A convergent discretization of the porous medium equation with fractional pressure

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Based on a joint work with:
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1 Introduction

2 Numerical scheme (Part 1)

3 Numerical scheme (Part 2)
Porous Medium Equation / Fast Diffusion Equation

\[ u_t - \Delta u^m = 0 \quad \text{in} \quad \mathbb{R}^N \times (0, T) \quad \text{(PME/FDE)} \]

Self Similar solutions:
\[ \mathcal{U}(x, t) = t^{-\alpha} F(|x|^t)^{1-\beta} \]

**Finite/Infinite** speed of propagation with \( m = 1 \) as borderline cases

**Slow Diffusion**: \( m > 1 \)
\[ F(y) \sim (R^2 - |y|^2)^{1/(m-1)} \]

**Fast Diffusion**: \( m < 1 \)
\[ F(y) \sim (R^2 + |y|^2)^{-1/(1-m)} \]

Reference: Vázquez’s book 2007 on the PME
Fractional porous medium models

\[ u_t - \Delta u^m = 0 \]

\[ \downarrow \]

\[ u_t + (-\Delta)^s u^m = 0 \]

de Pablo, Quirós, Rodríguez and Vázquez

Infinite speed of propagation for all \( m > 0 \).
Introduction

Fractional porous medium models

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deb Pablo, Quirós, Rodriguez
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**Infinite** speed of
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\[ u_t = \nabla \cdot \left( u^{m-1} \nabla u \right) \]

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Caffarelli and Vázquez (\( m = 2 \))
& Stan, del Teso, and Vázquez

\( m \in (1, \infty) \)
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Biler, Imbert, Karch and Monneau

Finite speed of propagation for all \( m > 1 \).
\[ u_t = \nabla \cdot \left( u^{m-1} \nabla (-\Delta)^{-\sigma} u \right) \]

Continuity equation: \[ u_t = -\nabla \cdot (u \mathbf{v}) \]

\[ u \equiv \text{density (mass per unit volume)}, \quad \mathbf{v} \equiv \text{flow velocity field}. \]

In this case

\[ \mathbf{v} = u^{m-2} \nabla (-\Delta)^{-\sigma} u \]
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Some observations:

- The velocity field $\mathbf{v}$ is degenerated if $m > 2$ and singular if $m < 2$. 

Consequences:

- Finite speed of propagation if $m > 2$ (also $m = 2$, Caffarelli and Vázquez 11').
- Infinite speed of propagation if $m \in (1, 2)$.
- No comparison principle (Caffarelli and Vázquez 11').
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**Some observations:**

- The velocity field \( \mathbf{v} \) is degenerated if \( m > 2 \) and singular if \( m < 2 \).
- The velocity field is nonlocal.

\[
\nabla (-\Delta)^{-\sigma} f(x) = \nabla^{1-2\sigma} f(x) = \int (f(x) - f(y)) \frac{x - y}{|x - y|^{N+2-2\sigma}} \, dy
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![Graphs showing different times](image)
\[ \partial_t u - \nabla \cdot (u^{m-1} \nabla (-\Delta)^{-\sigma} u) = 0, \]
\[ u(\cdot,0) = \mu_0. \]

\underline{Weak solution}

\[ \int_0^T \int_{\mathbb{R}^d} u\phi_t - \int_0^T \int_{\mathbb{R}^d} u^{m-1} \nabla (-\Delta)^{-\sigma} u \nabla \phi + \int_{\mathbb{R}^d} \phi(x,0)d\mu_0(x) = 0, \]
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\textbf{Properties (Stan, dT & Vázquez – 19')}

- Existence of nonnegative weak solutions for \( \mu_0 \in \mathcal{M}_+(\mathbb{R}^d) \).
- Conservation of mass.
- \( L^\infty \)-decay
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- \( L^1 - L^\infty \) smoothing effect: \( \|u(\cdot, t)\|_{L^\infty(\mathbb{R}^d)} \leq C_{N,s,m} t^{-\gamma} \mu_0(\mathbb{R}^d) \delta. \)
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- \( L^1 - L^\infty \) smoothing effect: \( \| u(\cdot, t) \|_{L^\infty(\mathbb{R}^d)} \leq C_{N, s, m} t^{-\gamma} \mu_0(\mathbb{R}^d)^\delta. \)
- \( (dT, Jakobsen) \) Tightness: For every \( \varepsilon, T > 0 \) there exists \( R = R(\varepsilon, \mu_0, T) > 0 \) s.t.

\[ \sup_{t \in [0, T]} \int_{|x| > R} u(x, t) dx < \varepsilon. \]
Cumulative density in dimension $d = 1$

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Let us consider the cumulative density function,

$$v(x, t) = \int_{-\infty}^{x} u(y, t) \, dy$$

so that $$v_x(x, t) = u(x, t).$$
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It satisfies

\[ (\partial_x v)_t = \partial_x \left( (\partial_x v)^{m-1} \partial_x (-\partial_{xx})^{-\sigma} (\partial_x v) \right) \]

\[ \quad \underbrace{\partial_x (-\partial_{xx})^{1-\sigma}} \]
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Integrating the equation

\[ \partial_t v + (\partial_x v)^{m-1} (-\Delta)^s v = 0, \]

with $s = 1 - \sigma$. 
\[ \partial_t v + (\partial_x v)^{m-1}(-\Delta)^s v = 0 \]

- The equation, for \( m = 2 \), was first studied in (Biller, Karch and Monneau 10’) as a model for nonlinear dislocation dynamics.
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- It is not in divergence form, and well-posedness has to be understood in the viscosity solution sense.
- A general class of nonlocal quasilinear equations including this one was studied in (Chasseigne and Jakobsen 17'):

\[
L[\phi](x) = \int_{|z|>0} (\phi(x + f(\nabla \phi(x))z) - \phi(x)) \frac{dz}{|z|^{d+2s}} = -f(\nabla \phi(x))^{2s} (-\Delta)^s \phi(x)
\]
\[ L[\phi] = (\partial_x \phi)^{m-1} (-\Delta)^s \phi \]

- Let us recall the definition of the \( p \)-Laplacian

\[ \Delta_p \phi = \nabla \cdot (|\nabla \phi|^{p-2} \nabla \phi) \]

that in dimension \( d = 1 \) can we expressed as

\[ \Delta_p \phi = |\partial_x \phi|^{p-2} \partial_{xx} \phi \]
$L[\phi] = (\partial_x \phi)^{m-1} (-\Delta)^s \phi$

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- (Chasseigne and Jakobsen 17') The operator \(L\) is a nonlocal version of the \(p\)-Laplacian. It does not coincide with other ones in the literature

(Variational) \((-\Delta)^s_p \phi(x) = \int \frac{|\phi(x) - \phi(x + z)|^{p-2} (\phi(x) - \phi(x + z))}{|z|^{d+sp}} dz\)

(Game th.) \(-\Delta^s_p \phi(x) = \sup_{|y|=1} \int_{C(x,y)} \frac{\phi(x) - \phi(x + z)}{|z|^{d+sp}} dz + \inf_{|\tilde{y}|=1} \int_{C(x,\tilde{y})} \frac{\phi(x) - \phi(x + z)}{|z|^{d+sp}} dz\)
\[ u_t - \partial_x \left( u^{m-1} \partial_x (-\partial_{xx})^{-\sigma} u \right) = 0 \] (FPE)

\[ v(x, t) = \int_{-\infty}^{x} u(y, t) \, dy \quad \downarrow \quad \uparrow \quad u = \partial_x v \]

\[ \partial_t v + (\partial_x v)^{m-1} (-\Delta)^s v = 0 \] (IP)

\[ u \geq 0 \quad \text{(nonnegative weak sol.)} \quad \iff \quad \partial_x v \geq 0 \quad \text{(nondecreasing viscosity sol.)} \]

---

By (Chasseigne and Jakobsen 17’):

- (IP) has existence and uniqueness of viscosity solutions.
- (IP) has comparison principle for sub and super solutions.

This seems to be a right context for numerical schemes!
\[ u_t - \partial_x \left( u^{m-1} \partial_x (-\partial_{xx})^{-\sigma} u \right) = 0 \]  \hspace{1cm} \text{(FPE)}

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2 Numerical scheme (Part 1)

3 Numerical scheme (Part 2)
How to discretize

\[ L[\phi] = (\partial_x \phi)^{m-1}(-\Delta)^s \phi \]

so that it preserves the monotonicity property:

If \( \psi(x) = \phi(x) \) and \( \psi \geq \phi \) then \( L[\psi](x) \geq L[\phi](x) \).
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- **Discretization of the fractional Laplacian:**

\[
-(-\Delta)^s \phi(x) = \text{P.V} \int_{|z|>0} (\phi(x+z) - \phi(x)) \frac{dz}{|z|^{1+2s}}
\]

\[
-(-\Delta)_h^s \phi(x) \sim \sum_{k \neq 0} (\phi(x+z_k) - \phi(x)) \omega_k \quad \text{with} \quad \omega_k = \omega_{-k} \geq 0.
\]

such that for \( \phi \in C^2 \cap C_b \)

\[
(-\Delta)_h^s \phi \xrightarrow{h \to 0} (-\Delta)^s \phi \quad \text{in} \quad L^\infty_{\text{loc}}.
\]
• **Discretization of $\partial_x$:** We think of $L[\phi] = (\partial_x \phi)^{m-1}(-\Delta)^s \phi$ as

$$L[\phi] = (\partial_x \phi)^{m-1} b(x, t),$$

with $b$ changing signs.
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with $b$ changing signs. Thus, we choose an UPWIND-type discretization:

$$\partial_x \phi(x) \sim D_h \phi(x) := \begin{cases} 
\frac{\phi(x + h) - \phi(x)}{h} & \text{if } (-\Delta^s_h \phi(x) \leq 0, \\
\frac{\phi(x) - \phi(x - h)}{h} & \text{if } (-\Delta^s_h \phi(x) > 0.}
\end{cases}$$
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\end{cases}$$

• Complete discretization

$$L_h[\phi](x) = (D_h \phi(x))^{m-1}(-\Delta)^s_h \phi(x)$$

• (Consistent) For $\phi \in C^2 \cap C_b$, $L_h[\phi] \xrightarrow{h \to 0} L[\phi]$ in $L^\infty_{loc}$

• (Monotone) If $\psi(x) = \phi(x)$ and $\psi \geq \phi$ then $L_h[\psi](x) \geq L_h[\phi](x)$

but only for nondecreasing functions $\phi$ and $\psi$!
Numerical scheme

Given discretization parameters \( h, \tau > 0 \), a uniform grids \( x_k = kh \) and \( t_j = j\tau \):

\[
V_k^{j+1} = V_k^j - \tau L_h V_k^j, \\
V_k^0 = \int_{-\infty}^{x_k} d\mu_0(x).
\]
Numerical scheme

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$$V_{k}^{0} = \int_{-\infty}^{x_k} d\mu_0(x).$$

Key result – Comparison for nondecreasing solutions: For $V^j, W^j$ nondecreasing,

$$V^j \leq W^j \implies V^{j+1} \leq W^{j+1}.$$

provided CFL-type conditions:

$$\tau \leq h^{2s+m-1} \quad (V,W \text{ bounded}) \quad \text{or} \quad \tau \leq h^{\max\{1,2s\}} \quad (V,W \text{ Lipschitz})$$
Theorem

Let \( \{V_0^i\}_{i \in \mathbb{Z}} \) and \( \{W_0^i\}_{i \in \mathbb{Z}} \) are nondecreasing and satisfying one of the following:

\[
|V_0^i|, |W_0^i| \leq M, \quad i \in \mathbb{Z}, \quad \text{and} \quad \tau \leq C_1 h^{2s+m-1},
\]

\[
|V_0^i|, |W_0^i| \leq M, \quad \frac{|V_{i+1}^0 - V_i^0|}{h}, \quad \frac{|W_{i+1}^0 - W_i^0|}{h} \leq L, \quad i \in \mathbb{Z}, \quad \text{and} \quad \tau \leq C_2 h^{\max\{2s,1\}},
\]

Then

(a) (Nondecreasing) \( V_i^j \leq V_{i+1}^j \).

(b) (\( \ell_\infty \)-stability) \( |V_i^j| \leq M \).

(c) (Nonnegative) If \( V_i^0 \geq 0 \) then \( V_i^j \geq 0 \).

(d) (\( \ell_\infty \)-contraction) \( \sup_{i \in \mathbb{Z}, j \in \mathbb{N}} |W_i^j - V_i^j| \leq \sup_{i \in \mathbb{Z}} |W_i^0 - V_i^0| \).

(e) (Lipschitz-stability) \( \sup_{i \in \mathbb{Z}, j \in \mathbb{N}} \frac{|V_{i+1}^j - V_i^j|}{h} \leq \sup_{i \in \mathbb{Z}} \frac{|V_{i+1}^0 - V_i^0|}{h} \).
Let $\overline{V}$ be an interpolation of $V$ (p.w. linear in space and p.w. constant in time).

**Theorem**

(a) (Continuous case) If $v_0 \in BUC(\mathbb{R}^d)$ then:

(i) (Convergence) $\overline{V}_h \xrightarrow{h \to 0} v$ locally uniformly in $\overline{Q}_T$.

(ii) (Limit) $v \in BUC(\mathbb{R}^d)$ is a viscosity solution of (IP).

(b) (Jump discontinuity) If $v_0(x) = \begin{cases} 0 & \text{if } x < a, \\ M & \text{if } x \geq a. \end{cases}$

(i) (Convergence) $\overline{V}_h \xrightarrow{h \to 0} v$ locally uniformly in $\overline{Q}_T \setminus \{(a,0)\}$.

(ii) (Limit) $v \in C(\overline{Q}_T \setminus \{(a,0)\})$ is a discontinuous viscosity solution of (IP).
Let $\overline{V}$ be an interpolation of $V$ (p.w. linear in space and p.w. constant in time).

**Theorem**

(a) (Continuous case) If $v_0 \in BUC(\mathbb{R}^d)$ then:

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(i) (Convergence) $\overline{V}_h \xrightarrow{h \to 0} v$ locally uniformly in $\overline{Q_T \setminus \{(a,0)\}}$.

(ii) (Limit) $v \in C_b(\overline{Q_T \setminus \{(a,0)\}})$ is a discontinuous viscosity solution of (IP).
Convergence by *half-relaxed limit method* *(Barles, Perthame, Souganidis).*

1. Uniform-in-$h$ $L^\infty$ bounds. Then

\[ v(x, t) := \liminf_{(y, \rho, h) \to (x, t, 0)} V_h(y, s) \quad \bar{v}(x, t) := \limsup_{(y, \rho, h) \to (x, t, 0)} V_h(y, s) \]

and $\underline{v} \leq \bar{v}$.

2. Monotonicity and consistency. Then

\[ \underline{v} \text{ discontinuous viscosity supersolution} \]
\[ \bar{v} \text{ discontinuous viscosity subsolution} \]

3. Comparison principle for *(IP)*. Then

\[ \bar{v} \leq \underline{v} \]

4. All together:

\[ \bar{v} = \underline{v} =: v = \lim_{h \to 0} V_h \]
1. Introduction

2. Numerical scheme (Part 1)

3. Numerical scheme (Part 2)
\begin{align*}
u_t - \partial_x \left( u^{m-1} \partial_x (-\partial_{xx})^{-\sigma} u \right) &= 0 \quad \text{(FPE)} \\
\nu(x, t) &= \int_{-\infty}^{x} u(y, t) \, dy \quad \downarrow \quad \uparrow \quad u = \partial_x \nu \\
\partial_t \nu + \left( \partial_x \nu \right)^{m-1} (-\Delta)^s \nu &= 0 \quad \text{(IP)}
\end{align*}
\[ u_t - \partial_x \left( u^{m-1} \partial_x (-\partial_{xx})^{-\sigma} u \right) = 0 \] (FPE)

\[ v(x, t) = \int_{-\infty}^{x} u(y, t) \, dy \quad \downarrow \quad \uparrow \quad u = \partial_x v \]

\[ \partial_t v + (\partial_x v)^{m-1} (-\Delta)^s v = 0 \] (IP)

Once a numerical method for (IP) is provided, the approximation for (FPE) is naturally given by numerical differentiation

\[ U_i^j = \frac{V_{i+1}^j - V_i^j}{h} \]

and $\overline{U}$ the corresponding p.w. constant interpolation.
Rubinstein-Kantorovich metric

\[ d_0(f_1, f_2) = \sup_{\varphi \in \text{Lip}_{1,1}(\mathbb{R})} \int_{\mathbb{R}} (f_1(x) - f_2(x))\varphi(x)dx, \]

where \( \text{Lip}_{1,1}(\mathbb{R}) = \{\varphi \in C(\mathbb{R}) : \|\varphi\|_{L^\infty} \leq 1, \|D\varphi\|_{L^\infty} \leq 1\} \) and \( D \) is the weak derivative.
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**Theorem**

\( \mu_0 \in \mathcal{M}_+(\mathbb{R}) \) and \( u \) the weak solution of (FPE).

**a** \((L^1 \text{ case})\) Assume \( \mu_0 \in L^1(\mathbb{R}) \) (resp. \( \mu_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}) \)), then we have

\[
\sup_{t \in [0,T]} d_0(\overline{U}_h(\cdot, t), u(\cdot, t)) \xrightarrow{h \to 0} 0.
\]
Rubinstein-Kantorovich metric

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**Theorem**

$\mu_0 \in \mathcal{M}_+(\mathbb{R})$ and $u$ the weak solution of (FPE).

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$$\sup_{t \in [0,T]} d_0(\overline{U}_h(\cdot, t), u(\cdot, t)) \rightarrow 0.$$

**b** (Fundamental solution) Assume $\mu_0 = M\delta_a$ then we have

$$\sup_{t \in [t_0, T]} d_0(\overline{U}_h(\cdot, t), u(\cdot, t)) \rightarrow 0.$$
Sketch of the proof:

- [Previous theorems] $\overline{V}(x, t) \rightarrow v(x, t) = \int_{-\infty}^{x} u(y, t)dy$ locally uniformly.

- [Interpolation choice] $\overline{V}(x, t) = \int_{-\infty}^{x} \overline{U}(y, t)dy$

- [Convergence I] For $\varphi \in \text{Lip}_{1,1}$ and $\partial_x \varphi$ with compact support:
  \[
  \int \varphi(\overline{U} - u)dx = \int \varphi_x \int_{-\infty}^{x} (\overline{U} - u)dydx = \int \varphi_x (\overline{V} - v)dx \xrightarrow{h \to 0} 0.
  \]

- [Equi-Tightness]
  \[
  \int \overline{U}(1 - \chi_R)dx < \varepsilon, \quad \text{and} \quad \int \overline{u}(1 - \chi_R)dx < \varepsilon.
  \]

- [Convergence II] For $\varphi \in \text{Lip}_{1,1}$:
  \[
  \int \varphi(\overline{U} - u)dx = \int \varphi \chi_R (\overline{U} - u)dx + \int \varphi(1 - \chi_R)(\overline{U} - u)dx \xrightarrow{h \to 0} 0.
  \]
Final comments

- The numerical method works in this specific framework just because:
  - $u \geq 0 \implies v$ nondecreasing.
  - Explicit scheme so we can upwind properly.
- For initial data $\mu_0 \in L^1 \cap L^\infty$ the explicit scheme is stable and convergent under nice a CFL-type condition $\tau \ll h \max\{1, 2^s\}$.
- When $\mu_0 \in L^1$, a very restrictive stability condition is needed $\tau \ll h^2 s + m - 1$.
- We can cover measure data (i.e. fundamental solution), since in the integrated variable it turns into a theory of discontinuous viscosity solutions.
- Numerical methods in dimensions $d > 1$?
- Numerical methods for sign-changing solutions?
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Thank you