Asymptotic stability of stationary states of a stochastic neural field in the form of a PDE

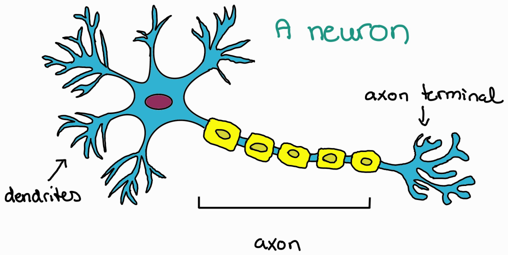
Susanne Solem

Workshop on nonlocal and nonlinear PDEs
Trondheim 24.05.2023

The neural field model under consideration

The probability density at time t, $\rho(t,x,s)$, of finding a neuron at $x \in \mathbb{T}^d$ with activity level $s \geqslant 0$ solves

$$\tau \frac{\partial \rho}{\partial t} = -\frac{\partial}{\partial s}\left(\left(\Phi_{\bar{\rho}} - s\right)\rho\right) + \sigma \frac{\partial^2 \rho}{\partial s^2},$$



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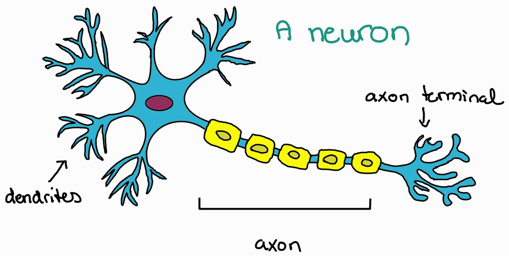

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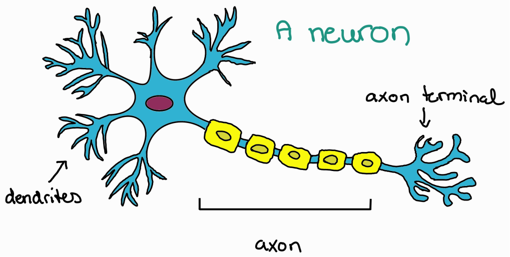

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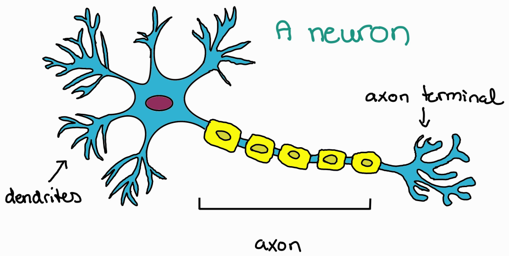
with $\bar{\rho}(t,x) = \int_0^{\infty} s\rho(t,x,s)\,\mathrm{d}s,$ and $\left(\Phi_{\bar{\rho}}\rho - \sigma\frac{\partial \rho}{\partial s}\right)(t,x,0) = 0.$



Why this model and why care about stationary states?

- Proposed as a model to study the effects of noise in grid cells.[1]

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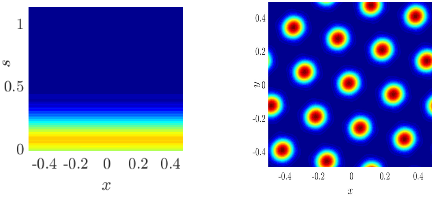

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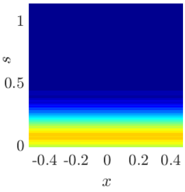
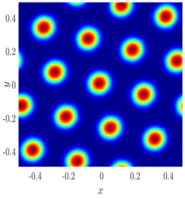
- Varying the noise level can shift the behaviour of the network from one state to another.
- Existence of bifurcation branches gives possible stationary states, their stability tells us which ones the model could settle into.





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If $B > 0$ and $\int_{\mathbb{T}^d} W(x)\,\mathrm{d}x < 0$, and $\Phi' \geqslant 0$, the homogeneous in space stationary states are unique.[1]

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Noise-induced apparition of patterns[2]

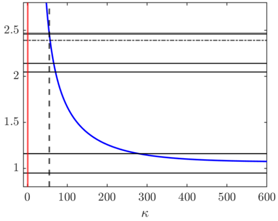
Let ρ_∞ denote the spatially homogeneous stationary state, and $\frac{1}{\sigma_0} > \frac{2|W_0|^2}{\pi B^2}$. Assume $\Phi'' > -C_{\sigma_0}$ and that $\exists k^* \in \mathbb{N}^d$ s.t.

$$\hat{W}_{k^*} = \frac{\sigma_0}{\Phi_0' M_\infty},$$

with

$$\Phi_0' = \Phi'\big(W_0 \bar{\rho}_\infty + B\big), \quad M_\infty = \int_0^{+\infty} (s - \bar{\rho}_\infty)^2 \rho_\infty(s) ds,$$

Then there exists spatially patterned bifurcation branches emanating from $(\rho_\infty^{\sigma_0}, \sigma_0)$.



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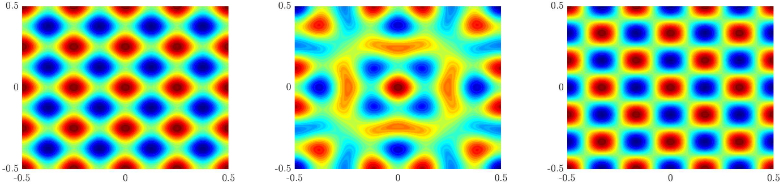
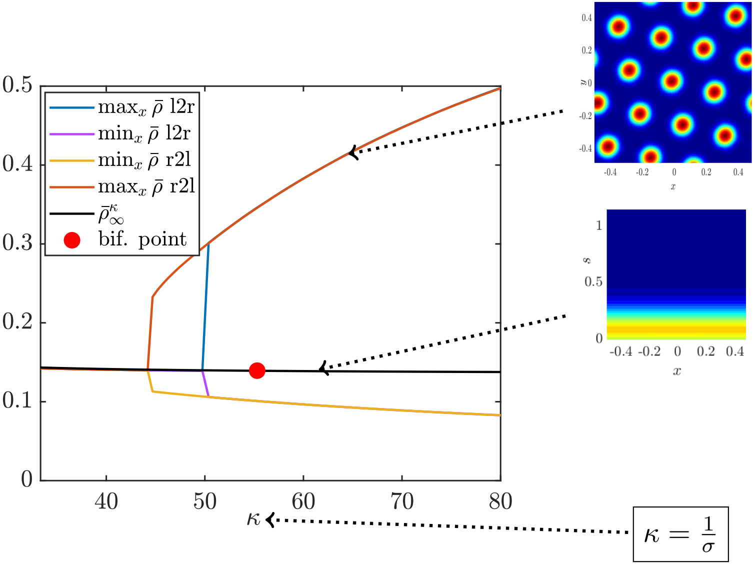
Possible patterns at the first three bifurcation points



Numerical bifurcation diagram[2]



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But we can say something about the spatially homogeneous stationary states, and *maybe* other states.

An optimal condition for linear stability in relative entropy[1]

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2. Carefully combine estimates for the time derivative of the relative entropy of each Fourier mode $(\hat{\rho} - \hat{\rho}_\infty)_k(s,t)$ of the perturbation $(\rho - \rho_\infty)(x,s,t)$, and the square of the Fourier modes of the perturbation of the mean, $(\hat{\bar{\rho}} - \hat{\bar{\rho}}_\infty)_k$.

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Resulted in exponential decay of the quantities

$$\int_0^{+\infty} \left(\frac{(\hat{\rho} - \hat{\rho}_\infty)_k}{\rho_\infty} \right)^2 \rho_\infty ds - \frac{\Phi_0'}{\sigma} \hat{W}_k ((\hat{\bar{\rho}} - \hat{\bar{\rho}}_\infty)_k)^2, \qquad k \in \mathbb{Z}^d,$$

which where shown to be positive under the condition

$$\hat{W}_k < \frac{\sigma}{\Phi_0' M_\infty}.$$

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An optimal condition for linear stability[1]

The spatially homogeneous stationary state ρ_∞, which depends on σ, is *linearly exponentially stable* in the L^1-norm of the relative entropy,

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when all the unnormalised Fourier modes of the component-wise symmetric W satisfy

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Optimal: replacing the inequality with an equality, this is exactly the condition leading to bifurcations.

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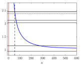

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Nonlinear stability

Goal: estimates for a combination of the time derivatives of the quantities

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Why? Linearizing $\Phi - \Phi_0$, we get

$$\int_{\mathbb{T}^d} \mathcal{E} - \frac{\mathcal{H}}{\sigma}\, dx \simeq \int_{\mathbb{T}^d} \int_0^\infty \left(\frac{\rho - \rho_\infty}{\rho_\infty}\right)^2 \rho_\infty \, ds dx - \frac{\Phi_0'}{\sigma} \int_{\mathbb{T}^d} W * (\bar{\rho} - \bar{\rho}_\infty)(\bar{\rho} - \bar{\rho}_\infty) dx.$$

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Nonlinear stability

Let Φ be C^2, and $W \in L^2(\mathbb{T}^d)$ be component-wise symmetric. Assume that that for a suitably small $\alpha > 0$,

$$\int_{\mathbb{T}^d} (1-\alpha) g^2(x) - \quad \frac{\Phi_g^\delta(x)}{\sigma} g(x)\, dx > 0, \quad g \in L^2(\mathbb{T}^d),$$

where

$$\Phi_g^\delta(x) = \Phi\big(M_\infty W * g(x) + W_0 \bar{\rho}_\infty + B\big) - \Phi_0.$$

Then, the L^1 norm of the relative entropy

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decays exponentially fast whenever ρ_0 is close enough to ρ_∞ in relative entropy.

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If Φ is linear, it reduces to the linear stability condition modulo α:

$$\hat{W}_k < \frac{\sigma(1-\alpha)}{M_\infty}$$

Stability of spatially dependent stationary states?

Let

$$M_\infty(x) = \int_0^{+\infty} (s - \bar{\rho}_\infty(x))^2 \rho_\infty(x,s) ds.$$

Assume that

$$\|\Phi'\|_\infty \|W\|_{L^2(\mathbb{T}^d)} \sup_{x\in\mathbb{T}^d} M_\infty^{1/2}(x) < \frac{\sigma}{2} \tilde{\gamma}(\rho_\infty)^{1/2},$$

where $\tilde{\gamma}(\rho_\infty) = \inf_{x\in\mathbb{T}^d} \gamma(\rho_\infty(x))$, and $\gamma(\rho_\infty(x))$ is the Poincaré constant for $\rho_\infty(x)$. Then,

$$\int_{\mathbb{T}^d} \int_0^{+\infty} \left(\frac{\rho(x,s,t) - \rho_\infty(x,s)}{\rho_\infty(x,s)} \right)^2 \rho_\infty(x,s) ds dx,$$

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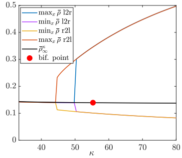
Summary and outlook

We

- know there exists spatially heterogeneous stationary states,[2]
- have almost optimal conditions for (local) stability of the spatially homogeneous stationary state, and
- have established stability of (possibly spatially heterogeneous) stationary states under somewhat restrictive conditions,

but what about

- existence and stability of the hexagonal states, or
- multistability?



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Thank you!

A preprint with J. Carrillo and P. Roux will be on arxiv very soon. (: