
- *M has a pole o and the sectional curvatures Sec_ω w.r.t. planes containing the radial direction w.r.t. o satisfy for all $x \in M \setminus \{o\}$.*

$$\mathrm{Sec}_\omega(x) \le -\frac{\psi''(r)}{\psi(r)} \qquad \text{with} \quad \int_0^\infty \frac{1}{\psi^{N-1}(r)} \int_0^r \psi^{N-1}(s)\,ds\,dr < \infty\,,$$

where $r := d(x,o)$ and $\psi : [0,\infty) \to [0,\infty)$ is regular and strictly positive for $r > 0$, with $\psi(0) = 0$ and $\psi'(0) = 1$.

Note: the second condition qualitatively singles out *quadratic growth* (to $-\infty$) of curvature.

Proposition (See e.g. Grigor'yan, Bull. AMS 99)

M is stochastically complete provided at least one of the following conditions holds:

Proposition (See e.g. Grigor'yan, Bull. AMS 99)

M is stochastically complete provided at least one of the following conditions holds:

- *M is parabolic.*

Proposition (See e.g. Grigor'yan, Bull. AMS 99)

M is stochastically complete provided at least one of the following conditions holds:

- *M is parabolic.*
- *For some $o \in M$ the function $r \mapsto \frac{r}{\log \mathcal{V}(o,r)}$ is not integrable at infinity, where $\mathcal{V}(o,r)$ is the volume of the geodesic ball of radius r centered at o. Note that this is true in particular if $\mathcal{V}(o,r) \le Ce^{ar^2}$ for suitable $C, a > 0$.*

Proposition (See e.g. Grigor'yan, Bull. AMS 99)

M is stochastically complete provided at least one of the following conditions holds:

- *M is parabolic.*
- *For some $o \in M$ the function $r \mapsto \frac{r}{\log \mathcal{V}(o,r)}$ is not integrable at infinity, where $\mathcal{V}(o,r)$ is the volume of the geodesic ball of radius r centered at o. Note that this is true in particular if $\mathcal{V}(o,r) \le Ce^{ar^2}$ for suitable $C, a > 0$.*
- *M has a pole o and the Ricci curvature Ric_o in the radial direction w.r.t. o satisfies*

$$\mathrm{Ric}_o(x) \ge -(N-1)\frac{\psi''(r)}{\psi(r)} \qquad \textit{with} \quad \int_0^\infty \frac{1}{\psi^{N-1}(r)} \int_0^r \psi^{N-1}(s)\, ds\, dr = \infty\,,$$

where $\psi : [0,\infty) \to [0,\infty)$ is a regular function as above.

Extensions

In a recent work (G., Ishige, Muratori, Punzo, preprint 2023) we show that the result of the second part are much more general.

Extensions

In a recent work (G., Ishige, Muratori, Punzo, preprint 2023) we show that the result of the second part are much more general.

In fact, denote by $\mathfrak{L}$ the class of all functions $\phi : [0, +\infty) \to \mathbb{R}$ satisfying the following assumptions:

Extensions

In a recent work (G., Ishige, Muratori, Punzo, preprint 2023) we show that the result of the second part are much more general.

In fact, denote by $\mathfrak{L}$ the class of all functions $\phi : [0, +\infty) \to \mathbb{R}$ satisfying the following assumptions:

$$\begin{gathered} \phi \text{ is continuous in } [0,+\infty) \text{ and locally Lipschitz in } (0,+\infty)\,; \\ \phi \text{ is strictly increasing,} \end{gathered} \tag{13}$$

Extensions

In a recent work (G., Ishige, Muratori, Punzo, preprint 2023) we show that the result of the second part are much more general.

In fact, denote by $\mathfrak{L}$ the class of all functions $\phi : [0, +\infty) \to \mathbb{R}$ satisfying the following assumptions:

$$\begin{gathered} \phi \text{ is continuous in } [0, +\infty) \text{ and locally Lipschitz in } (0, +\infty)\,; \\ \phi \text{ is strictly increasing,} \end{gathered} \tag{13}$$

and, for any $\psi \in \mathfrak{C}$, we will then consider the following nonlinear elliptic equation:

$$\Delta W = \psi(W) \qquad \text{in } M\,. \tag{14}$$

Extensions

In a recent work (G., Ishige, Muratori, Punzo, preprint 2023) we show that the result of the second part are much more general.

In fact, denote by $\mathfrak{L}$ the class of all functions $\phi : [0, +\infty) \to \mathbb{R}$ satisfying the following assumptions:

$$\begin{gathered} \phi \text{ is continuous in } [0,+\infty) \text{ and locally Lipschitz in } (0,+\infty)\,; \\ \phi \text{ is strictly increasing,} \end{gathered} \tag{13}$$

and, for any $\psi \in \mathfrak{C}$, we will then consider the following nonlinear elliptic equation:

$$\Delta W = \psi(W) \qquad \text{in } M\,. \tag{14}$$

No concavity assumption is assumed!

Similarly, denote by $\mathfrak{L}$ the class of all functions $\phi : [0, +\infty) \to \mathbb{R}$ satisfying the following assumptions:

Similarly, denote by $\mathfrak{L}$ the class of all functions $\phi : [0, +\infty) \to \mathbb{R}$ satisfying the following assumptions:

$$\begin{gathered} \phi \text{ is continuous in } [0, +\infty) \text{ and locally Lipschitz in } (0, +\infty)\,; \\ \phi \text{ is strictly increasing.} \end{gathered} \tag{15}$$

Similarly, denote by $\mathfrak{L}$ the class of all functions $\phi : [0, +\infty) \to \mathbb{R}$ satisfying the following assumptions:

$$\begin{gathered} \phi \text{ is continuous in } [0,+\infty) \text{ and locally Lipschitz in } (0,+\infty)\,; \\ \phi \text{ is strictly increasing.} \end{gathered} \tag{15}$$

For any $\phi \in \mathfrak{L}$, $T \in (0, +\infty]$ and $u_0 \in L^\infty(M)$, with $u_0 \geq 0$, we will then consider the following nonlinear parabolic Cauchy problem:

Similarly, denote by $\mathfrak{L}$ the class of all functions $\phi : [0, +\infty) \to \mathbb{R}$ satisfying the following assumptions:

$$\begin{gathered} \phi \text{ is continuous in } [0,+\infty) \text{ and locally Lipschitz in } (0,+\infty)\,; \\ \phi \text{ is strictly increasing.} \end{gathered} \tag{15}$$

For any $\phi \in \mathfrak{L}$, $T \in (0, +\infty]$ and $u_0 \in L^\infty(M)$, with $u_0 \geq 0$, we will then consider the following nonlinear parabolic Cauchy problem:

$$\begin{cases} u_t = \Delta\phi(u) & \text{in } M \times (0, T)\,, \\ u = u_0 & \text{on } M \times \{0\}\,. \end{cases} \tag{16}$$

No convexity assumption is assumed! The porous medium equation is included

Our result in the stationary setting is the following:

Our result in the stationary setting is the following:

Theorem 9

Let M be a noncompact Riemannian manifold. Let $\psi \in \mathfrak{C}$. Then the following properties are equivalent:

Our result in the stationary setting is the following:

Theorem 9

Let M be a noncompact Riemannian manifold. Let $\psi \in \mathfrak{C}$. Then the following properties are equivalent:

(a) *M is stochastically incomplete;*

Our result in the stationary setting is the following:

Theorem 9

Let M be a noncompact Riemannian manifold. Let $\psi \in \mathfrak{C}$. Then the following properties are equivalent:

(a) *M is stochastically incomplete;*

(b) *The nonlinear elliptic equation* (14) *admits infinitely many nonnegative, nontrivial, bounded solutions;*

Our result in the stationary setting is the following:

Theorem 9

Let M be a noncompact Riemannian manifold. Let $\psi \in \mathfrak{C}$*. Then the following properties are equivalent:*

(a) *M is stochastically incomplete;*

(b) *The nonlinear elliptic equation* (14) *admits infinitely many nonnegative, nontrivial, bounded solutions;*

(c) *The nonlinear elliptic equation* (14) *admits a nonnegative, nontrivial, bounded subsolution.*

In the evolutionary setting we have what follows:

In the evolutionary setting we have what follows:

Theorem 10

Let M be a noncompact Riemannian manifold. Let $\phi \in \mathfrak{L}$, $T \in (0, +\infty]$ and $u_0 \in L^\infty(M)$, with $u_0 \geq 0$. Then the following properties are equivalent:

In the evolutionary setting we have what follows:

Theorem 10

Let M be a noncompact Riemannian manifold. Let $\phi \in \mathfrak{L}$, $T \in (0, +\infty]$ and $u_0 \in L^\infty(M)$, with $u_0 \geq 0$. Then the following properties are equivalent:

(a) *M is stochastically incomplete;*

(b) *The Cauchy problem* (16) *admits infinitely many nonnegative solutions in $L^\infty(M \times (0, T))$;*

In the evolutionary setting we have what follows:

Theorem 10

Let M be a noncompact Riemannian manifold. Let $\phi \in \mathfrak{L}$, $T \in (0, +\infty]$ and $u_0 \in L^\infty(M)$, with $u_0 \geq 0$. Then the following properties are equivalent:

(a) *M is stochastically incomplete;*

(b) *The Cauchy problem* (16) *admits infinitely many nonnegative solutions in $L^\infty(M \times (0, T))$;*

(c) *The Cauchy problem* (16) *admits at least two nonnegative solutions in $L^\infty(M \times (0, T))$.*

In the evolutionary setting we have what follows:

Theorem 10

Let M be a noncompact Riemannian manifold. Let $\phi \in \mathfrak{L}$, $T \in (0, +\infty]$ and $u_0 \in L^\infty(M)$, with $u_0 \geq 0$. Then the following properties are equivalent:

- (a) *M is stochastically incomplete;*
- (b) *The Cauchy problem* (16) *admits infinitely many nonnegative solutions in $L^\infty(M \times (0, T))$;*
- (c) *The Cauchy problem* (16) *admits at least two nonnegative solutions in $L^\infty(M \times (0, T))$.*

THANKS FOR YOUR ATTENTION!