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Fakultät für Mathematik

Robust nonlocal trace and extension problems

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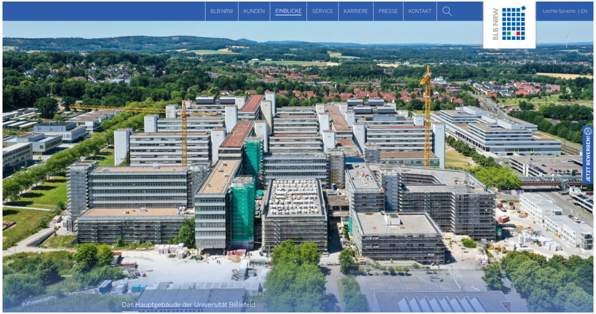
Based on a joint work with Florian Grube (Bielefeld), arXiv:2305.05735

Contents

I. Origin of the project



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Florian Grube

I. Exterior Dirichlet value problems

Goal today

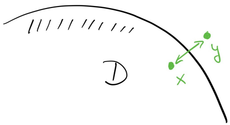
Goal: Robust trace spaces for nonlinear problems in Lipschitz domains
Application: Solve, for a natural and large class of exterior Dirichlet data g,

$$
\begin{aligned}
(-\Delta)_p^s u(x) &= f \quad \text{in } D\,, \\
u &= \mathbf{g} \quad \text{in } \mathbb{R}^d \setminus D\,.
\end{aligned}
\qquad \text{(DP)}
$$

where

$$(-\Delta)_p^s u(x) = (1-s)\,\text{pv.}\int_{\mathbb{R}^d} |u(y)-u(x)|^{p-2}(u(x)-u(y))\frac{\mathrm{d}y}{|x-y|^{d+sp}}$$

1. Focus is on the behavior of g near ∂D.
2. Away from $\overline{D}$, the object $(-\Delta)_p^s u$ makes sense already if $u \in L^{p-1}(\mathbb{R}^d; \frac{\mathrm{d}x}{(1+|x|)^{d+sp}})$.
3. The stronger assumption $u \in L^p(\mathbb{R}^d; \frac{\mathrm{d}x}{(1+|x|)^{d+sp}})$ is natural in the variational context.



I. Exterior Neumann-type value problems

So far, Neumann-type conditions have not been well understood for nonlocal operators.

Given, a Lipschitz domain $D \subset \mathbb{R}^d$, a natural goal is to study solutions $u : (0, T) \times \mathbb{R}^d \to \mathbb{R}$ to the *parabolic nonlocal Neumann problem*:

$$\begin{cases} \partial_t u - \mathcal{L} u = f(t,x) & \text{in } (0,T) \times D, \\ \mathcal{N} u = \mathbf{h(t,x)} & \text{in } (0,T) \times \overline{D}^c, \\ u(0,x) = u_0(x) & \text{in } D, \end{cases} \tag{1}$$

where $\mathcal{L}$ is a nonlocal operator, e.g., fractional Laplace operator and $\mathcal{N} u$ is the "nonlocal normal derivative of u" associated to D, J.

Here again: **function spaces** necessary for analysis and stochastic anlaysis.

Upcoming [joint with Soobin Cho]: probabilistic interpretation of $u(t,x)$ in linear case

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II. The space $V_\nu(D|\mathbb{R}^d)$ for general measures ν

Assumptions:

- $\nu : \mathbb{R}^d \setminus \{0\} \to [0, \infty)$ is symmetric with

$$\int \min\{1, |h|^2\}\nu(h)\mathrm{d}h < \infty\,. \tag{L}$$

- $D \subset \mathbb{R}^d$ open and bounded.

Definition

$V_\nu(D|\mathbb{R}^d)$ is the vector space of all measurable $u : \mathbb{R}^d \to \mathbb{R}$ with

$$|u|^2_{V_\nu(D|\mathbb{R}^d)} = \iint\limits_{D\mathbb{R}^d} \big(u(y) - u(x)\big)^2 \nu(x-y)\mathrm{d}y\mathrm{d}x < \infty\,. \tag{2}$$

We endow the vector space $V_\nu(D|\mathbb{R}^d)$ with the norm $\|\cdot\|_{V_\nu(D|\mathbb{R}^d)}$ given by

$$\|u\|^2_{V_\nu(D|\mathbb{R}^d)} = \|u\|^2_{L^2(D)} + |u|^2_{V_\nu(D|\mathbb{R}^d)}\,. \tag{3}$$

II. Quadratic forms, $V_{\nu,0}(D|\mathbb{R}^d)$, and $H_\nu(D)$

Given functions $u, v \in V_\nu(D|\mathbb{R}^d)$, we define a bilinear form $\mathcal{E}$ by

$$\mathcal{E}(u,v) = \frac{1}{2} \iint\limits_{(D^c \times D^c)^c} \big(u(x) - u(y)\big)\big(v(x) - v(y)\big)\, \nu(x-y) \mathrm{d}x \, \mathrm{d}y \, .$$

Note $|u|^2_{V_\nu(D|\mathbb{R}^d)} \leq \mathcal{E}(u,u) \leq 2|u|^2_{V_\nu(D|\mathbb{R}^d)}$ for any u.

Definition

$$\begin{aligned} V_{\nu,0}(D|\mathbb{R}^d) &:= \{u \in V_\nu(D|\mathbb{R}^d) \mid u = 0 \text{ a.e. on } \mathbb{R}^d \setminus D\} \\ H_\nu(D) &:= \Big\{u \in L^2(D) |\, \mathcal{E}_D(u,u) < \infty \Big\} \end{aligned}$$

with norm $\|u\|^2_{H_\nu(D)} = \|u\|^2_{L^2(D)} + \mathcal{E}_D(u,u)$, where

$$\mathcal{E}_D(u,v) = \iint\limits_{DD} \big(u(x) - u(y)\big)\big(v(x) - v(y)\big)\, \nu(x-y) \mathrm{d}x\mathrm{d}y \, .$$

II. Density in energy spaces for $p = 2$

Theorem: Let ν satisfy (L) with full support and let $D \subset \mathbb{R}^d$ be open.

1. $C^\infty(D) \cap H_\nu(D)$ is dense in $H_\nu(D)$.
2. If ∂D is compact & continuous, then $C_c^\infty(\overline{D})$ is dense in $H_\nu(D)$.
3. If ∂D is compact & continuous, then $C_c^\infty(D)$ is dense in $V_{\nu,0}(D|\mathbb{R}^d)$.
4. If ∂D is compact and Lipschitz, then $C_c^\infty(\mathbb{R}^d)$ is dense in $V_\nu(D|\mathbb{R}^d)$ with respect to $\|\cdot\|_{V_\nu(D|\mathbb{R}^d)}$ and $|||\cdot|||_{V_\nu(D|\mathbb{R}^d)}$ with

$$|||u|||^2_{V_\nu(D|\mathbb{R}^d)} = \|u\|^2_{L^2(\mathbb{R}^d)} + |u|^2_{V_\nu(D|\mathbb{R}^d)} .$$

Theorem taken from Foghem/MK [FK22]

References:
First statement (Meyers-Serrin) and second statement: [FG20] and [DK21]
Third statement: [FSV15] for a special choice of ν and in [FG20], [BGPR20a] for the general case.
Fourth assertion: [FKV20]

II. Exterior Neumann-type derivatives

Let $\mathcal{N}u$ be the "nonlocal normal derivative of u" associated to D and ν, see [DROV17].

$$\mathcal{N}u(y) = \int_D (u(y) - u(z))\nu(z-y)dz, \quad y \in \overline{D}^c. \tag{4}$$

Lemma

Let $f \in L^2(D)$. Assume $u \in V_\nu(D|\mathbb{R}^d)$ minimizes the functional $v \mapsto \frac{1}{2}\mathcal{E}(v,v) - \int_D fv$ in the space $V_\nu(D|\mathbb{R}^d)$. Then $\mathcal{N}u = 0$ in D^c.

1. $\mathcal{N}u = 0$ on D^c constitutes a *natural* complement condition, analogous to $\partial_n u = 0$ on ∂D in the classical case. No regularity of D is needed, though.
2. Gauss-Green type formula: For $u \in C_b^2(\mathbb{R}^d)$, $v \in C_b^1(\mathbb{R}^d)$ we have

$$\int_D -\mathcal{L}u(x)\, v(x)\mathrm{d}x = \mathcal{E}(u,v) - \int_{D^c} \mathcal{N}u(y)v(y)\mathrm{d}y.$$

In particular, by choosing $v = 1$ one deduces $\int_D \mathcal{L}u(x)\mathrm{d}x = \int_{D^c} \mathcal{N}u(y)\mathrm{d}y$.

II. Approximation of divergence and normal derivative

Lemma [HK23]

The classical divergence theorem in bounded C^1-domains follows from the Fubini theorem with the help of nonlocal operators and rescaling.

Key ideas in the proof:

- notion of nonlocal divergence and nonlocal normal derivative
- choice of localizing sequence ν_ε
- Fubini provides a trivial nonlocal "divergence theorem"
- $\varepsilon \to 0$

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III. Exterior Dirichlet data for $p > 1$

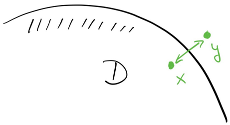
Task: Robust trace spaces for rather general domains and nonlinear problems.
Application: Solve, for a natural and large class of exterior Dirichlet data g,

$$\begin{aligned} (-\Delta)^s_p u(x) &= f \quad \text{in } D\,, \\ u &= \mathbf{g} \quad \text{in } \mathbb{R}^d \setminus D\,. \end{aligned} \tag{DP}$$

where

$$(-\Delta)^s_p u(x) = (1-s)\,\text{pv.}\int_{\mathbb{R}^d} |u(y)-u(x)|^{p-2}(u(x)-u(y))\frac{\mathrm{d}y}{|x-y|^{d+sp}}$$

1. Focus is on the behavior of g near ∂D.
2. Away from $\overline{D}$, the object $(-\Delta)^s_p u$ makes sense already if $u \in L^{p-1}(\mathbb{R}^d; \frac{\mathrm{d}x}{(1+|x|)^{d+sp}})$.
3. The stronger assumption $u \in L^p(\mathbb{R}^d; \frac{\mathrm{d}x}{(1+|x|)^{d+sp}})$ is natural in the variational context.



III. Energy spaces for $p > 1$

Definition

For a bounded Lipschitz domain $D \subset \mathbb{R}^d$ and $1 \leq p < \infty$ we define

$$V^{s,p}(D \,|\, \mathbb{R}^d) := \{u : \mathbb{R}^d \to \mathbb{R} \text{ measurable } \,|\, [u]_{V^{s,p}(D \,|\, \mathbb{R}^d)} < \infty\},$$

$$[u]^p_{V^{s,p}(D \,|\, \mathbb{R}^d)} := \frac{1-s}{p} \iint_{D \times \mathbb{R}^d} \frac{|u(x) - u(y)|^p}{|x-y|^{d+s\,p}} \, dx \, dy.$$

We endow this space with the norm $\|u\|^p_{V^{s,p}(D \,|\, \mathbb{R}^d)} := \|u\|^p_{L^p(D)} + [u]^p_{V^{s,p}(D \,|\, \mathbb{R}^d)}$.

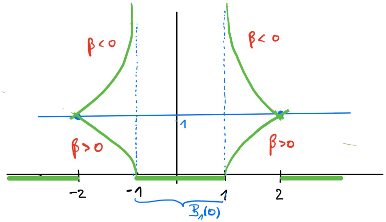
Lemma (Dyda/MK [DK19])

Assume D is bounded and $u \in V^{s,p}(D \,|\, \mathbb{R}^d)$. Then $u \in L^p(\mathbb{R}^d; \frac{\mathrm{d}x}{(1+|x|)^{d+sp}})$

III. Understanding the space $V^{s,p}(D \,|\, \mathbb{R}^d)$

Choose $D = B_1(0) \subset \mathbb{R}^d$. Let $\beta \in \mathbb{R}$. We study an example of $g \in V^{s,p}(B_1 \,|\, \mathbb{R}^d)$.
Assume $g : \mathbb{R}^d \to \mathbb{R}$ is defined by

$$g(x) = \begin{cases} 0, & |x| < 1\,, \\ (|x|-1)^\beta, & 1 \le |x| \le 2\,, \\ 0, & |x| > 2\,. \end{cases}$$



$$\iint\limits_{B_1\,\mathbb{R}^d} \frac{\big(g(y)-g(x)\big)^2}{|y-x|^{d+2s}}\mathrm{d}y\mathrm{d}x = \iint\limits_{B_1\,(B_2\setminus B_1)} \frac{\big((|y|-1)^\beta\big)^2}{|y-x|^{d+2s}}\mathrm{d}y\mathrm{d}x = \int\limits_{B_2\setminus B_1} (|y|-1)^{2\beta}\Big(\int\limits_{B_1} |y-x|^{-d-2s}\mathrm{d}x\Big)\mathrm{d}y$$

$$\asymp \int\limits_{B_2\setminus B_1} (|y|-1)^{2\beta-2s}\mathrm{d}y \qquad g \in V^{s,p}(B_1 \,|\, \mathbb{R}^d) \quad \text{iff } 2\beta - 2s > -1 \quad \Leftrightarrow \quad \beta > s - \frac{1}{2}$$

III. Problem of trace space

Let $D \subset \mathbb{R}^d$ be a bounded domain with some regularity of ∂D. Assume $s \in (0,1)$, $1 \le p < \infty$.

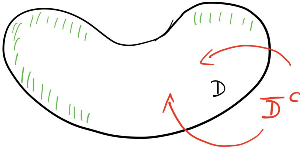
Main Aim

We search for a space of functions on D^c, say $\mathcal{T}^{s,p}(D^c)$, and a map

$$\mathrm{Tr}_s : V^{s,p}(D \,|\, \mathbb{R}^d) \to \mathcal{T}^{s,p}(D^c), \qquad u \mapsto u|_{D^c} \,,$$

which is continuous and linear with a continuous right inverse

$$\mathrm{Ext}_s : \mathcal{T}^{s,p}(D^c) \to V^{s,p}(D \,|\, \mathbb{R}^d), \qquad g \mapsto \mathrm{ext}(g) \,.$$



Maximum goal

There is (μ_s) such that for any $u \in W^{1,p}(\mathbb{R}^d)$, as $s \to 1^-$,

$$\|\mathrm{Tr}\, u\|_{L^p(D^c;\,\mu_s)} \to \|\gamma u\|_{L^p(\partial D)} \,, \qquad [\mathrm{Tr}\, u]_{\mathcal{T}^{s,p}(D^c)} \to [\gamma u]_{W^{1-1/p,p}(\partial D)} \,.$$

III. Earlier attempts on nonlocal trace spaces

(1) $H^{1/2}(\partial D)$ as trace space of weighted $L^2(D)$, [TD17] $\Rightarrow$ Peridynamics. See follow-ups.
(2) Definition of $\mathcal{T}^{s,p}(D^c)$ in [DK19]. Traces and extensions. Not robust as $s \to 1-$, though.
(3) $\mathcal{T}^{s,2}(D^c)$ together with traces and extensions in [BGPR20a]. Douglas identity. Not explicit. Some extensions in [BGPR20b].
(4) [GH22]: explicit definition of $\mathcal{T}^{s,2}(D^c)$. Robust extensions and traces for $p = 2$, $\partial D \in C^{1,1}$.

Features of current project [GK23]: $\mathbf{p > 1}$ and $\mathbf{p = 1}$, **∂D Lipschitz, Whitney covering**

An abstract definition of a trace space is always possible.

Definition (abstract)

We define $\mathcal{T}^{s,p}(D^c)$ as the space of restrictions to $\mathbb{R}^d \setminus D$ of functions of $V^{s,p}(D\,|\,\mathbb{R}^d)$. That is,

$$\mathcal{T}^{s,p}(D^c) = \{v : D^c \to \mathbb{R} \text{ meas. such that } v = u|_{D^c} \text{ with } u \in V^{s,p}(D\,|\,\mathbb{R}^d)\},$$
$$\|v\|_{\mathcal{T}^{s,p}(D^c)} = \inf\{\|u\|_{V^{s,p}(D\,|\,\mathbb{R}^d)} : \; u \in V^{s,p}(D\,|\,\mathbb{R}^d) \text{ with } v = u|_{D^c}\}.$$

III. Trace spaces

We define measures μ_s on Borel sets of $\mathbb{R}^d$ by

$$\mu_s(\mathrm{d}x) := \mathbb{1}_{D^c}(x)\,(1-s)d_x^{-s}\,(1+d_x)^{-d-s(p-1)}\,\mathrm{d}x \tag{5}$$

on $\mathbb{R}^d$, $s \in (0,1)$, $p \geq 1$ where $d_x := \operatorname{dist}(x, \partial D)$ for $x \in \mathbb{R}^d$.

We introduce the trace spaces

$$\mathcal{T}^{s,p}(D^c) := \{g : D^c \to \mathbb{R} \text{ measurable} \mid \|g\|_{\mathcal{T}^{s,p}(D^c)} < \infty\},$$

$$\|g\|^p_{\mathcal{T}^{s,p}(D^c)} := \|g\|^p_{L^p(D^c;\,\mu_s)} + [g]^p_{\mathcal{T}^{s,p}(D^c)},$$

$$[f]^p_{\mathcal{T}^{s,p}(D^c\,|\,D^c)} := \iint\limits_{D^c\,D^c} \frac{|f(x)-f(y)|^p}{((|x-y|+d_x+d_y)\wedge 1)^{d+s(p-2)}}\mu_s(\mathrm{d}x)\mu_s(\mathrm{d}y)$$

III. Trace spaces

Recall

$$\mu_s(\mathrm{d}x) := 1_{D^c}(x)\,(\mathbf{1}-\mathbf{s})\mathbf{d}_{\mathbf{x}}^{-\mathbf{s}}\,(1+d_x)^{-d-s(p-1)}\,\mathbf{dx}$$

Let $f : [a,b] \to \mathbb{R}$ be continuous. Assume $\varepsilon > 0$.

Let $\delta > 0$ such that $|f(a) - f(x)| < \varepsilon$ for all $x \in [a, a+\delta]$. Then

$$\begin{aligned}\int_a^b f(x)\,(1-s)(x-a)^{-s}\,\mathrm{d}\mathbf{x} &= (1-s)\int_a^{a+\delta} f(x)\,(x-a)^{-s}\,\mathrm{d}x + \underbrace{(1-s)\int_{a+\delta}^b f(x)\,(x-a)^{-s}\,\mathrm{d}x}_{\to 0 \text{ as } s\to 1} \\ &\lesseqgtr (f(a)\pm\varepsilon)\,(1-s)\int_a^{a+\delta}(x-a)^{-s}\,\mathrm{d}x + \mathcal{O}(1-s) \\ &= (f(a)\pm\varepsilon)\delta^{1-s} + \mathcal{O}(1-s) \quad\longrightarrow\quad (f(a)\pm\varepsilon) \text{ as } s\to 1.\end{aligned}$$

Thus, the measure $(1-s)(x-a)^{-s}\,\mathrm{d}\mathbf{x}$ converges to $\delta_{\{a\}}$ as $s \to 1$.

III. Trace spaces

Theorem (Grube/MK)

Let $D \subset \mathbb{R}^d$ be a bounded Lipschitz domain, $s \in (0,1)$, $1 < p < \infty$. Then the trace map

$$\mathrm{Tr}_s : V^{s,p}(D \,|\, \mathbb{R}^d) \to \mathcal{T}^{s,p}(D^c), \qquad u \mapsto u|_{D^c}$$

is continuous and linear and there exists a continuous right inverse

$$\mathrm{Ext}_s : \mathcal{T}^{s,p}(D^c) \to V^{s,p}(D \,|\, \mathbb{R}^d), \qquad g \mapsto \mathrm{ext}(g)\,.$$

Theorem (Grube/MK)

Let $D \subset \mathbb{R}^d$ be a Lipschitz domain, $s \in (0,1)$, $1 < p < \infty$. For any $u \in W^{1,p}(\mathbb{R}^d)$, as $s \to 1^-$,

$$\|\mathrm{Tr}\, u\|_{L^p(D^c;\, \mu_s)} \to \|\gamma u\|_{L^p(\partial D)}\,, \qquad [\mathrm{Tr}\, u]_{\mathcal{T}^{s,p}(D^c)} \to [\gamma u]_{W^{1-1/p,p}(\partial D)}\,.$$

III. Well-posedness for exterior value problems

Corollary (Grube/MK)

Let $D \subset \mathbb{R}^d$ be a bounded Lipschitz domain, $s_\star \leq s < 1$, $1 < p < \infty$. Let $g \in \mathcal{T}^{s,p}(D^c)$ and $f \in V^{s,p}(D \,|\, \mathbb{R}^d)' \supset L^{p'}(D)$. Then there exists a unique solution $u \in V^{s,p}(D \,|\, \mathbb{R}^d)$ to problem (DP). Moreover, there is a constant $c > 0$, depending only on $p, D, s_\star$ such that

$$\|u\|_{V^{s,p}(D \,|\, \mathbb{R}^d)} \leq c\left(\|g\|_{\mathcal{T}^{s,p}(D^c)} + \|f\|_{V^{s,p}(D \,|\, \mathbb{R}^d)'}\right). \tag{6}$$

Proof: Let $V_g^{s,p}(D|\mathbb{R}^d)$ be the set of all functions v of the form $v = \mathrm{Ext}_s\, g + v_0$ with $v_0 \in V_0^{s,p}(D \,|\, \mathbb{R}^d)$. This set is a closed convex subset of $V^{s,p}(D \,|\, \mathbb{R}^d)$. Let $I : V_g^{s,p}(D|\mathbb{R}^d) \to \mathbb{R}$ be defined by

$$I(v) = \frac{1-s}{p} \iint\limits_{D\,\mathbb{R}^d} \frac{|v(y) - v(x)|^p}{|x-y|^{d+s\,p}} \mathrm{d}x\mathrm{d}y - f(v).$$

The functional I is strictly convex and weakly lower semicontinuous on the reflexive, separable Banach space $V_g^{s,p}(D|\mathbb{R}^d)$.

III. Well-posedness for exterior value problems

Since

$$|f(v)| \leq \|f\|_{V^{s,p}(D\,|\,\mathbb{R}^d)'} \|v\|_{V^{s,p}(D\,|\,\mathbb{R}^d)} \leq \delta \|v\|^p_{V^{s,p}(D\,|\,\mathbb{R}^d)} + (p')^{-1}(\delta p)^{-1/(p-1)} \|f\|^{p'}_{V^{s,p}(D\,|\,\mathbb{R}^d)'}$$

for every $\delta \in (0,1)$, we can apply the Poincaré-Friedrichs inequality to the function $v - \mathrm{Ext_s}(g)$ to obtain

$$I(v) \geq \frac{1}{2p}[v]^p_{V^{s,p}(D\,|\,\mathbb{R}^d)} + c_1^{-1}\|v\|^p_{L^p(D)} - c_1\|f\|^{p'}_{V^{s,p}(D\,|\,\mathbb{R}^d)'} - c_1 \|\mathrm{Ext_s}\, g\|^p_{V^{s,p}(D\,|\,\mathbb{R}^d)}$$

for some constant c_1 depending on p and the Poincare-constant. Thus, the functional I is coercive in the sense that $I(v) \to +\infty$ for $\|v\|_{V^{s,p}(D\,|\,\mathbb{R}^d)} \to +\infty$. We have shown that I attains a unique minimizer u on the set $V^{s,p}_g(D|\mathbb{R}^d)$. It is now straightforward to show that the function u solves problem (DP). The claimed estimate follows from $I(u) \leq I(\mathrm{Ext_s}\, g)$, and the above estimate. $\square$

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IV. Details of the proofs.

Analogous to the measure μ_s from (5), we define for $s \in (0,1)$ the measure

$$\tau_s(\mathrm{d}x) = \frac{1-s}{d_x^s} 1_D(x)\mathrm{d}x \tag{7}$$

on Borel subsets of $\mathbb{R}^d$. Recall that $d_x = \mathrm{dist}(x, \partial D)$.

Theorem (Approximate trace inequality)

Let $D \subset \mathbb{R}^d$ be a bounded Lipschitz domain, $1 < p_\star < p^\star < \infty$ and $s_\star \in (0,1)$ there exists a constant $C = C(d, D, p_\star, p^\star, s_\star) > 0$ such that for every $s \in (s_\star, 1)$, $p_\star \leq p \leq p^\star$ and $u \in W^{s,p}(D)$

$$\int_D |u(x)|^p \, \tau_s(\mathrm{d}x) + \int_D \int_D \frac{|u(x)-u(y)|^p}{((|x-y| + d_x + d_y) \wedge 1)^{d+s(p-2)}} \tau_s(\mathrm{d}y)\tau_s(\mathrm{d}x) \leq C \, \|u\|^p_{W^{s,p}(D)} \, . \tag{8}$$

IV. Details of the proofs.

Lemma

Let $D \subset \mathbb{R}^d$ be a bounded Lipschitz domain with a localization radius $r_0 > 0$.
(1) For every s the measure τ_s is a doubling measure.
(2) There exists $C = C(d, D) > 0$ such that for any $s \in (0,1)$, $0 < r \leq r_0/2$, and $x \in D$

$$\tau_s(B_r(x)) \leq C\, r^{d-s}.$$

Let $u \in W^{s,p}(\mathbb{R}^d)$.

$$\int_D \int_D \frac{|u(x) - u(y)|^p}{((|x-y| + d_x + d_y) \wedge 1)^{d+s(p-2)}} \tau_s(\mathrm{d}y)\tau_s(\mathrm{d}x)$$

$$\leq 2\sum_{n=0}^{\infty} 2^{ns(p-1)} \iint\limits_{\substack{D\times D \\ 2^{-n-1} \leq |x-y| < 2^{-n}}} |u(x) - u(y)|^p \frac{(\tau_s \otimes \tau_s)(\mathrm{d}(y,x))}{|x-y|^{d-s}} + \iint\limits_{\substack{D\times D \\ 1 \leq |x-y|}} |u(x) - u(y)|^p \, \tau_s(\mathrm{d}y)\tau_s(\mathrm{d}x)$$

IV. Details of the proofs.

Define $H := L^p(D \times D, |x-y|^{-d+s}\,\tau_s(\mathrm{d}y)\tau_s(\mathrm{d}x))$ and, for $1 < q \leq \infty$, $\beta > 0$, the space

$$\ell^{\beta,q} := \{(h_n)_n \mid h_n \in H\},$$
$$\|(h_n)\|_{\ell^{\beta,q}} := \left\| \left(2^{n\beta} \|h_n\|_H \right)_n \right\|_{\ell^q(\mathbb{N})}.$$

Then

$$\textbf{[Blue Term]} = \left\| \left((u(x) - u(y)) 1_{2^{-n-1} \leq |x-y| < 2^{-n}} \right)_n \right\|^p_{\ell^{s-s/p,p}}. \tag{9}$$

Define the linear map

$$Tf(x,y) := \left((f(x) - f(y)) 1_{2^{-n-1} \leq |x-y| < 2^{-n}} \right)_n, \quad f : \mathbb{R}^d \to \mathbb{R}.$$

Denoe by $H^{\alpha,s}$ the Bessel potential space.

Key observation: $T : H^{\alpha,s} \to \ell^{\beta,\infty}$ with $\beta = \alpha - s/p$ is continuous.

IV. Details of the proofs.

Lemma (Chapter V, Lemma C in [JW84])

Let $D \subset \mathbb{R}^d$ be a bounded connected Lipschitz domain, $0 < s_\star \le s < 1$ and $1 < p_\star \le p \le p^\star < \infty$. We set

$$\alpha_0 := s\frac{1+p}{2p}, \quad \alpha_1 := 1 + \frac{s}{2p}. \tag{10}$$

and $\beta_i := \alpha_i - s/p$ for $i \in \{0,1\}$. There exists a constant $C = C(d, D, p_\star, p^\star, s_\star) > 0$ such that for all $0 < r \le r_0/2$ and $f \in L^p(\mathbb{R}^d)$ we have

$$\iint\limits_{\substack{D\times D\\ |x-y|<r}} \left|G_{\alpha_i} * f(x) - G_{\alpha_i} * f(y)\right|^p \tau_s(\mathrm{d}y)\tau_s(\mathrm{d}x) \le C r^{p\beta_i} \left\|f\right\|^p_{L^p(\mathbb{R}^d)}, \tag{11}$$

$$\int\limits_D \left|G_{\alpha_i} * f(x)\right|^p \tau_s(\mathrm{d}x) \le C \left\|f\right\|^p_{L^p(\mathbb{R}^d)}. \tag{12}$$

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V. The case $p = 1$

Theorem

Let $D \subset \mathbb{R}^d$ be a bounded Lipschitz domain, $s \in (0,1)$. Then the trace operator

$$\mathrm{Tr}_s : V^{s,1}(D|\mathbb{R}^d) \to L^1(D^c; \mu_s(\mathrm{d}x)), \qquad u \mapsto u|_{D^c}$$

is continuous and linear. There exists a continuous linear right inverse

$$\mathrm{Ext}_s : \mathcal{T}^{s,1}(D^c) \to V^{s,1}(D|\mathbb{R}^d), \qquad g \mapsto \mathrm{ext}(g)\,.$$

The continuity constants of the linear trace and extension operator only depend on D and a lower bound on s. In addition, the norm of the extension operator in dimension $d = 1$ also depends on a lower bound on $1 - s$.

V. The case $p = 1$

The theorem is analogous to the local setting, i.e., for $s = 1$. $L^1(D^c; \mu_s)$ replaces $L^1(\partial D)$.

There exists a *nonlinear* bounded extension operator from $L^1(\partial D)$ to $BV(D)$, see e.g. [MSS18, Theorem 1.2]. A continuous extension map of integrable functions on ∂D to a function of bounded variation in D cannot be linear [Pee79].

If we restrict ourselves to the Besov space $B^0_{1,1}(\partial D) \subset L^1(\partial D)$, then a continuous linear extension to functions $BV(D)$ that is right inverse to the trace map exists, see [MSS18, Theorem 1.1].

The trace embedding $V^{s,1}(D \,|\, \mathbb{R}^d) \to \mathcal{T}^{s,1}(D^c)$ cannot be continuous. Consider the sequence of functions

$$u_n(x) := \begin{cases} 0 & , x \in D \\ n^{1-s} & , x \in D^c, \operatorname{dist}(x, \partial D) < 1/n \\ 0 & , x \in D^c, \operatorname{dist}(x, \partial D) \geq 1/n \end{cases}$$

for $n \in \mathbb{N}$. One easily sees that $\|u_n\|_{V^{s,1}(D \,|\, \mathbb{R}^d)} \asymp \|u_n\|_{L^1(D^c;\, \mu_s)} \asymp 1$ but a simple calculation yields $[u_n]_{\mathcal{T}^{s,1}(D^c)} \asymp \ln(n) \to \infty$ as $n \to \infty$.

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