Differential games and Zubov's Method

Oana Silvia Serea

Western Norway University of Applied Sciences, Bergen, Norway and Université de Perpignan Via Domitia, France

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Zubov's method '64 is a classical method for computing Lyapunov functions and domains of attraction for differential equations $\dot{x} = f(x)$ with a locally asymptotically stable equilibrium $x^* \in \mathbb{R}^N$.

Zubov's main result states that for a suitable function $g : \mathbb{R}^N \to \mathbb{R}$

$$\nabla W(x) f(x) = -g(x)(1 - W(x))\sqrt{1 + \|f(x)\|}, \tag{1}$$

has a unique differentiable solution $W : \mathbb{R}^N \to [0, 1]$ with $W(x^*) = 0$, which characterizes the domain of attraction $\mathcal{D}$ of x^* via $\mathcal{D} = \{x \in \mathbb{R}^N \mid W(x) < 1\}$ and which is a Lyapunov function on $\mathcal{D}$.

- deterministic case Grune& Wirth '00, Camilli & al. '01a, Malisoff '05 and Camilli & al.'08
- stochastic case Camilli & Grune '03
- Sontang '83 (integral version of (1))
- numerical methods Camilli & Grune '01b and Giesel '07

Outline

$$(2) \qquad \begin{cases} i) & \dot{x}(s) = f(x(s), u(s), v(s)) \ \text{ for almost all } \ s \geq t \\ ii) & x(0) = x. \end{cases}$$

Here $f : \mathbb{R}^n \times U \times V \to \mathbb{R}^n$ is continuous and bounded and we assume that $f(x, u, v)$ is locally Lipschitz in x uniformly in u and v.

U and V are compact and of finite dimension and the controls $u(\cdot) : [0, \infty) \to U$ and $v(\cdot) : [0, \infty) \to V$ are measurable functions.

We define

$$\mathcal{U} = \{u(\cdot) : [0, \infty) \to U, \ \text{measurable}\},$$
$$\mathcal{V} = \{v(\cdot) : [0, \infty) \to V, \ \text{measurable}\}$$

as the respective sets of control functions.

We investigate the situation in which the first player wants to control the system asymptotically to the origin $x = 0$ while the second player tries to avoid this. For this reason, we will usually interpret $u(\cdot)$ as a control function while we consider $v(\cdot)$ as a (time varying) perturbation.

In order to obtain a Zubov type characterization of this situation, to any solution $x(\cdot, x, u(\cdot), v(\cdot))$ of (2) with initial value x we associate a payoff which depends on $u(\cdot)$ and $v(\cdot)$ and is denoted by

$$J(x, u(\cdot), v(\cdot)) \tag{3}$$

Definition (Elliot-Kalton-Varayia strategies (cf. Elliot & Kalton '72))

We say that a map $\alpha : \mathcal{V} \to \mathcal{U}$ is a *nonanticipative strategy* (for the first player) if it satisfies the following condition:
For any $v_1, v_2 \in \mathcal{V}$ which coincide almost everywhere on $[0, s]$ for an $s \geq 0$, the images $\alpha(v_1)$ and $\alpha(v_2)$ also coincide almost everywhere on $[0, s]$.
Nonanticipative strategies $\beta : \mathcal{U} \to \mathcal{V}$ (for the second player) are defined similarly.
The set of nonaticipating strategies α for the first player are denoted by Γ and the respective set for the second player is denoted by Δ.

Definition

The lower value function for the game (2), (3) is given by

$$V^-(x) = \inf_{\alpha \in \Gamma} \sup_{v \in \mathcal{V}} J(x, \alpha(v), v) \tag{4}$$

and the upper value function is defined by

$$V^+(x) = \sup_{\beta \in \Delta} \inf_{u \in \mathcal{U}} J(x, u, \beta(u)) \tag{5}$$

for all $x \in \mathbb{R}^N$.

Question: The existence of the value, i.e., one has

$$V^+ = V^-.$$

Friedman '70, , Elliot & Kalton '72, Breakwell '77, Bercovitz '86, Evans & Souganidis '84, Krasovskii & Subbotin '87...

We assume that $x = 0$ is an equilibrium (or fixed point) for some control $u_0 \in U$ and all $v \in V$, i.e., there exists $u_0 \in U$ such that $f(0, u_0, v) = 0$ for any $v \in V$.

The equilibrium 0 could be replaced by a more general compact set $A \subset \mathbb{R}^n$.

Definition

i) We call a point $x \in \mathbb{R}^n$ lower asymptotically controllable to 0 if there exists a nonanticipative strategy $\alpha_x(\cdot) \in \Gamma$ such that for any perturbation $v(\cdot) \in \mathcal{V}$ the corresponding solution $x(t, x, \alpha_x(v), v)$ of (2) satisfies $x(t, x, \alpha_x(v), v) \to 0$ for $t \to \infty$. The domain of lower asymptotic null-controllability $\mathcal{D}^-$ is the collection of all points that are lower asymptotically controllable to 0.

ii) We call a point $x \in \mathbb{R}^n$ upper asymptotically controllable to 0 if for any nonanticipative strategy $\beta(\cdot) \in \Delta$ there exists a control $u_{x,\beta}(\cdot) \in \mathcal{U}$ such that the corresponding solution $x(t, x, u_{x,\beta}, \beta(u_{x,\beta}))$ of (2) satisfies $x(t, x, u_{x,\beta}, \beta(u_{x,\beta})) \to 0$ for $t \to \infty$. The domain of upper asymptotic null-controllability $\mathcal{D}^+$ is the collection of all points that are lower asymptotically controllable to 0.

Definition

i) We call a point $x \in \mathbb{R}^n$ uniformly lower asymptotically controllable to 0 if there exist a function $\theta(t) \to 0$ as $t \to \infty$ and $\alpha_x(\cdot) \in \Gamma$ such that for any $v(\cdot) \in \mathcal{V}$ we have that $x(t, x, \alpha_x(v), v)$ satisfies $\|x(t, x, \alpha_x(v), v)\| \leq \theta(t)$ for all $t > 0$. The domain of uniform lower asymptotic null-controllability $\mathcal{D}_0^-$ is the collection of all points that are lower uniformly asymptotically controllable to 0.

ii) We call a point $x \in \mathbb{R}^n$ uniformly upper asymptotically controllable to 0 if there exists a function $\theta(t) \to 0$ as $t \to \infty$ such that for any $\beta(\cdot) \in \Gamma$ there exists $u_{x,\beta}(\cdot) \in \mathcal{U}$ with the property that $x(t, x, u_{x,\beta}, \beta(u_{x,\beta}))$ satisfies $\|x(t, x, u_{x,\beta}, \beta(u_{x,\beta}))\| \leq \theta(t)$ for all $t > 0$. The domain of uniform upper asymptotic null-controllability $\mathcal{D}_0^+$ is the collection of all points that are upper uniformly asymptotically controllable to 0.

(H^-) There exists an open ball $B(0,r)$ and η such that for any $x \in B(0,r)$ there is a nonanticipative strategy $\alpha_x(\cdot) \in \Gamma$ such that for any perturbation $v(\cdot) \in \mathcal{V}$ the solution $x(t,x,\alpha_x(v),v)$ exists for all $t \geq 0$ and satisfies

$$\|x(t,x,\alpha_x(v),v)\| \leq \eta(\|x\|,t)\,, \quad \forall t \geq 0\,.$$

(H^+) There exists an open ball $B(0,r)$ and η such that for any $x \in B(0,r)$ and for any nonanticipative strategy $\beta \in \Delta$ there is a control $u_{x,\beta}(\cdot) \in \mathcal{U}$ for which the solution $x(t,x,u_{x,\beta},\beta(u_{x,\beta}))$ exists for all $t \geq 0$ and satisfies

$$\|x(t,x,u_{x,\beta},\beta(u_{x,\beta}))\| \leq \eta(\|x\|,t)\,, \quad \forall t \geq 0\,.$$

Note that $(H^{\pm})$ immediately imply the inclusions $B(0,r) \subseteq \mathcal{D}_0^{\pm}$.

There are several ways to ensure $(H^{\pm})$. For instance, (H^{+}) respectively (H^{-}) holds whenever $\min_u \max_v \langle f(x,u,v), x\rangle < 0$ or, respectively, $\max_v \min_u \langle f(x,u,v), x\rangle < 0$ for all $x \in B(0,r)$, $x \neq 0$, because in this case the Euclidean norm of the solutions is strictly decreasing.

The following example shows that $\mathcal{D}_0^-$ may be strictly larger than $\mathcal{D}_0^+$.

Example

Consider the 1d control system

$$\dot{x}(t) = -x(t) + u(t)v(t)x(t)^3 =: f(x(t), u(t), v(t))$$

with $x \in \mathbb{R}$ and $U = V = \{-1, 1\}$.
For $|x| \leq 1/2$ one easily sees that f satisfies $f(x, u, v) \leq -3x/4$ if $x \geq 0$ and $f(x, u, v) \geq -3x/4$ if $x \leq 0$, regardless of how u and v are chosen. Thus, for all $u(\cdot) \in \mathcal{U}$ and $v(\cdot) \in \mathcal{V}$ we obtain $|x(t, x, u(\cdot), v(\cdot))| \leq e^{-3t/4}|x|$. This implies both (H^-) and (H^+) with $\eta(r, t) = e^{-3t/4}r$. Furthermore, for all $|x| < 1$ one sees that $f(x, u, v) < 0$ if $x > 0$ and $f(x, u, v) > 0$ for $x < 0$ for all $u \in U$ and $v \in V$. Hence, all solutions starting in some $|x| < 1$ converge to 0 which immediately implies the inclusions $(-1, 1) \subseteq \mathcal{D}_0^+$ and $(-1, 1) \subseteq \mathcal{D}_0^-$.

Example

Now we investigate $\mathcal{D}_0^-$. We define a nonanticipating strategy $\alpha \in \Gamma$ by

$$\alpha(v)(t) = -v(t).$$

This implies

$$f(x, \alpha(v)(t), v(t)) = -x - x^3 \tag{6}$$

for all $v(\cdot) \in \mathcal{V}$ and all $t \geq 0$. Since (6) is a globally asymptotically stable vector field, choosing $\alpha_x = \alpha$ for all x implies $\mathcal{D}_0^- = \mathbb{R}$.

Example

On the other hand, defining $\beta \in \Delta$ via

$$\beta(u)(t) = u(t)$$

implies

$$f(x, u(t), \beta(u)(t)) = -x + x^3 \tag{7}$$

for all $u(\cdot) \in \mathcal{U}$. Thus, for all $x > 1$ the corresponding solution $x(t, x, u, \beta(u))$ diverges to ∞ and for all $x < 1$ the solution diverges to $-\infty$. Thus, all these points cannot belong to $\mathcal{D}_0^+$. Since $x = 1$ and $x = -1$ are equilibria of (7) the corresponding solutions satisfy $x(t, \pm 1, u, \beta(u)) = \pm 1$ and thus do not converge to 0, either, hence $\pm 1 \notin \mathcal{D}_0^+$. Since above we already showed $(-1, 1) \subseteq \mathcal{D}_0^+$ we thus obtain $\mathcal{D}_0^+ = (-1, 1)$.

Hence, summarizing we obtain that both (H^-) and (H^+) are satisfied for this example and that $\mathcal{D}_0^- = \mathbb{R} \neq (-1, 1) = \mathcal{D}_0^+$.

For each $u(\cdot) \in \mathcal{U}$ and $v(\cdot) \in \mathcal{V}$ we define

$$(8) \qquad t(x, u(\cdot), v(\cdot)) = \inf \{ t \geq 0 \text{ such that } x\,(t; x, u(\cdot), v(\cdot)) \in B\,(0, r) \}\,.$$

For this time we define

$$(9) \qquad t^{-}(x) = \inf_{\alpha \in \Gamma} \sup_{v \in \mathcal{V}} t(x, \alpha(v), v) \quad \text{and} \quad t^{+}(x) = \sup_{\beta \in \Delta} \inf_{u \in \mathcal{U}} t(x, u, \beta(u))$$

for all x.

Lemma

Assume $(H^{\pm})$. *Then the following identities hold*

$$i)\ \mathcal{D}_0^- = \left\{x \in \mathbb{R}^n \mid t^-(x) < \infty\right\} =: dom(t^-),$$
$$ii)\ \mathcal{D}^- = \{x \in \mathbb{R}^n \mid \exists \alpha_x(\cdot) \in \Gamma \text{ such that } t(x, \alpha_x(v), v) < \infty, \forall v(\cdot) \in \mathcal{V}\},$$
$$iii)\ \mathcal{D}_0^+ = \left\{x \in \mathbb{R}^n \mid t^+(x) < \infty\right\} =: dom(t^+),$$
$$iv)\ \mathcal{D}^+ = \{x \in \mathbb{R}^n \mid \forall \beta(\cdot) \in \Delta, \exists u_\beta(\cdot) \in \mathcal{U} \text{ such that } t(x, u_\beta, \beta(u_\beta)) < \infty\}.$$

We continue with characterizing properties of $\mathcal{D}_0^-$ and $\mathcal{D}_0^+$. For doing this we introduce the following definition (see for instance Krasovskii & Subbotin '88, Cardaliaguet '96).

Definition

i) A set M is called discriminant domain for the dynamics (2) if for any $x \in M$ there exists a nonanticipative strategy $\alpha_x(\cdot) \in \Gamma$ such that for any perturbation $v(\cdot) \in \mathcal{V}$ the corresponding solution $x(t, x, \alpha_x(v), v)$ of (2) satisfies $x(t, x, \alpha_x(v), v) \in M$ for all $t \geq 0$.
ii) A set N is called leadership domain for the dynamics (2) if for any $x \in N$ and for any nonanticipative strategy $\beta(\cdot) \in \Delta$ there exists a control $u_{x,\beta}(\cdot) \in \mathcal{U}$ such that the corresponding solution $x\,(t; x, u_{x,\beta}, \beta(u_{x,\beta}))$ of (2) satisfies $x\,(t; x, u_{x,\beta}, \beta(u_{x,\beta})) \in N$ for all $t \geq 0$.

Proposition

Under the Assumptions $(H^{\pm})$ *the following properties hold.*

1) $clB\,(0,r) \subseteq \mathcal{D}_0^-$
2) *The set* $\mathcal{D}_0^-$ *is open, connected and a discriminant domain for* (2)
3) *The set* $\mathcal{D}^-$ *is pathwise connected and a discriminant domain for* (2)
4) $cl\mathcal{D}^-$ *and* $cl\mathcal{D}_0^-$ *are discriminant domains for* (2)
5) $t^-(x_n) \to \infty$ *for any sequence of points with* $x_n \to x \in \partial\mathcal{D}_0^-$ *or* $\|x_n\| \to \infty$ *as* $n \to \infty$

Proposition

6) $clB(0,r) \subseteq \mathcal{D}_0^+$

7) *The set* $\mathcal{D}_0^+$ *is open, connected and a leadership domain for* (2)

8) *The set* $\mathcal{D}^+$ *is pathwise connected and a leadership domain for* (2)

9) $cl\mathcal{D}^+$ *and* $cl\mathcal{D}_0^+$ *are leadership domains for* (2)

10) $t^+(x_n) \to \infty$ *for any sequence of points with* $x_n \to x \in \partial\mathcal{D}_0^+$ *or* $\|x_n\| \to \infty$ *as* $n \to \infty$

Remark

If f does not depend on u, then the identities $\mathcal{D}^- = \mathcal{D}^+ =: \mathcal{D}$ and $\mathcal{D}_0^- = \mathcal{D}_0^+ =: \mathcal{D}_0$ hold. In this case, the inclusion $\mathcal{D} \subseteq cl\mathcal{D}_0$ was shown.

The proof of this equality, however, does not carry over to our case in which we have an additional dependence on α_x in t^- and have $\beta(u)$ instead of v in t^+. Thus, it is an open question whether the inclusions $\mathcal{D}^- \subseteq cl\mathcal{D}_0^-$ and $\mathcal{D}^+ \subseteq cl\mathcal{D}_0^+$ hold in our game theoretical setting.

We can alternatively establish a characterization via an integral cost $J(x, u, v)$. In order to define the integral cost, consider a bounded, continuous function $g : \mathbb{R}^N \times U \times V \to \mathbb{R}$ satisfying:

i) $g(0, u_0, v) = 0$ and $g(x, u, v) > 0$ for all $u \in U$ and $v \in V$ if $x \neq 0$;

ii) there exists a constant $g_0 > 0$ such that $\inf_{x \notin B(0,r), u \in U, v \in V} g(x, u, v) \geq$

iii) for every $R > 0$ there exists a constant L_R such that

$$\|g(x, u, v) - g(y, u, v)\| \leq L_R \|x - y\|$$

holds for all $\|x\|, \|y\| \leq R$ and all $u \in U, v \in V$.

Here, in the first condition $u_0 \in U$ is the control value for which $x = 0$ is an equilibrium, i.e., for which $f(0, u_0, v) = 0$ holds for all $v \in V$.

Using this g we define the integral cost

$$J(x,u,v) := \int_0^\infty g\left(x(s,x,u,v),u(s),v(s)\right) ds$$

For this cost, we define the lower value function

$$V^-(x) = \inf_{\alpha\in\Gamma} \sup_{v\in\mathcal{V}} J(x,\alpha(v),v)$$

and its Kruzkov transformation

$$W^-(x) := 1 - e^{-V^-(x)} = \inf_{\alpha\in\Gamma} \sup_{v\in\mathcal{V}} \left\{1 - e^{-J(x,\alpha(v),v)}\right\}$$

We have the dynamic programming principle, i.e., for each $t > 0$

$$V^-(x) = \inf_{\alpha\in\Gamma} \sup_{v\in\mathcal{V}} \left\{ J_t(x,\alpha(v),v) + V^-(x(t,x,\alpha(v),v)) \right\}$$

and

$$W^-(x) = \inf_{\alpha\in\Gamma} \sup_{v\in\mathcal{V}} \left\{ 1 - G_t(x,\alpha(v),v) + G_t(x,\alpha(v),v) W^-(x(t,x,\alpha_x(v),v)) \right\}$$

where we used the abbreviations
(10)

$$J_t(x,u,v) := \int_0^t g\left(x(s,x,u,v),u(s),v(s)\right) ds \quad \text{and} \quad G_t(x,u,v) := e^{-J_t(x,u,v)}.$$

Analogously, we define the upper value function

$$V^+(x) = \sup_{\beta\in\Delta} \inf_{u\in\mathcal{U}} J(x,u,\beta(u))$$

and its Kruzkov transformation, i.e.

$$W^+(x) := 1 - e^{-V^+(x)} = \sup_{\beta\in\Delta} \inf_{u\in\mathcal{U}} \left\{1 - e^{-J(x,u,\beta(u))}\right\}$$

Again, V^+ and W^+ satisfy the dynamic programming principle, i.e., for each $t > 0$ we have

$$V^+(x) = \sup_{\beta\in\Delta} \inf_{u\in\mathcal{U}} \left\{J_t(x,u,\beta(u)) + V^+(x(t,x,u,\beta(u)))\right\}$$

and

$$W^+(x) = \sup_{\beta\in\Delta} \inf_{u\in\mathcal{U}} \left\{G_t(x,u,\beta(u)) + G_t(x,u,\beta(u))W^+(x(t,x,u,\beta(u)))\right\}$$

with $J_t(x,u,v)$ and $G_t(x,u,v)$ from (10).

We have that $V^+ \geq 0$ and $W^+ \in [0,1]$, because $g \geq 0$. Moreover,

$$\begin{aligned} V^\pm(x) = 0 \quad &\Leftrightarrow \quad W^\pm(x) = 0 \\ V^\pm(x) \in (0,\infty) \quad &\Leftrightarrow \quad W^\pm(x) \in (0,1) \\ V^\pm(x) = \infty \quad &\Leftrightarrow \quad W^\pm(x) = 1 \end{aligned}$$

We investigate the relation between $\mathcal{D}_0^{\pm}$ and $V^{\pm}$/ $W^{\pm}$ and the continuity of $V^{\pm}/W^{\pm}$.
To this end, we make the following additional assumption on g.

$$\begin{cases} \text{there exists } \gamma \text{ such that for each } x \in B(0,r) \text{ and } \alpha_x \text{ and } u_x \text{ from} \\ (H^-) \text{ and } (H^+)\text{, respectively, the inequalities} \\ \qquad g(x(t,x,\alpha_x(v),v),\alpha_x(v)(t),v(t)) \leq e^{-t}\gamma(\|x\|) \\ \text{and} \\ \qquad g(x(t,x,u_x,\beta(u_x)),u_x(t),\beta(u_x)(t)) \leq e^{-t}\gamma(\|x\|) \\ \text{hold for all } v \in \mathcal{V} \text{ and } \beta \in \Delta\text{, respectively, and almost all } t \geq 0. \end{cases}$$

Proposition

The following properties hold.

1) $V^{-}(x) < \infty$ *iff* $x \in \mathcal{D}_0^{-}$
2) $V^{-}(x) = 0$ *iff* $x = 0$
3) V^{-} *is continuous on* $\mathcal{D}_0^{-}$
4) $V^{-}(x_n) \to \infty$ *for any sequence of points with* $x_n \to x \in \partial\mathcal{D}_0^{-}$ *or* $\|x_n\| \to \infty$ *as* $n \to \infty$
5) $W^{-}(x) < 1$ *iff* $x \in \mathcal{D}_0^{-}$
6) $W^{-}(x) = 0$ *iff* $x = 0$
7) W^{-} *is continuous on* $\mathbb{R}^N$
8) $W^{-}(x_n) \to 1$ *for any sequence of points with* $x_n \to x \in \partial\mathcal{D}_0^{-}$ *or* $\|x_n\| \to \infty$ *as* $n \to \infty$

Proposition

9) $V^+(x) < \infty$ *iff* $x \in \mathcal{D}_0^+$
10) $V^+(x) = 0$ *iff* $x = 0$
11) V^+ *is continuous on* $\mathcal{D}_0^+$
12) $V^+(x_n) \to \infty$ *for any sequence of points with* $x_n \to x \in \partial\mathcal{D}_0^+$ *or* $\|x_n\| \to \infty$ *as* $n \to \infty$
13) $W^+(x) < 1$ *iff* $x \in \mathcal{D}_0^+$
14) $W^+(x) = 0$ *iff* $x = 0$
15) W^+ *is continuous on* $\mathbb{R}^N$
16) $W^+(x_n) \to 1$ *for any sequence of points with* $x_n \to x \in \partial\mathcal{D}_0^+$ *or* $\|x_n\| \to \infty$ *as* $n \to \infty$

V^- and W^- are viscosity solutions of the equations

(11) $$H^-(x, \nabla V(x)) = 0 \text{ for } x \in \mathcal{D}_0^-$$

and, respectively,

(12) $$\widetilde{H}^-(x, W(x), \nabla W(x)) = 0 \text{ for } x \in \mathbb{R}^N.$$

Here, $H^-, \tilde{H}^- : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$, are given by

$$H^-(x,p) = \min_{v\in V} \max_{u\in U} \{-\langle p, f(x,u,v)\rangle - g(x,u,v)\}$$

and, respectively,

$$\widetilde{H}^-(x,r,p) = \min_{v\in V} \max_{u\in U} \{-\langle p, f(x,u,v)\rangle - (1-r)g(x,u,v)\}.$$

Lions '85, Dupuis & Ishii '90, Crandall & al. '92, Barles '94, Barron & Jensen '90 and Frankowska '93

Secondly, V^+ and W^+ are viscosity solutions of the equations

(13) $$H^+(x, \nabla V(x)) = 0 \text{ for } x \in \mathcal{D}_0^+$$

and, respectively,

(14) $$\widetilde{H}^+(x, W(x), \nabla W(x)) = 0 \text{ for } x \in \mathbb{R}^N.$$

Here, $H^+, \tilde{H}^+ : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$, are given by:

$$H^+(x,p) = \max_{u \in U} \min_{v \in V} \{-\langle p, f(x,u,v)\rangle - g(x,u,v)\}$$

and, respectively,

$$\widetilde{H}^+(x,r,p) = \max_{u \in U} \min_{v \in V} \{-\langle p, f(x,u,v)\rangle - (1-r)g(x,u,v)\}.$$

Abbreviate

$$
\begin{aligned}
L_T(x_0, u, v) \quad = \quad & -\int_0^T g(x(t), u(t), v(t)) \exp\left(-\int_0^t g(x(\tau), u(\tau), v(\tau)) d\tau \right) dt \\
& + \exp\left(-\int_0^T g(x(\tau), u(\tau), v(\tau)) d\tau \right) W(x(T))
\end{aligned}
$$

with $x(t) = x(t, x_0, u, v)$. Let $\Omega \subset \mathbb{R}^N$ be an open and bounded set and define

$$
t_\Omega(x_0, u, v) := \inf\{t \geq 0 \,|\, x(t) \notin \Omega\}.
$$

Proposition

i) Let W be a continuous supersolution of (14) *in* $\mathbb{R}^N$. *We have*

$$W(x_0) = \sup_{\beta\in\Delta}\ \inf_{u\in\mathcal{U}}\ \sup_{t\in[0,t_\Omega(x_0,u,\beta(u))]} L_t(x_0,u,\beta(u))$$

ii) Let W be a continuous subsolution of (14) *in* $\mathbb{R}^N$. *We have*

$$W(x_0) = \sup_{\beta\in\Delta}\ \inf_{u\in\mathcal{U}}\ \inf_{t\in[0,t_\Omega(x_0,u,\beta(u))]} L_t(x_0,u,\beta(u))$$

Proposition

iii) Let W be a continuous supersolution of (12) *in* $\mathbb{R}^N$*. We have*

$$W(x_0) = \inf_{\alpha \in \Gamma} \sup_{v \in \mathcal{V}} \sup_{t \in [0, t_\Omega(x_0, \alpha(v), v)]} L_t(x_0, \alpha(v), v)$$

(iv) Let W be a continuous subsolution of (12) *in* $\mathbb{R}^N$*. We have*

$$W(x_0) = \inf_{\alpha \in \Gamma} \sup_{v \in \mathcal{V}} \inf_{t \in [0, t_\Omega(x_0, \alpha(v), v)]} L_t(x_0, \alpha(v), v)$$

Theorem

i) Let O be an open set and let $V : O \to \mathbb{R}$ be a continuous viscosity solution of (11) or, respectively, (13). Suppose that V satisfies

$$V(0) = 0 \quad \text{and} \quad V(y) \to \infty \text{ for } y \to x \in \partial O \text{ and for } \|y\| \to \infty.$$

Then $O = \mathcal{D}_0^-$ and $V = V^-$ or, respectively, $O = \mathcal{D}_0^+$ and $V = V^+$.
ii) The function W^- is the unique continuous and bounded viscosity solution of (12) on $\mathbb{R}^N$ with $W^-(0) = 0$.
iii) The function W^+ is the unique continuous and bounded viscosity solution of (14) on $\mathbb{R}^N$ with $W^-(0) = 0$.
Moreover, under Isaacs' condition, i.e., $H^- = H^+$, we have that $V^- = V^+$, $W^- = W^+$ and consequently $\mathcal{D}_0^- = \mathcal{D}_0^+$.

Corollary

The unique continuous and bounded viscosity solutions W^- and W^+ of (12) *and* (14) *characterize $\mathcal{D}_0^-$ and $\mathcal{D}_0^+$ via*

$$\mathcal{D}_0^- = \{x \in \mathbb{R}^N \,|\, W^-(x) < 1\} \quad \text{and} \quad \mathcal{D}_0^+ = \{x \in \mathbb{R}^N \,|\, W^+(x) < 1\}.$$

In our next result we investigate the relation between the Assumptions (H^-) and (H^+).

Theorem

The Assumptions (H^-) and (H^+) satisfy the following properties.

(i) *Assumption (H^+) implies (H^-).*

(ii) *If the condition*

$$\max_{v\in V}\min_{u\in U}\langle p, f(x,u,v)\rangle = \min_{u\in U}\max_{v\in V}\langle p, f(x,u,v)\rangle \quad \text{for all} \quad p\in\mathbb{R}^N \tag{15}$$

holds, then (H^+) is equivalent to (H^-).

Remark

(i) The proof uses the fact that V^+ is a Lyapunov functions for (2). Indeed, besides the characterization of the controllability domains, its ability to deliver Lyapunov functions is another main feature of Zubov's method. More precisely, one can show the following Lyapunov function properties for $V^\pm$ and $W^\pm$, each $x \in \mathcal{D}_0^\pm$ and each $\delta > 0$.

(a) *There exists $\alpha_x \in \Gamma$ such that for all $v \in \mathcal{V}$ and all $t \geq 0$ the inequality*

$$V^-(x(t, x, \alpha_x(v), v) \leq \varphi(t, (1+\delta)V^-(x))$$

holds.

(b) *There exists $\alpha_x \in \Gamma$ such that for all $v \in \mathcal{V}$ and all $t \geq 0$ the inequality*

$$W^-(x(t, x, \alpha_x(v), v) \leq \theta(t, (1+\delta)W^-(x))$$

holds.

(c) For each $\beta \in \Delta$ there exists $u_{x,\beta} \in \mathcal{U}$ such that for all $t \geq 0$ the inequality

$$V^+(x(t, x, u_{x,\beta}, \beta(u_{x,\beta})) \leq \varphi(t, (1 + \delta)V^+(x))$$

holds.

(d) For each $\beta \in \Delta$ there exists $u_{x,\beta} \in \mathcal{U}$ such that for all $t \geq 0$ the inequality

$$W^+(x(t, x, u_{x,\beta}, \beta(u_{x,\beta})) \leq \theta(t, (1 + \delta)W^+(x))$$

holds.

Here the function $\varphi(t, r)$ is constructed as in Grune & Serea '11 and $\theta(t, s) = 1 - e^{-\varphi(t, -\ln(1-s))}$. These functions are independent of δ and converge to 0 monotonically for $t \to \infty$ for all $r \geq 0$ and all $s \in [0, 1)$, respectively.

Under Isaacs' condition (15) the existence and the regularity of the value was studied in by Petrov '70 under the following "transversality or Petrov" condition

$$H^{+}(x, n_x) = H^{-}(x, n_x) < 0$$

for all $x \in B(0, r)$ and all n_x exterior normal to $B(0, r)$ at x, which is stronger than our assumption (H^{+}).

The function η for (H^-) obtained in Grune & Serea '11 will in general be different from the function η in (H^+), and vice versa in (ii). This is due to the fact that in Zubov's method the the function V^+ does not contain information about the rate of controllability η from (H^+). We conjecture that a game theoretic extension of alternative Lyapunov function constructions (like the one in Grune '02) may be used to obtain identical η in both assumptions.

Corollary

Assume that Isaacs' condition (15) *holds and that* (H^+) *and thus also* (H^-) *is satisfied. Then* $\mathcal{D}_0^- = \mathcal{D}_0^+$ *holds.*

In Example 7 we showed by direct arguments that for the vector field

$$f(x, u, v) = -x + uvx^3$$

with $x \in \mathbb{R}$ and $U = V = \{-1, 1\}$ that the controllability domains satisfy $\mathcal{D}_0^- = \mathbb{R} \neq \mathcal{D}_0^+ = (-1, 1)$. Since in Example 7 we showed that both (H^-) and (H^+) hold, Corollary 21 implies that Isaacs' condition (15) must be violated. Indeed, e.g. for $p = 1$ we get

$$\langle p, f(x, u, v)\rangle = -x + uvx^3$$

implying

$$\max_{v \in V} \min_{u \in U} \langle p, f(x, u, v)\rangle = -x - x^3$$

but

$$\min_{u \in U} \max_{v \in V} \langle p, f(x, u, v)\rangle = -x + x^3,$$

hence (15) does not hold.

In fact, for this example and the choice $g(x) = x^2$ the solutions to our equations are readily computable (e.g., using MAPLE) and we obtain

$$W^-(x) = \frac{\sqrt{1+x^2}-1}{\sqrt{1+x^2}}$$

and

$$W^+(x) = \begin{cases} 1-\sqrt{1-x^2}, & |x| \leq 1 \\ 1, & |x| > 1 \end{cases}$$

For $x \in [-5, 5]$, these functions are plotted in Figure 1. Using Corollary 18, these explicit formulas and the plots confirm our computations from Example 7, i.e., $\mathcal{D}_0^- = \mathbb{R}$ and $\mathcal{D}_0^+ = (-1, 1)$.

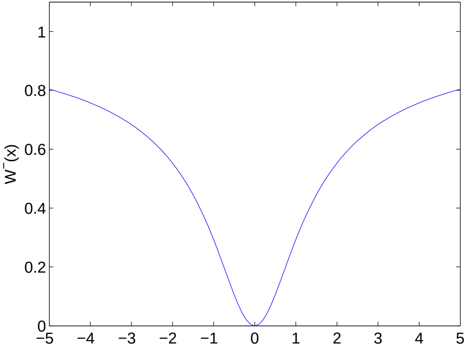

Figure: $W^{-}(x)$ for Example 7.

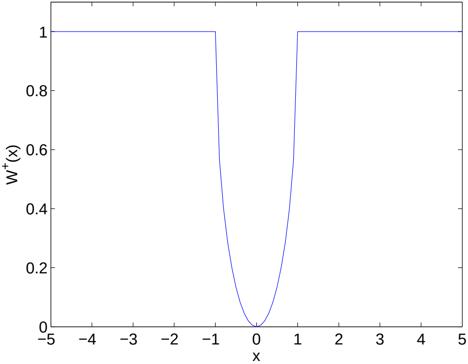

Figure: $W^{+}(x)$ for Example 7.

- Under suitable assumptions we can obtain a way of characterizing $\mathcal{D}_0^{\pm}$ by looking at the minimal time to reach the ball $B(0, r)$. More precisely, $\mathcal{D}_0^{-}$ and $\mathcal{D}_0^{+}$ are open connected, contain $B(0, r)$ and coincide with the domains of the value functions.
- We show that we can alternatively establish a characterization via an integral cost using Zubov type PDE. We formulate the Hamilton-Jacobi equations corresponding to the respective upper and lower value functions — the min-max and max-min Zubov equations — and show the relation between these value functions and the domains of controllability $\mathcal{D}_0^{\pm}$. Finally, under Isaacs' condition $\mathcal{D}_0^{-}$ and $\mathcal{D}_0^{+}$ coincide.

- We provide uniqueness results for the upper and the lower value of the game and for general choices of g. We prove that the game theoretic generalization of the Zubov PDE always has a continuous viscosity solution. This implies the existence of a continuous Lyapunov function from which robustness properties of the asymptotic controllability. Moreover, it implies rigorous convergence results for numerical approximations.
- We studied linearization techniques for controlled piecewise deterministic Markov processes and application to Zubov's method (Goreac & Serea '12)

- The lower and upper domains of null controllability can be defined with respect to nonanticipative strategies with delay. An open question is the equivalence with the definition we employ in Grune & Serea '11. Moreover, the stochastic case is to be considered.
- Generalization of the Zubov method to algebraic (stochastic) differential equations based on Serea, Tambue & Tsafack '23 .