A Lagrange-Galerkin scheme for first order mean field games

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Introduction

First order (or deterministic) Mean Field Games (MFGs) were first introduced in Lasry-Lions'07 in the following form

$$\left.\begin{array}{rcl} -\partial_t v + H(x, D_x v) & = & F(x, m(t)) \quad \text{in } [0,T]\times\mathbb{R}^d, \\ v(T,x) & = & G(x, m(T)) \quad \text{in } \mathbb{R}^d, \\ \partial_t m - \text{div}\big(D_p H(x, Dv) m\big) & = & 0 \quad \text{in } [0,T]\times\mathbb{R}^d, \\ m(0,\cdot) & = & m_0^* \quad \text{in } \mathbb{R}^d. \end{array}\right\} \quad \text{(MFG)}$$

- The *Hamiltonian* $H\colon \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ is given by

$$H(x,p) = \sup_{a\in\mathbb{R}^d} \{\langle a, p\rangle - L(x,a)\}, \quad \text{where } L : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}.$$

- F, $G\colon \mathbb{R}^d \times \mathcal{P}_1(\mathbb{R}^d) \to \mathbb{R}$ and $m_0^* \in L^p(\mathbb{R}^d)$ for some $p \in]1,\infty[$.

- When the Hamiltonian H is coercive, the existence of solutions to (MFG) has been studied in Lasry-Lions'07 and in Cardaliaguet-Hadikhanloo'17.
- If H is not coercive, the existence question has been studied in Achdou-Mannucci-Marchi-Tchou'20 and in Cannarsa-Mendico'20.
- The notion of MFG equilibria can be stated in terms of probability measures over the set of paths $C([0,T];\mathbb{R}^d)$.
  - The existence of equilibria for this relaxed, also called Lagrangian, form can be shown under some rather general assumptions on the data.
  - Under some regularity assumptions on the data, then a solution to (MFG) can be obtained from a relaxed equilibrium.

Concerning the numerical approximation of solutions to (MFG):

- In the coercive case:
  - In Camilli-S.'12, for $H(x,p) = |p|^2/2$, a semi-discrete SL scheme is proposed and convergence is shown.
  - A fully-discrete version proposed in Carlini-S.'14, for $H(x,p) = |p|^2/2$, is shown to converge when $d = 1$.
  - Extensions to the second order case have been studied in Carlini-S'15-18 and to the case of fractional and non-local operators in Chowdhury-Ersland-Jakobsen'22.
  - An approximating MFG with discrete time and finite state space is proposed in Hadikhanloo-S.'19. Convergence is obtained in general dimensions.
- In the non-coercive case:
  - See Gianatti-S'22 and Gianatti-S-Zorkot'23.

Assumptions

In what follows, $C > 0$ denotes a generic constant.

- L is of class C^2, and for all x, $a \in \mathbb{R}^d$, we have

$$\begin{aligned} L(x,a) &\leq C(|a|^2+1), \\ |D_x L(x,a)| &\leq C(|a|^2+1), \\ C|b|^2 &\leq D^2_{aa}L(x,a)(b,b), \\ D^2_{xx}L(x,a)(y,y) &\leq C(|a|^2+1)|y|^2. \end{aligned}$$

These assumptions on L imply that H has quadratic growth and

$$|D_p H(x,p)| \leq C(1+|p|) \quad \text{for all } x,\ p \in \mathbb{R}^d.$$

A typical example is $H(x,p) = a(x)|p|^2 + \langle b(x), p\rangle$, with a and b of class C^2_b and a bounded from below by a strictly positive constant.

- F and G are bounded, continuous and, for every $\mu \in \mathcal{P}_1(\mathbb{R}^d)$,

$$|F(x,\mu) - F(y,\mu)| + |G(x,\mu) - G(y,\mu)| \leq C|x-y|,$$
$$F(x+y,\mu) - 2F(x,\mu) + F(x-y,\mu) \leq C|y|^2,$$
$$G(x+y,\mu) - 2G(x,\mu) + G(x-y,\mu) \leq C|y|^2.$$

Notice that no differentiability is supposed on F and G. Thus, we can consider functionals of the form

$$F(x,\mu) = \min\{|x-\bar{x}|^2, R\} + f_0(x,\mu) \quad \text{for } \bar{x} \in \mathbb{R}^d,\ R > 0.$$

- m_0^* has compact support and $m_0^* \in L^p(\mathbb{R}^d)$ (for some $p \in]1,\infty]$).

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Approximation of the HJB equation

Let $\mu \in C([0,T];\mathcal{P}_1(\mathbb{R}^d))$ and consider the HJB equation

$$\begin{aligned} -\partial_t v + H(x, Dv) &= F(x, \mu(t)) \quad \text{in } [0,T]\times\mathbb{R}^d, \\ v(T,x) &= G(x, \mu(T)) \quad \text{in } \mathbb{R}^d. \end{aligned}$$

If $v[\mu]$ denotes its solution, then for every $(t,x)\in[0,T]\times\mathbb{R}^d$,

$$v[\mu](t,x) = \inf \int_t^T \Big(L(\gamma(s),\alpha(s)) + F(\gamma(s),\mu(s)) \Big) \mathrm{d}s + G(\gamma(T),\mu(T))$$
$$\text{s.t.} \quad \dot\gamma(s) = -\alpha(s) \quad \text{in }]s,T[, \quad \gamma(t) = x.$$

Proposition

The value function is uniformly bounded, and the following hold:

$$\text{(Lip)} \qquad \big|v[\mu](t,x) - v[\mu](t,y)\big| \leq C|x-y|,$$
$$\text{(SC)} \qquad v[\mu](x+y,\mu) - 2v[\mu](x,\mu) + v[\mu](x-y,\mu) \leq C|y|^2.$$

Using the properties above for $v[\mu]$, one can show the existence of $m[\mu]$ solving

$$\partial_t m - \text{div}(D_p H(x, D_x v)m) = 0 \quad \text{in }]0,T[\times\mathbb{R}^d, \quad m(0) = m_0^*$$

and such that

- $m[\mu](t,\cdot)$ has a compact support, independent of μ.
- *"The mass does not concentrate too much in finite time"*

$$\|m[\mu](t,\cdot)\|_{L^p} \leq C\|m_0^*\|_{L^p}.$$

As in Carlini-S'14, given $(\Delta t, \Delta x)$ we consider the following SL scheme for the HJB equation:

$$\begin{aligned} v_{k,i} &= \inf_{a\in\mathbb{R}^d} \left[\Delta t L(x_i, a) + I^1[v_{k+1,\cdot}](x_i - \Delta t a)\right] + \Delta t F(x_i, \mu(t_k)), \\ v_{N,i} &= G(x_i, \mu(T)), \end{aligned}$$

where, given ϕ defined on $\mathcal{G}_{\Delta x} = \{x_i = \Delta x \,|\, i \in \mathbb{Z}^d\}$,

$$I^1[\phi](x) = \sum_{i\in\mathbb{Z}^d} \beta_i^1(x)\phi(x_i) \quad \text{for all } x \in \mathbb{R}^d,$$

with $\{\beta_i^1 \,|\, i \in \mathbb{Z}^d\}$ being a Q_1-basis on the regular mesh $\mathcal{G}_{\Delta x}$.

This scheme preserves:

- The Lipschitz property (Lip).
- The semiconcavity (SC).

We set

$$v^{\Delta t,\Delta x}[\mu](t,x) = I^1[v_{k,\cdot}](x) \quad \text{if } t \in [t_k, t_{k+1}[,\ x \in \mathbb{R}^d,$$

and, given $\varepsilon > 0$ and a standard mollifier ρ_ε, we set $\Delta = (\Delta t, \Delta x, \varepsilon)$ and

$$v^{\Delta}[\mu](t,x) = \Big(\rho_\varepsilon * v^{\Delta t,\Delta x}[\mu](t,\cdot)\Big)(x).$$

- $v^{\Delta}[\mu]$ preserves the Lipschitz property.
- The following semi-concavity estimate holds:

$$\langle D^2_{xx} v^{\Delta}[\mu](t,x)y, y\rangle \leq C\left(1 + \frac{(\Delta x)^2}{\varepsilon^4}\right)|y|^2.$$

- Under suitable assumptions on the parameters, if $\mu_n \to \mu$ and $\Delta_n \to 0$, then $v^{\Delta_n}[\mu_n] \to v[\mu]$ uniformly over compact sets, and $D_x v^{\Delta_n}[\mu_n] \to D_x v[\mu]$ a.e.

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Approximation of the continuity equation

We now focus on the discretization of the continuity equation

$$\partial_t m - \text{div}(D_p H(x, D_x v^\Delta[\mu])m) = 0 \quad \text{in }]0,T[\times\mathbb{R}^d, \quad m(0) = m_0^*,$$

Since v^Δ is smooth w.r.t. the state, this equation has a unique solution

$$m^\Delta[\mu](t,\cdot) = \Phi^\Delta[\mu](0,t,\cdot)\sharp m_0^*,$$

where, for $s \leq t$, $\Phi^\Delta[\mu](s,t,x)$ is the the solution, at time t, of the ODE:

$$\dot\gamma(r) = -D_p H\Big(\gamma(r), D_x v^\Delta[\mu](r,\gamma(r))\Big) \quad \text{in }]s,T[, \quad \gamma(s) = x.$$

Equivalently, for every $0 \leq s \leq t \leq T$, and φ, integrable w.r.t. $m^\Delta[\mu](s)$,

$$\int_{\mathbb{R}^d} \varphi(x) \mathrm{d}m^\Delta[\mu](t)(x) = \int_{\mathbb{R}^d} \varphi\big(\Phi^\Delta[\mu](s,t,x)\big) \mathrm{d}m^\Delta[\mu](s)(x). \qquad (*)$$

- Approximate $\Phi^{\Delta}[\mu](t_k, t_{k+1}, x)$ by

$$\Phi_k^{\Delta}[\mu](x) = x - \Delta t D_p H\big(x, D_x v^{\Delta}[\mu](t_k, x)\big).$$

- Let $\{\beta_i\}_{i\in\mathbb{Z}^d}$ be a FE basis and approximate $m^{\Delta}[\mu](t_k)$ by

$$\mathrm{m}^{\Delta}[\mu](t_k, x) = \sum_{i\in\mathbb{Z}^d} m_{k,i}\beta_i(x)$$

- Using this approximation and taking $\varphi = \beta_j$ in $(*)$, we get

$$\sum_{i\in\mathbb{Z}^d} m_{k+1,i}\int_{\mathbb{R}^d}\beta_i(x)\beta_j(x)\mathrm{d}x = \sum_{i\in\mathbb{Z}^d} m_{k,i}\int_{\mathbb{R}^d}\beta_j(\Phi_k^{\Delta}[\mu](x))\beta_i(x)\mathrm{d}x.$$

- In what follows, we take

$$\beta_i = \beta_i^0 = \mathbb{I}_{E_i}, \quad \text{where } E_i = [x_i - \Delta x/2, x_i + \Delta x/2]^d.$$

This yields the following LG scheme

$$
\begin{aligned}
m_{k+1,i} &= \frac{1}{(\Delta x)^d} \sum_j m_{k,j} \int_{E_j} \beta_i^0(\Phi_k^\Delta[\mu](x))\mathrm{d}x, \\
m_{0,i} &= \frac{1}{(\Delta x)^d} \int_{E_i} m_0^*(x)\mathrm{d}x.
\end{aligned} \tag{LG}
$$

- Since

$$
\int_{E_j} \beta_i^0(\Phi_k^\Delta[\mu](x))\mathrm{d}x = \mathcal{L}^d\Big(\Phi_k^\Delta[\mu]^{-1}(E_i) \cap E_j\Big),
$$

this scheme coincides with the one proposed in Piccoli and Tosin.[1]

Given a solution to (LG), if $t \in [t_k, t_{k+1})$, set

$$
\mathsf{m}^\Delta[\mu](t,x) = \left(\frac{t_{k+1}-t}{\Delta t}\right) \sum_{i\in\mathbb{Z}^d} m_{k,i}\beta_i^0(x) + \left(\frac{t-t_k}{\Delta t}\right) \sum_{i\in\mathbb{Z}^d} m_{k+1,i}\beta_i^0(x).
$$

[1] B. Piccoli and A. Tosin. Time-evolving measures and macroscopic modeling of pedestrian flow. *Arch. Ration. Mech. Anal.* 2011

The approximation $\mathsf{m}^{\Delta}[\mu]$ satisfies

- $\mathsf{m}^{\Delta}[\mu] \in C([0,T];\mathcal{P}_1(\mathbb{R}^d))$.
- There exists $C^* > 0$ such that $\sup(\mathsf{m}^{\Delta}[\mu](t,\cdot)) \subseteq \overline{B}(0,C^*)$.
- The map $[0,T] \ni t \mapsto \mathsf{m}^{\Delta}[\mu](t,\cdot) \in \mathcal{P}_1(\mathbb{R}^d)$ is Lipschitz continuous.
- If $\Delta x = O(\Delta t)$ and $\Delta t = O(\varepsilon^2)$ then

$$\|\mathsf{m}^{\Delta}[\mu](t,\cdot)\|_{L^p} \leq C\|m_0^*\|_{L^p}.$$

The proof of the L^p-stability mainly relies on the following facts:

- $\Delta t/\varepsilon$ small enough $\Rightarrow$ $\Phi_k^{\Delta}[\mu]$ is one-to-one.
- The estimate on $D^2_{xx}v^{\Delta}[\mu](t_k,\cdot)$ implies that

$$\left|\det\left(D_x\Phi_k^{\Delta}[\mu](x)\right)\right|^{-1} \leq 1 + C\Delta t.$$

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Approximation of the MFG system

- The MFG system is discretized as follows:

$$\text{Find } \mu \text{ such that } \mu = \mathsf{m}^{\Delta}[\mu]. \qquad (\text{MFG})^{\Delta}$$

  Using the Brouwer's fixed point theorem, one shows that $(\text{MFG})^{\Delta}$ admits at least one solution.

- Convergence holds in general state dimensions.

Theorem

Let $\Delta_n = (\Delta t_n, \Delta x_n, \varepsilon_n) \in]0, \infty[^3$, *denote by* m_n *a solution to* $(\text{MFG})^{\Delta_n}$, *and set* $v_n = v^{\Delta_n}[\mathsf{m}_n]$.
Assume that, as $\Delta_n \to 0$, $\Delta x_n = o(\Delta t_n)$ *and* $\Delta t_n = O(\varepsilon_n^2)$. *Then, up to some subsequence,* (v_n, m_n) *converges to a solution* (v^*, m^*) *to* (MFG).

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Numerical results

- In order to implement the scheme, we follow Morton-Priestley-Süli'88 by considering the following approximation

$$\Phi_k^{\Delta}[\mu](x) \sim x - \Delta t D_p H\big(x_i, D_x v^{\Delta}[\mu](t_k, x_i)\big) \quad \text{if } x \in E_i$$

  to obtain, surprisingly, that

$$\int_{E_j} \beta_i^0(\Phi_k^{\Delta}[\mu](x))\mathrm{d}x = \beta_i^1(\Phi_k^{\Delta}[\mu](x_j)),$$

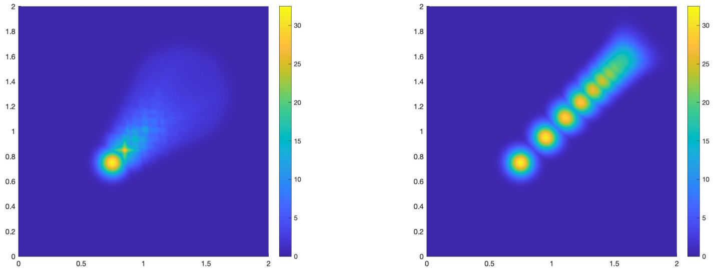
  and, hence, the LG scheme implemented with this approximation coincides with the scheme proposed in Carlini-S'14.

- In the numerical test below, we take $d = 2$, $T = 1$,

$$m_0^*(x) = \frac{\nu(x)}{\int_{[0,2]^2} \nu(x)\mathrm{d}x} \quad \text{with } \nu(x) = e^{\frac{-|x-x_0|^2}{0.01}} \text{ and } x_0 = (0.75, 0.75),$$

$$H(x,p) = |p|^2/2, \quad F(x,m) = \gamma \min\{R, |x - x_f|^2\} + (\rho_\sigma * m)(x), \quad G = 0,$$

with $x_f = (1.75, 1.75)$. In the figures below, we display the distributions for $\gamma = 1$ and $\gamma = 3$.



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