Nonlinear Diffusion Equations driven by Fractional Operators

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Workshop "On nonlocal and nonlinear PDEs"

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Outline

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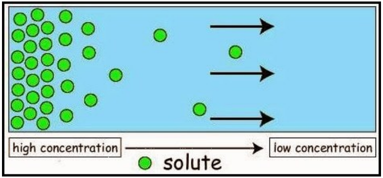
Diffusion

Diffusion

Diffusion equations describe how a continuous medium (say, a population) spreads to occupy the available space.

- Models come from all kinds of applications: fluids, chemicals, bacteria, animal populations, the momentum of a viscous (Newtonian) fluid diffuses, there is diffusion in the stock market,...



Diffusion of particles in a water solution

- So the question is : what is diffusion for a mathematician? how to analyze diffusion mathematically?
  This question has received two quite different answers in recent history.

The two ways to diffusion

The two answers:

- First direction: Is diffusion more or less related to random walk ? This is a correct answer, and this approach leads to Brownian motion and Stochastic Processes, with the famous Ito equation:

$$dx = b\,dt + \frac{1}{2}\sigma dW.$$

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- Understanding this double way has been the source of much effort and the work goes on today.
- Here we will follow the way of Analysis with PDEs, inaugurated by Joseph Fourier (1807, 1822) in an apparently different context, Heat Propagation.

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add: Geometry, Physics, Biology.

The heat equation semigroup and Gauss

- When heat propagates in free space the natural problem is the initial value problem

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$$G(x,t) = (4\pi t)^{-n/2} \exp\left(-|x|^2/4t\right). \tag{2}$$

Note that G has very nice analytical properties for $t > 0$, but note that $G(x,0) = \delta(x)$, a Dirac mass. G works as a kernel (Green, Gauss).

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- Representation. The maps $S_t : u_0 \mapsto u(t) := u_0 * G(\cdot, t)$ form a linear continuous semigroup of contractions in all L^p spaces $1 \le p \le \infty$.
(This is pure Functional Analysis, XXth century)

- Question of Existence and uniqueness. It is very well known in other spaces different from L^1.

The heat equation semigroup II

- Question of Regularity. The maximum principle applies and differentiation of the equation reproduces the equations. Solutions are C^{∞} inside the domain. Free boundaries do not exist in this simple setting.

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- **Asymptotic behaviour as $t \to \infty$, convergence to the Gaussian**. Under very mild conditions on u_0 it is proved that

$$\lim_{t\to\infty} \|u(x,t) - M\, G(x,t)\|_1 = 0, \tag{3}$$

$$\lim_{t\to\infty} t^{n/2}(u(x,t) - M\, G(x,t)) = 0 \tag{4}$$

uniformly, if $M = \int u_0(x)\, dx$. For convergence in L^p less is needed. This is the famous Central Limit Theorem in its continuous form (Probability).

Heat equation graphs. Conflicting views

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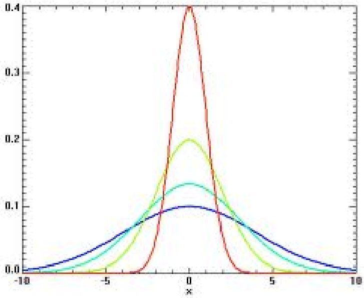
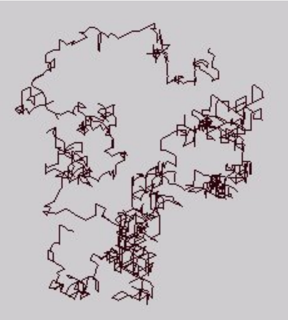
- The comparison of ordered dissipation vs underlying chaos





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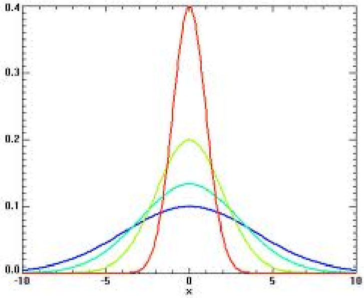

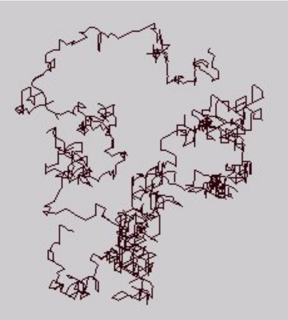

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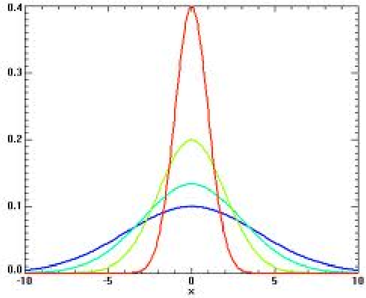

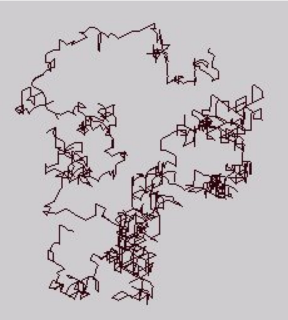

Left, the evolution to a nice Gaussian

Right, a sample of random walk, origin of Brownian motion

Looking for good models. The linear heat Equations

The Heat Equation and the parabolic families of related PDEs

$$u_t = \sum_{ij} a_{ij}\partial_i\partial_j u + \sum_i b_i\partial_i u + cu + f$$

and

$$u_t = \sum_{ij} \partial_i(a_{ij}\partial_j u) + \sum_i \partial_i(b_i u) + cu + f$$

(where (a_{ij}) is a positive definite matrix, possible variable with space and time) are a powerful tool in advanced mathematics.

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The HE and the Parabolic Equation Models have produced a huge number of concepts, techniques and connections for pure and applied science. We talk about the Gaussian function, separation of variables, Fourier analysis, spectral decomposition, Dirichlet forms, Maximum Principles, Brownian motion, generation of semigroups, functional inequalities, positive operators in Banach spaces, entropy dissipation, ...

In that sense these equations serve as a basic training tool for students of different orientations.

Looking for good models. Nonlinear equations

- Let us take a step forward and expand the family of diffusive models in a difficult direction, that of including nonlinearities.
- Indeed, the heat example and the linear models are not representative enough, since many models of science are nonlinear in a form that is very non-linear. A general model of nonlinear diffusion takes the divergence form

$$\boxed{\partial_t H(u) = \nabla \cdot \vec{\mathcal{A}}(x, u, Du) + \mathcal{B}(x, t, u, Du)}$$

with monotonicity conditions on H and $\nabla_p \vec{\mathcal{A}}(x, t, u, p)$ and structural conditions on $\vec{\mathcal{A}}$ and $\mathcal{B}$. Posed in the 1960s (Serrin et al.)

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- In this generality the mathematical theory is too rich to admit a simple description. This includes the big areas of Nonlinear Diffusion and Reaction Diffusion, where I have been working.

Specific nonlinear heat flows

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- Typical nonlinear diffusion: Stefan Problem (phase transition between two fluids like ice and water), ♡

  Hele-Shaw Problem (potential flow in a thin l♡ayer between solid plates),

  Porous Medium Equation: $u_t = \Delta(u^m)$,

  Evolution p-Laplacian Eqn: $u_t = \nabla \cdot (|\nabla u|^{p-2} \nabla u)$. ♡

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- Finally, recall that "elliptic and parabolic problems go together well". Example: Obstacle problems. Caffarelli 1977. ♡

Maestro LUIS CAFFARELLI and the ABEL PRIZE

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• Really big news. For over 40 years Luis Caffarelli has been the driving force of a very large group of researchers all over the world, a truly global endeavor with different capitals, the last one Austin, Texas. A group driven to solve the problems posed by the previous types of applied nonlinear PDEs, specially with Free Boundaries and Degeneracies. In the last 15 years problems with nonlocal operators.

• I was his friend, collaborator and I followed him on land and over the seas in quest of a good problem a good theorem and great conversation.

El gran Luis

2010

2015

Den lille store Luis

2023

2023

Nonlocal operators. Fractional diffusion ♡

- Replacing Laplacians by fractional Laplacians is motivated by the need to represent anomalous diffusion. In probabilistic terms, it replaces next-neighbour interaction of Random Walks and their limit, the Brownian motion, by long-distance interaction rates. This leads in probability to the family of Lévy flights[1]. The main mathematical models in Analysis are the Fractional Laplacians that have special symmetry and invariance properties. More general nonlocal operators can now be considered
- The Basic Stationary and Evolution Equations

$$\boxed{(-\Delta)^s u = f(x,u)} \quad \boxed{u_t + (-\Delta)^s u = 0}$$

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- Historical: Intense work in Stochastic Processes since many decades,

  On the other side, the operator called fractional Laplacian was known in Harmonic Analysis since the the work of Riesz in the 1930s

  But research in Analysis of nonlinear PDEs did not start in force until less than two decades ago. I knew from the work done by and around Prof. Caffarelli in Texas, in particular his seminal work with L. Silvestre[2].

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Recalling the fractional Laplacian operator

- **Different formulas for fractional Laplacian operator**.
  We assume that the space variable $x \in \mathbb{R}^n$, and the fractional exponent is $0 < s < 1$. First, pseudo differential operator given by the Fourier transform:

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- Singular integral operator:

$$\boxed{(-\Delta)^s u(x) = C_{n,s} \int_{\mathbb{R}^n} \frac{u(x)-u(y)}{|x-y|^{n+2s}}\, dy}$$

With this definition, it is the inverse of the Riesz integral operator $(-\Delta)^{-s}u$. This one has kernel $C_1|x-y|^{n-2s}$, which is not integrable in $\mathbb{R}^n$ though it is locally integrable. Basic analysis done $\sim$ 1970 (E. Stein, N. Landkoff).

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- Take the random walk for Lévy processes:

$$u_j^{n+1} = \sum_k P_{jk} u_k^n$$

  where P_{ik} denotes the transition function which has a power tail (i.e, power decay with the distance $|i-k|$). In the limit you get an operator A as the infinitesimal generator of a Levy process:

The fractional Laplacian operator II

if X_t is the isotropic α-stable LÃ©vy process we have

$$Au(x) = \lim_{h\to 0} 1/h\,\mathbb{E}(u(x) - u(x + X_h))$$

- The α-harmonic extension: Find first the solution of the $(n+1)$ problem

  $$\nabla \cdot (y^{1-\alpha}\nabla v) = 0 \quad (x,y) \in \mathbb{R}^n \times \mathbb{R}_+; \quad v(x,0) = u(x), \quad x \in \mathbb{R}^n.$$

  Then, putting $\alpha = 2s$ we have

  $$(-\Delta)^s u(x) = -C_\alpha \lim_{y\to 0} y^{1-\alpha} \frac{\partial v}{\partial y}$$

  When $s = 1/2$ i.e. $\alpha = 1$, the extended function v is harmonic (in $n+1$ variables) and the operator is the Dirichlet-to-Neumann map on the base space $x \in \mathbb{R}^n$. It was proposed in PDEs by Caffarelli and Silvestre, 2007.

- The semigroup formula in terms of the linear heat flow generated by Δ:

  $$(-\Delta)^s f(x) = \frac{1}{\Gamma(-s)} \int_0^\infty \left(e^{t\Delta} f(x) - f(x)\right) \frac{dt}{t^{1+s}}.$$

A detailed text expanding my previous work and a basic theory of nonlinear and nonlocal diffusion :

J. L. Vázquez. *The mathematical theories of diffusion. Nonlinear and fractional diffusion*,

♣ appeared as Chapter 5 of Springer Lecture Notes in Mathematics 2186, “Nonlocal and Nonlinear Diffusions and Interactions: New Methods and Directions”, CIME Summer Course in Cetraro, Italy, 2016; pp. 205–278.

Volume Authors: J.A. Carrillo, M. del Pino, A. Figalli, G. Mingione, JLV.
Editors: M. Bonforte, G. Grillo.

SECOND PART

𝔖*ummary:* A glimpse on recent work

in the world of nonlinear evolution flows

where semigroups do not have a Green function

but they have attracting asymptotics

that solve unusual elliptic problems

Gagliardo seminorms

- Let us present some recent contributions of the author. In the first topic we present today, the nonquadratic and nonlocal choice is quite simple. The nonlocal energy functional

$$\boxed{\mathcal{J}_{p,s}(u) = \frac{1}{p}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}\,dxdy\,.} \qquad (5)$$

is a power-like functional with nonlocal kernel of the s-Laplacian type that has attracted a great deal of attention in recent years.

It is just the p-power of the Gagliardo seminorm, used in the definition of the $W^{s,p}$ spaces (fractional Sobolev, Slobodeckii or Gagliardo spaces)

$$[u]^p_{s,p} = p\mathcal{J}_{p,s}(u), \qquad \|u\|^p_{s,p} = \int |u|^p\,dx + p\mathcal{J}_{p,s}(u).$$

We may consider it for exponents $0 < s < 1$ and $1 < p < \infty$, in $N \geq 1$. [3]

[3] Di Nezza - Palatucci - Valdinoci '12 – Hitchhiker's guide to the fractional Sobolev spaces.

Gagliardo seminorms and subdifferential operator

- The nonquadratic and nonlocal energy functional can be written as

$$\mathcal{J}_{p,s}(u) = \frac{1}{p} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \left(\frac{|u(x) - u(x+z)|}{|z|^s} \right)^p \frac{dz}{|z|^N} \, dx. \tag{6}$$

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  is a power-like functional with nonlocal kernel of the s-Laplacian type that has attracted a great deal of attention in recent years. It is just the p-power of the Gagliardo seminorm, used in the definition of the $W^{s,p}$ spaces

- Its subdifferential (or Euler-Lagrange operator) $\mathcal{L}_{s,p}$ is the nonlinear operator defined a.e. by the formula

$$\mathcal{L}_{s,p}(u):=P.V.\int_{\mathbb{R}^n}\frac{\Phi(u(x,t)-u(y,t))}{|x-y|^{N+sp}}\,dy\,, \tag{7}$$

  where $\Phi(z)=|z|^{p-2}z$. Called s-fractional p-Laplacian operator. By general theory $\mathcal{L}_{s,p}$ is a maximal monotone operator in $L^2(\mathbb{R}^n)$ with dense domain.

I. Gradient flow in the superlinear case

- In a first paper : The evolution fractional p-Laplacian equation in $\mathbb{R}^n$. Fundamental solution and asymptotic behaviour [4] we study the corresponding gradient flow, i.e., the evolution equation

$$\boxed{\partial_t u + \mathcal{L}_{s,p} u = 0\,,} \tag{8}$$

posed in the Euclidean space $x \in \mathbb{R}^n$, $N \geq 1$, for $t > 0$. We will often refer to it as the EFPL equation (evolution fractional p-Laplacian equation). We put initial datum

$$\lim_{t\to 0} u(x,t) = u_0(x), \tag{9}$$

where in principle $u_0 \in L^2(\mathbb{R}^n)$. It is not difficult to prove that this Cauchy problem is well-posed in all $L^q(\mathbb{R}^n)$ spaces, $1 \leq q < \infty$. This parallels what is known in the case of bounded domains. In fact, the problem generates a continuous nonlinear semigroup in any $L^q(\mathbb{R}^n)$ space, $1 \leq q < \infty$, a nonexpansive semigroup. We define the class of *continuous strong solutions* that correspond to L^2 and L^1 initial data and derive its main properties in detail as regularity and boundedness.

Remark. The study in bounded domain is different, see e.g. paper by Mazón et al. + the JLV paper in JDE (both in 2016). That theory is different from what follows.

[4] JLV, Nonlinear Analysis, 2020.

Selfsimilar fundamental solutions. Solitons of the Problem

Theorem 1 (Existence and uniqueness)

Let $p > 2$*. For every given mass* $M > 0$ *there exists a unique self-similar solution of Problem* (8)-(9) *with* u_0 *a Dirac delta. It has the form*

$$U(x,t;M) = M^{sp\beta}t^{-\alpha}F(M^{-(p-2)\beta}x\,t^{-\beta})\,, \qquad (10)$$

with self-similarity exponents

$$\alpha = \beta N, \quad \beta = \frac{1}{N(p-2)+sp}\,. \qquad (11)$$

The difficulty lies in finding the profile $F(r)$*. It is a continuous, positive, radially symmetric (*$r = |x|\,t^{-\beta}$*), and decreasing function such that* $F(r) \approx r^{-(N+sp)}$ *as* $r \to \infty$*.*

So there is a known fractional diffusive rate for the fundamental solutions of this semigroup. And it has a definite asymptotic shape that is NOT Gaussian.

The graphics for different diffusion rates

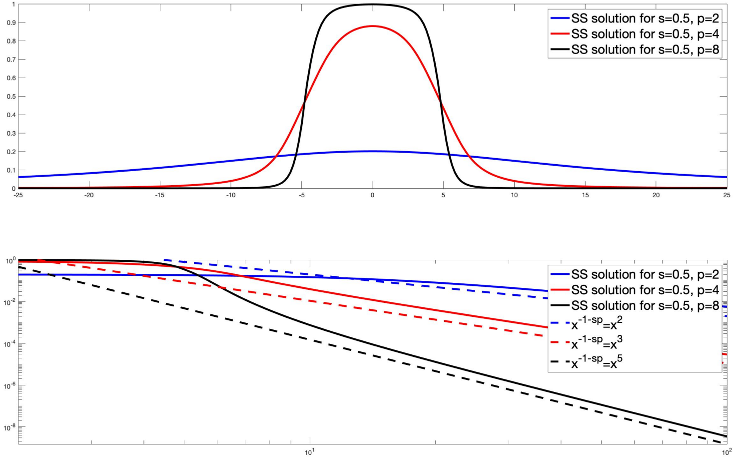

Figure: Self-similar fundamental solutions for different p, with $s = 0.5$. The profiles are computed in dimension $N = 1$.

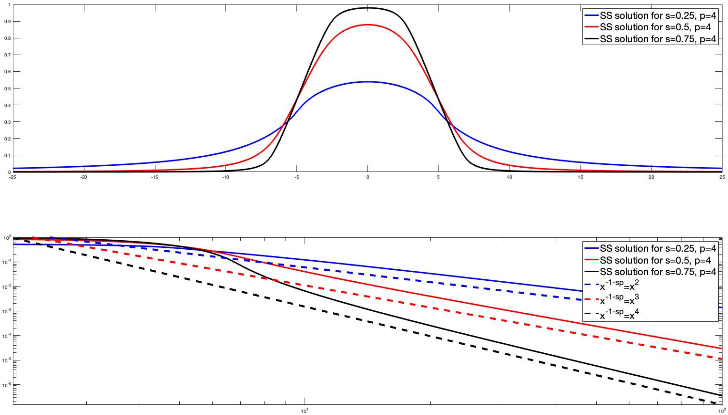

Figure: Self-similar fundamental solutions for different s, with $p = 4$.

Computed graphics. The second picture in each figure shows clearly the predicted decay with exponent $1 + sp$ using the logarithmic scale. Also to be remarked the flat behaviour of the profile near the origin for large values of p.

Asymptotics

The fundamental solution is the key to the study of the long-time behaviour of our problem with general initial data, since it represents the *intermediate asymptotics,* as in Barenblatt's Self-similarity book [5]. This is the sharp asymptotic result we obtain.

Theorem 2 (Asymptotic behaviour)

Let u be a solution of Problem (8)-(9) *with initial data* $u_0 \in L^1(\mathbb{R}^n)$ *of integral M, and let* U_M *be the fundamental solution with that mass. Then,*

$$\lim_{t\to\infty} \|u(t) - U_M(t)\|_1 = 0\,. \tag{12}$$

We also have the L^∞*-estimate*

$$\lim_{t\to\infty} t^{\alpha} \|u(\cdot,t) - U_M(\cdot,t)\|_\infty = 0\,. \tag{13}$$

There is no restriction on the sign of the solution. By interpolation, we can are easily obtain rates in all L^q spaces, $1 < q < \infty$. Of course, for $M = 0$ we just say that $\|u(\cdot,t)\|_1$ goes to zero.

[5] G.I. Barenblatt. Similarity, self-similarity, and intermediate asymptotics (1978, 1979, 1996).

Details

- The construction we do here needs the standard toolbox plus special tricks the so-called standard evolution tools that have to be proved are
  - ▷ the L^1-L^∞ effect,
  - ▷ conservation of mass,
  - ▷ derivative estimates that imply compactness,
  - ▷ scaling transformations, i. e., group invariance
  - ▷ strict positivity
  - ▷ and a Lyapunov functional to measure dissipation and prove uniqueness
- The most novel tool is the existence of a sharp upper barrier, which is an explicit function that mimicks the decay behaviour that will be guessed a priori and is a supersolution to the equation. See more detail below.

  Together they produce the main theorems.

  We also get an important global Harnack inequality.Regularity questions for this flow are still in process.

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  Together they produce the main theorems.

  We also get an important global Harnack inequality.Regularity questions for this flow are still in process.
- The limits $s = 1$ and $p = 2$ give the expected results: plain p-Laplacian, resp. linear fractional equation.

"Fast range" of the fractional p-Laplacian equation

- There is a companion paper, dealing with the sublinear case $1 < p < 2$. [6]
- The "fast" or "superdiffusive" case $1 < p < 2$ may look like an extension of the study for $p > 2$, and the papers begin in a similar way, but experts know better: fast diffusion is very tricky, *the faster the trickier.*

[6] JLV; The fractional p-Laplacian evolution equation in $\mathbb{R}^N$ in the sublinear case. Calc. Var. PDES 60 (2021); arXiv:2011.01521.

[7] See e.g. JLV; Smoothing and decay estimates for nonlinear diffusion equations. Oxford University Press, 2006.

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- The analysis of the most classical fast diffusion model (Fast Porous Medium Equation) shows that there is a nice theory for powers not far from the linear case, and a much more complicated theory farther away from linearity[7]. Here, this stranger behaviour happens for $1 < p < p_c$, where $p_c = 2N/(N+s)$.
- The work we are presenting now takes large space to explain, about 60 pages of the journal. It only covers systematically the good range $p_c < p < 2$ (i.e., the case close to 2).
- Recall that we want the linear case $p = 2$ to be recovered as a limit when $p \to 2$. Indeed, such limits work well and fit with the linear fractional case done before.

[6] JLV; The fractional p-Laplacian evolution equation in $\mathbb{R}^N$ in the sublinear case. Calc. Var. PDES 60 (2021); arXiv:2011.01521.

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Ideas of proof. Renormalized flow (proper rescaling)

It is interesting to interpret this asymptotic Theorem in terms of the rescaled variables defined as follows. We apply a zooming to the original solution according to the self-similar exponents α and β fixed above. More precisely, the change uses the formulas

$$u(x,t) = (t+a)^{-N\beta} v(y,\tau) \quad y = x\,(t+a)^{-\beta}, \quad \tau = \log(t+a), \tag{14}$$

with $\beta = (N(p-2)+sp)^{-1}$, and any constant $a > 0$, we mostly use $a = 1$. Here, τ is called the *new time* or logarithmic time. The formulas imply that $v(y,\tau)$ is a solution of the corresponding PDE:

$$\partial_\tau v + \mathcal{L}_{s,p} v - \beta \nabla \cdot (y\, v) = 0\,, \tag{15}$$

called the renormalized flow. This transformation is also called *continuous-in-time rescaling* to mark the difference with the scaling transformations with fixed parameter.

• Note that the mass of the v solution at new time $\tau \ge \tau_0$ equals that of the u at the corresponding time $t \ge 0$. Important physical conserved quantities are also conserved upon renormalization.

Renormalized flow II

• In our way of proof of the asymptotic theorem, we rephrase it as saying that the rescaled solution $v(y,\tau)$ converges to the equilibrium state $F_M(y)$ of the flow equation (15) in all L^q-norms, $1 \le q \le \infty$. Indeed, F_M solves the nonlinear and nonlocal elliptic equation

$$\partial_\tau v = 0, \qquad \mathcal{L}_{s,p} v - \beta \nabla \cdot (y\, v) = 0\,. \tag{16}$$

We prove that F_M attracts along this renormalized flow all finite-mass solutions with the same mass. The corresponding results for the standard p-Laplacian, were proved by Kamin-V (1988) and V (2003).

• We work on the renormalized flow to obtain a stationary upper bound in the form of an explicit supersolution or barrier. It is there that we have to choose the options in fast diffsion, (i) $p_c < p < p_1$ or (ii) $p_1 < p < 2$, and one in slow diff $p > 2$, to actually construct the supersolution that decays as needed (otherwise, it does not work). We the use a fine L^1 Lyapunov functional that needs delicate estimates.

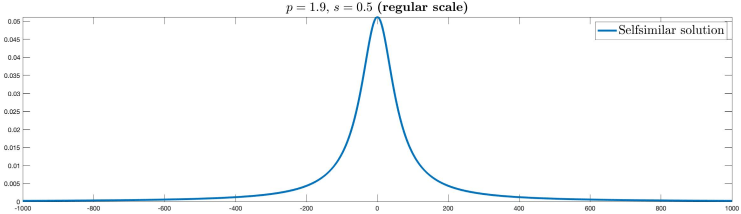
Self-similarity and asymptotics. Related results

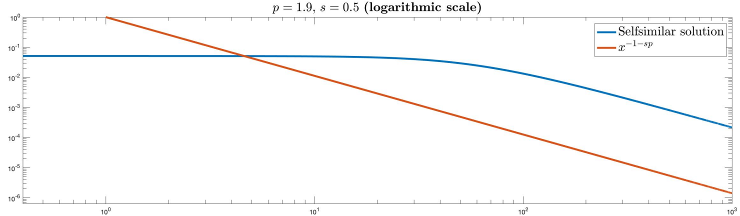
- Fundamental solutions are a basic ingredient in the theory of the heat equation (Gaussian profile) and other linear parabolic equations posed in the whole space. They also work for the linear fractional heat equation (JLV, 2018).
- In nonlinear diffusion there is no representation theorem, but fundamental solutions have been constructed in many cases. First, by Barenblatt et al. for porous medium and p-Laplacian equation, and then the doubly nonlinear equation. Solutions then were self-similar and explicit.
- Asymptotic attraction theorems for the previous items are proved at different levels in the standard local case. They are difficult for Fast diffusion (Carrillo-V 2003, Blanchet et al, 2009). This means $p < 2$ or $m < 1$
- For nonlinear fractional diffusion, I have constructed the fundamental solution for FPME, Model 1 (with Caffarelli, 2011), and also for FPME model 2 (JLV, 2014). Asymptotic theorems are proved, each needs a different method. Rates of convergence are obtained only in special cases using Functional Inequalities for the associated entropies. Like in [8]. The problem for the s-p Laplacian evolution is open.

[8] Carrillo; Huang; Santos; VÃ¡zquez. Exponential convergence towards stationary states for the 1D porous medium equation with fractional pressure (2015)

Numerics

The following figures show the profiles of the self-similar fundamental solutions in the two ranges of *s* and *p*. They were computed by numerically integrating the evolution equation starting with smooth initial data with compact support.





Numerics II

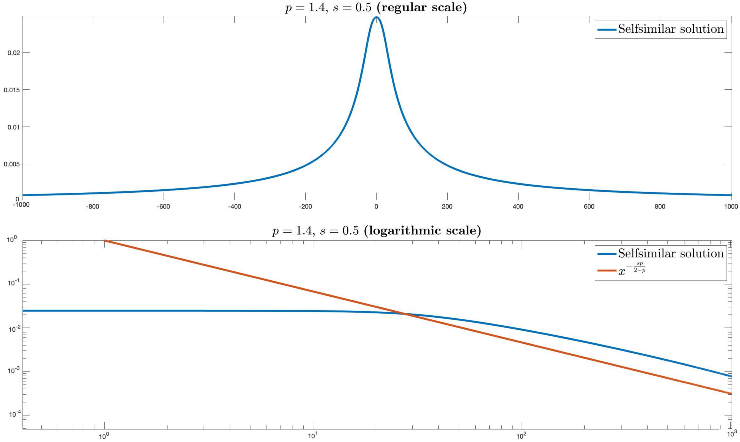


Figure: Self-similar fundamental solution for $s = 0.5$ and $p = 1.4 < p_1$. The second graphic in logarithmic scale shows clearly the decay with exponent $-sp/(2-p)$.

THE LAST FAMILY OF MODELS

New type of *p*-Laplacians

Tug-of-war games in Probab. and PDEs

The infinity Laplacian

- The so-called infinity Laplacian is defined for smooth functions as the operator

$$\Delta_\infty u = |\nabla u|^{-2} \sum_{i,j} \partial_i u \, \partial^2_{ij} u \, \partial_j u = \langle D^2 u \frac{Du}{|Du|}, \frac{Du}{|Du|} \rangle. \qquad (17)$$

  Informally, this is the second derivative of u in the direction of the gradient of u. **Note**. Observe that with the square gradient normalization, Δ_∞ coincides with the ordinary Laplacian Δ in the one-dimensional case.

- The idea is to pass to the limit $p \to \infty$ in the standard p-Laplacian operator.

$$\Delta_p u = \nabla \cdot (|\nabla u|^{p-2} \nabla u). \qquad (18)$$

  Note the lack of power agreement. Renormalize using the factor $|\nabla u|^{2-p}$.

- The solutions of $\Delta_\infty u = 0$ in an open domain $U \subset \mathbb{R}^n$ were studied by G. Aronsson in 1967, 1968. The type and existence of solutions depend on the type of boundary data $u = g$ on ∂U. Aronsson showed the equivalence of the classical solutions of this Dirichlet problem with the solutions of the Problem of Absolute Minimizing Extension such that the norm is kept, $Lip_U(u) = Lip_Y(g)$ where $Y = \partial U$. Jensen proved the general uniqueness of Lipschitz extensions minimizing the sup norm of the gradient, in 1993.

- **Note 2**. In the study of the homogenous equation $\Delta_\infty u = 0$, the normalizing factor $|\nabla u|^{-2}$ is sometimes omitted; however, it is important to include it in the non-homogenous equation $\Delta_\infty u = f$ or in parabolic problems $u_t = \Delta_\infty u$.

The tug of war approach

- In a famous paper appeared in J. Amer. Math. Soc. (2009), Y. Peres, O. Schramm, S. Sheffield, and D. B. Wilson introduced the derivation of the infinity Laplacian by means of a stochastic process called *Tug-of-war* . More precisely, the standard infinity Laplace equation is solved by the value function for a two-players random turn “tug-of-war” game.
  The game is as follows: a token is initially placed at a position $x_0 \in \Omega$, and every turn a fair coin is tossed to choose which of the players plays. This player moves the token to any point in the ball of radius $\varepsilon > 0$ around the current position. If, eventually, iterating this process, the token reaches a point $x_e \in \partial\Omega$, the players are awarded (or penalized) $g(x_e)$ (payoff function).
- The article provides a game theoretic point of view for understanding the infinity Laplacian equation and a new proof of Jensen's influential uniqueness result, this time based on a probabilistic approach. Recall that there is an old probabilistic interpretation for the harmonic functions, $\Delta u = 0$, based on Brownian motion (Wiener-Lévy process). Cabré likes that! There is no need for two players.
- Two basic references for infinity Laplacians:
  M. G. Crandall, *A visit with the ∞-Laplace equation.* Lecture Notes in Math., 1927, Springer, Berlin, 2008.
  P. Lindqvist. *Notes on the infinity Laplace equation.* Springer, 2016.

The tug of war, parabolic setting

- The parabolic problem was studied by several authors. We have been motivated in the work below by two papers of Portilheiro and JLV, one in CPDE (2021) and CPDE (2013) dealing respectively with

$$u_t = \Delta_\infty(u^m)$$

both in bounded domains and the whole space, and the second in CPDE (2013) dealing with $u_t = \Delta^h_\infty u$, where Δ^h_∞ is the h-homogeneous operator associated with the infinity-Laplacian,

$$\Delta^h_\infty u = |\nabla u|^{h-3} \sum_{i,j} \partial_i u \, \partial^2_{ij} u \, \partial_j u \qquad h > 1.$$

In the papers viscosity solutions were used to show well-posedness.

The study of this model in $\mathbb{R}^n$ also showed an interesting large time behaviour: solutions with bounded L^1 data arrive at an asymptotic radial shape with a one-dimensional profile.

The infinity fractional Laplacian. Elliptic problem

- Bjorland, Caffarelli and Figalli introduced in a paper in CPAM (2012) equations involving the so-called *infinity fractional Laplacian* as a model for a nonlocal version of the "tug-of-war" game. Following their explantation, instead of flipping a coin at every step, every player chooses a direction and it is an s-stable Levy process that chooses both the active player and the distance to travel. The corresponding operator is the infinity fractional Laplacian, with symbol Δ^s_∞, it is a nonlinear integro-differential operator given by

$$\Delta^s_\infty \phi(x) := C_s \sup_{|y|=1} \inf_{|\tilde{y}|=1} \int_0^\infty \left(\phi(x+\eta y) + \phi(x-\eta\tilde{y}) - 2\phi(x)\right) \frac{\mathrm{d}\eta}{\eta^{1+2s}}, \qquad (19)$$

  where $s \in (1/2, 1)$. The constant $C_s > 0$ is irrelevant here. There are alternative definitions. Note that the integration is one dimensional, along straight lines. Sup and inf are shown to be taken in the direction of the gradient. The operator does not look like a limit of s-p-Laplacian as introduced before. The infinity limit is tricky.

- In their paper the authors study two stationary problems involving the infinity fractional Laplacian posed in bounded space domains, namely, a Dirichlet problem and a double-obstacle problem.

  Uniqueness and comparison of viscosity solutions are widely open problems.

The parabolic equation for the infinity fractional Laplacian

- Lastly, I want to announce the results of a new paper: "Evolution Driven by the Infinity Fractional Laplacian", a collaboration by Félix del Teso, Jorgen Endal, Espen Jakobsen, and J.L.V, appeared in Arxiv paper .
- We consider the evolution problem

$$\partial_t u(x,t) = \Delta_\infty^s u(x,t), \qquad x \in \mathbb{R}^n, t > 0, \qquad (20)$$

$$u(x,0) = u_0(x), \qquad x \in \mathbb{R}^n, \qquad (21)$$

  with $s \in (1/2, 1)$. We consider spatial dimension $n \geq 2$, since for $n = 1$ the operator $-\Delta_\infty^s$ is just the usual linear fractional Laplacian operator $(-\Delta)^s$ of order s, and equation (20) is just the well-known fractional heat equation, well studied in the literature. Note that for $n \geq 2$ the operator is nonlinear, so a new theory is needed.
- Firstly, we develop an existence theory of suitable viscosity solutions for the parabolic problem (20)–(21), based on approximation with monotone schemes. We show that the obtained class of solutions enjoys a number of good properties. As in the elliptic case (BjCaFi 2012), we lack a general comparison and uniqueness result in the context of viscosity solutions.

Project Coauthors

Espen, Felix, Jorgen, Juan Luis in June in Bergen, Norway

The parabolic equation II

- However: we are able to prove an important comparison theorem relating two types of solutions, classical and viscosity ones.

Theorem 3 (Comparison with smooth solutions)

• *Let* $u \in C_b^\infty(\mathbb{R}^n \times [0,\infty))$ *be a pointwise solution with initial data* u_0.
• *Let* $\overline{u}, \underline{u} \in BUC(\mathbb{R}^n \times [0,\infty)$ *be constructed viscosity solutions with initial data* $\overline{u}_0, \underline{u}_0$.
If $\underline{u}_0 \le u_0 \le \overline{u}_0$ *in* $\mathbb{R}^n$.

$$\text{Then } \underline{u} \le u \le \overline{u} \text{ in } \mathbb{R}^n \times (0,\infty).$$

Moreover, we show that for smooth, radially symmetric functions and nonincreasing along the radius, the operator $-\Delta_\infty^s$ reduces to the classical fractional Laplacian $(-\Delta)^s$ in dimension 1. Note that no similar reduction applies to more general functions, even in the radial case (we produce a radial counterexample).

Classical solutions

Theorem 4

Assume that $\phi \in C^{1,1}(x) \cap B(\mathbb{R}^n)$ is radial, i.e., $\phi(x) = \Phi(|x|)$ for all $x \in \mathbb{R}^n$, where $\Phi : \mathbb{R} \to \mathbb{R}$ is such that $\Phi(-r) = \Phi(r)$ for all $r \geq 0$ and Φ is radially nonincreasing, i.e., $\Phi(r_1) \geq \Phi(r_2)$ if $0 \leq r_1 \leq r_2$. Then,

$$\Delta_\infty^s \phi(x) = -(-\partial_{rr}^2)^s \Phi(|x|).$$

- **Proof:** Assume $\nabla\phi(x) \neq 0$. Recall that $v := \frac{\nabla\phi(x)}{|\nabla\phi(x)|} = \pm\frac{x}{|x|}$.

$$\begin{aligned}\Delta_\infty^s \phi(x) &= C_s \int_0^\infty \left(\phi\left(x+\eta v\right) + \phi\left(x-\eta v\right) - 2\phi(x)\right)\frac{\mathrm{d}\eta}{\eta^{1+2s}} \\ &= C_s \int_0^\infty \left(\Phi\left(r+\eta\right) + \Phi\left(r-\eta\right) - 2\Phi(r)\right)\frac{\mathrm{d}\eta}{\eta^{1+2s}} = -(-\partial_{rr}^2)^s \Phi(|x|).\end{aligned}$$

Now let $\nabla\phi(x) = 0$. Note that $\sup_{z\in\partial B_\eta(x)}\{\phi(z)\} = \phi(x - \eta\frac{x}{|x|})$. Then

$$\sup_{|y|=1} \int_0^\infty \left(\phi\left(x+\eta y\right) - \phi(x)\right)\frac{\mathrm{d}\eta}{\eta^{1+2s}} = \int_0^\infty \left(\phi\left(x - \eta\frac{x}{|x|}\right) - \phi(x)\right)\frac{\mathrm{d}\eta}{\eta^{1+2s}}. \qquad \blacksquare$$

Classical solutions

Theorem 5

Let $u_0 \in C_b^\infty(\mathbb{R}^n)$ *be radially symmetric and radially nonincreasing.*

- *Then there exists a pointwise solution* $u \in C_b^\infty(\mathbb{R}^n \times [0,\infty))$.
- *Explicit form for the solution: Let* $U_0 : \mathbb{R} \to \mathbb{R}$ *be defined by* $U_0(r) := u_0(|x|)$ *with* $r = |x|$ *and* $U(-r) := U(r)$*, then*

$$u(x,t) = (P_s(\cdot,t) * U_0)(r) = \int_{-\infty}^{\infty} P_s(r-y,t)U_0(y)\mathrm{d}y \qquad \text{for all } |x| = r,$$

where P_s *is the fundamental solution of the one dimensional fractional heat equation. .*

Classical solutions

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Comments:

• Now we can compare with radially decreasing solutions of the one dimensional fractional heat equation.

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where P_s is the fundamental solution of the one dimensional fractional heat equation. .

Comments:

• Now we can compare with radially decreasing solutions of the one dimensional fractional heat equation.

• The assumption “radially nonincreasing" cannot removed. We can build counterexamples where

$$\Delta_\infty^s \phi(x) \neq -(-\partial_{rr}^2)^s \Phi(|x|).$$

The global Harnack result

- Having established the existence of a large class of classical solutions, and the comparison of classical and smooth solutions, and recalling the convergence of solutions of the 1D fractional heat equations to its fundamental solution, $P_s(r,t)$,

Theorem 6 (Global Harnack principle)

Let $u \in BUC(\mathbb{R}^n \times [0,\infty))$ be a viscosity solution of (20)–(21), as constructed by the approximation scheme, with initial data $u_0 \in BUC(\mathbb{R}^n)$ such that $u_0 \not\equiv 0$ and

$$0 \le u_0(x) \le (1+|x|^2)^{-\frac{1+2s}{2}} \qquad \text{for all } |x| \ge R \ge 1.$$

Then, for all $\tau > 0$,

$$C_1 P_s(|x|,t) \le u(x,t) \le C_2 P_s(|x|,t) \qquad \text{for all } (x,t) \in \mathbb{R}^n \times [\tau,\infty),$$

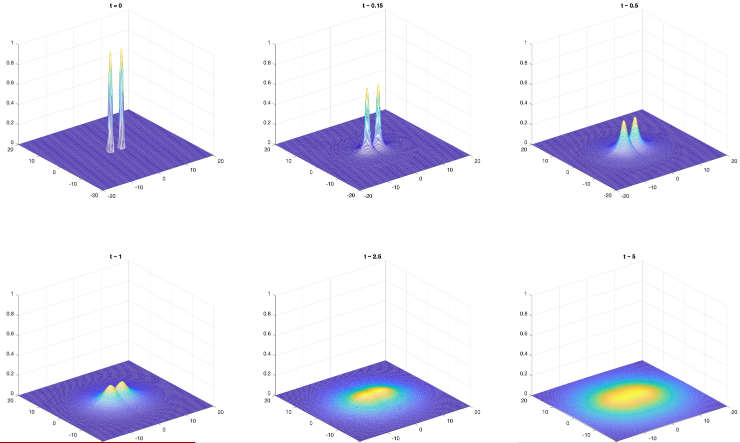
where $C_1, C_2 > 0$ are constants depending only on s, R, u_0. Moreover, for all $\tau > 0$,

$$\boxed{\tilde{C}_1 \frac{t}{(t^{\frac{1}{s}} + |x|^2)^{\frac{1+2s}{2}}} \le u(x,t) \le \tilde{C}_2 \frac{t}{(t^{\frac{1}{s}} + |x|^2)^{\frac{1+2s}{2}}}}$$

for all $(x,t) \in \mathbb{R}^n \times [\tau,\infty)$, where $\tilde{C}_1, \tilde{C}_2 > 0$ are constants dep. on s, R, and u_0.

Numerics

Finally, we propose a fully discrete and monotone finite-difference scheme, and support our theoretical results with numerical evidence, see paper. To our knowledge, there is no previous study of numerical implementations of the infinity fractional Laplacian operator, either in the elliptic or parabolic setting.



The saga of diffusive evolutions and its stationary states goes on unstoppable. It is very active in many countries (like USA, France, Spain, and Norway).

In her Abel Laudatio lecture on May 24 Professor Sylvia Serfaty talked about "From diffusions to fluid equations: the question of regularity"in honor of Luis Caffarelli' seminal work that inspired so many of us.

The Abel lectures: University of Oslo, Georg Sverdrups hus, Oslo.

Geometry appears as free boundaries, but that is another tale you have probably heard a lot in Zurich and/or Barcelona. Many open problems. Professor Alessio Figalli talked in Oslo about "From elastic membranes to ice melting".

◊

Thank you for your attention

Tusen takk, Muchas gracias

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