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On nonlocal and nonlinear PDEs
Trondheim, Norway, 24-26 May 2023

Norwegian University of Science and Technology (NTNU)

24/5/2023, 13:30-14:20

Asymptotic profiles for inhomogeneous classical and nonlocal heat equations

THE PROBLEM

$$\partial_t^\alpha u + (-\Delta)^\beta u = f \quad \text{in } \mathbb{R}^N \times \mathbb{R}_+, \qquad u(\cdot, 0) = u_0 \quad \text{in } \mathbb{R}^N$$

- $\alpha \in (0,1]$, $\beta \in (0,1]$
- $u_0 \in L^1(\mathbb{R}^N)$
- $f \in L^1_{\text{loc}}([0,\infty); L^1(\mathbb{R}^N))$

Large-time behaviour

- ❑ Decay/growth **rates**
- ❑ **Profiles**

Applications:

- Anomalous diffusion
- Materials with memory
- Long-range effects

Innocent looking **linear** problem, **BUT**...

Cortázar-Q-Wolanski: JFA 2021, ...

THE OPERATORS

Caputo fractional derivative: $\alpha \in (0,1)$

$$\partial_t^\alpha u(x,t) = \frac{1}{\Gamma(1-\alpha)} \partial_t \int_0^t \frac{u(x,\tau) - u(x,0)}{(t-\tau)^\alpha} \, d\tau$$

Nonlocal: memory effects

Fractional Laplacian: $\beta \in (0,1)$

$$(-\Delta)^\beta u(x,t) = \int_{\mathbb{R}^N} \left(u(x,t) - \frac{u(x-y,t) + u(x+y,t)}{2} \right) |y|^{-N-2\beta} \, \mathrm{d}y$$

- $\mathcal{F}[(-\Delta)^\beta \phi](\eta) = |\eta|^{2\beta} \mathcal{F}[\phi](\eta)$

Nonlocal: long-range effects

HOMOGENEOUS PROBLEM

$\alpha = 1$

$$\partial_t u + (-\Delta)^\beta u = 0 \quad \text{in } \mathbb{R}^N \times \mathbb{R}_+, \qquad u(\cdot, 0) = u_0 \in L^1(\mathbb{R}^N)$$

$$u(\cdot, t) = Z(\cdot, t) * u_0$$

Fundamental solution: Z

$$Z(x, t) = t^{-\frac{N}{2\beta}} F(x / t^{\frac{1}{2\beta}}),$$

$\beta = 1: \ F(\eta) = (4\pi)^{-\frac{N}{2}} \mathrm{e}^{-|\eta|^2/4}$

$F > 0$

$F \in C^\infty(\mathbb{R}^N)$

$F \in L^p(\mathbb{R}^N),\ 1 \le p \le \infty$

→ Smoothing effects

Asymptotic simplification:

$$\lim_{t\to\infty} t^{\frac{N}{2\beta}(1-\frac{1}{p})}\|u(\cdot,t) - MZ(\cdot,t)\|_{L^p(\mathbb{R}^N)} = 0, \quad M = \int_{\mathbb{R}^N} u_0$$

⬇

$$K \subset \mathbb{R}^N \text{compact:} \quad \lim_{t\to\infty} \|t^{\frac{N}{2\beta}} u(\cdot,t) - MF(0)\|_{L^\infty(K)} = 0$$

HOMOGENEOUS PROBLEM

$\alpha \in (0,1)$

$$u(\cdot,t) = Z(\cdot,t) * u_0$$

Fundamental solution: Z

Fourier spatial variable: $\partial_t^\alpha \widehat{Z}(\eta,t) = -|\eta|^{2\beta}\widehat{Z}(\eta,t), \qquad \widehat{Z}(\eta,0) = 1$

$$\widehat{Z}(\eta,t) = E_\alpha(-|\eta|^{2\beta}t^\alpha), \qquad E_\alpha\!: \text{ Mittag-Leffler of order } \alpha$$

$$Z(x,t) = t^{-N\theta}F(xt^{-\theta}), \quad \theta = \alpha/(2\beta), \qquad F\!: \text{ radial Fox } H\text{-function}$$

$F \notin C^1$ → Mild solution

HOMOGENEOUS PROBLEM

$\alpha \in (0,1)$

$F > 0$ smooth outside the origin

$$F(\xi) \to \kappa > 0 \qquad \text{as } |\xi| \to 0,\ N < 2\beta$$

$$F(\xi)/E_N(\xi) \to \kappa \quad \text{as } |\xi| \to 0,\ N \geq 2\beta$$

$$E_N(\xi) = \begin{cases} |\xi|^{2\beta-N}, & N > 2\beta \\ -\ln|\xi|, & N = 2\beta \end{cases}$$

$$|\xi|^{N+2\beta} F(\xi) \leq C, \qquad |\xi| \geq 1$$

Memory

HOMOGENEOUS PROBLEM

$\alpha \in (0,1)$

$$F \in L^p(\mathbb{R}^N) \quad \Longleftrightarrow \quad p \in \begin{cases} [1,\infty], & N < 2\beta \\ [1,\infty), & N = 2\beta \\ [1,p_c), & N > 2\beta \end{cases}$$

$$p_c = N/(N-2\beta)$$

p subcritical

Smoothing effect: L^1–L^p

Asymptotic simplification:

p subcritical ⟶ $$\lim_{t\to\infty} t^{N\theta(1-\frac{1}{p})}\|u(\cdot,t)-MZ(\cdot,t)\|_{L^p(\mathbb{R}^N)}$$

[Kemppainen-Siljander-Vergara-Zacher, 2016], [Cortázar-Q-Wolanski, 2021]

p **not subcritical**: $u_0 \in L^p(\mathbb{R}^N)$ ⟶ $u(\cdot,t) \in L^p(\mathbb{R}^N)$

BUT

$Z(\cdot,t) \notin L^p(\mathbb{R}^N)$ ⟶ Z cannot give the large-time behaviour

AVOID THE ORIGIN ! (in selfsimilar variables)

p supercritical ⟹ $\lim_{t\to\infty} t^{N\theta(1-\frac{1}{p})} \|u(\cdot,t) - MZ(\cdot,t)\|_{L^p(\{|x|\geq \nu t^\theta\})}$

Outer regions:

- **Diffusive** scale: $|x| \asymp t^\theta$
- Most of the mass is here

HOMOGENEOUS PROBLEM

$\alpha \in (0,1)$

K compact $\Rightarrow$ $$\lim_{t\to\infty} \|t^{\alpha} u(\cdot,t) - \kappa I_\beta[u_0]\|_{L^\infty(K)} = 0$$

$$I_\beta[u_0](x) = \int_{\mathbb{R}^N} \frac{u_0(x-y)}{|y|^{N-2\beta}}\,dy$$

"Weak" asymptotic simplification (**memory**)

HOMOGENEOUS PROBLEM: RATES

$\alpha \in (0,1)$

RATES:

$$\|u(\cdot,t)\|_{L^p} \asymp \begin{cases} t^{-\frac{N\alpha}{2\beta}\left(1-\frac{1}{p}\right)} & \text{diffusive scales} \\ t^{-\alpha} & \text{compact sets} \end{cases}$$

Critical p:

$$\|u(\cdot,t)\|_{L^p(\mathbb{R}^N)} \asymp \begin{cases} t^{-\frac{N\alpha}{2\beta}\left(1-\frac{1}{p}\right)} & p \text{ subcritical} \\ t^{-\alpha} & \text{otherwise} \end{cases}$$

Critical dimension phenomenon

[Kemppainen-Siljander-Vergara-Zacher, 2016]

$$\|u(\cdot,t)\|_{L^p(\mathbb{R}^N)} \asymp \begin{cases} t^{-\frac{N\alpha}{2\beta}\left(1-\frac{1}{p}\right)} & N \le 2\beta p/(p-1) \\ t^{-\alpha} & \text{otherwise} \end{cases}$$

THE INHOMOGENEOUS PROBLEM

$$\partial_t^\alpha u - \Delta u = f \quad \text{in } \mathbb{R}^N \times \mathbb{R}_+, \qquad u(\cdot,0)=0$$

$$\|f(\cdot,t)\|_{L^1(\mathbb{R}^N)} \asymp (1+t)^{-\gamma} \quad \text{for some } \gamma \in \mathbb{R}$$

"Duhamel": $$u(x,t) = \int_0^t \int_{\mathbb{R}^N} Y(x-y,t-s) f(y,s)\,\mathrm{d}y\mathrm{d}s$$

$$Y = \partial_t^{1-\alpha} Z$$

INHOMOGENEOUS PROBLEM: PRECEDENTS

$\alpha = 1,\ \beta = 1$

$$f \in L^1([0,\infty); L^1(\mathbb{R}^N)):$$

$$\lim_{t\to\infty} \|u(\cdot,t) - M_\infty Z(\cdot,t)\|_{L^1(\mathbb{R}^N)} = 0$$

$$M_\infty := \int_0^\infty \int_{\mathbb{R}^N} f(x,t)\,\mathrm{d}x\mathrm{d}t < \infty$$

[Biler-Guedda-Karch, 2004], [Dolbeault-Karch, 2006]

$$f \in L^1_{\mathrm{loc}}([0,\infty); L^1(\mathbb{R}^N)),\ f \notin L^1([0,\infty); L^1(\mathbb{R}^N)) + \text{extra conditions:}$$

$$\lim_{t\to\infty} h(t)\|u(\cdot,t) - M(t) Z(\cdot,t)\|_{L^1(\mathbb{R}^N)} = 0$$

$$M(t) := \int_0^t \int_{\mathbb{R}^N} f(x,t)\,\mathrm{d}x\mathrm{d}t = \int_{\mathbb{R}^N} u(x,t)\,\mathrm{d}x$$

[Dolbeault-Karch, 2006]

INHOMOGENEOUS PROBLEM

$\alpha = 1$

$\gamma > 1$, p subcritical:

$$\lim_{t\to\infty} t^{\frac{N}{2\beta}\left(1-\frac{1}{p}\right)} \|u(\cdot,t) - M_\infty Z(\cdot,t)\|_{L^p(\mathbb{R}^N)} = 0$$

Why p subcritical?

Space-time convolution:

$$\|Z(\cdot,t)\|_{L^p(\mathbb{R}^N)} = Ct^{-\frac{N}{2\beta}(1-\frac{1}{p})}$$

$$Z \in L^1_{\text{loc}}([0,\infty); L^p(\mathbb{R}^N)) \quad \Longleftrightarrow \quad p \text{ subcritical}$$

INHOMOGENEOUS PROBLEM

$\alpha = 1$

$\gamma = 1$, p subcritical:

$$\lim_{t\to\infty} \frac{t^{N\theta(1-\frac{1}{p})}}{\log t} \|u(\cdot,t) - M(t)Z(\cdot,t)\|_{L^p(\mathbb{R}^N)} = 0$$

$$M(t) := \int_0^t \int_{\mathbb{R}^N} f(x,t)\,\mathrm{d}x\mathrm{d}t$$

$\gamma < 1$, p subcritical:

$$\lim_{t\to\infty} t^{-1+\gamma+N\theta(1-\frac{1}{p})} \Big\|u(\cdot,t) - \int_0^t M_f(s)Z(\cdot,t-s)\,\mathrm{d}s\Big\|_{L^p(\mathbb{R}^N)} = 0$$

$$M_f(t) := \int_{\mathbb{R}^N} f(y,t)\,\mathrm{d}y$$

INHOMOGENEOUS PROBLEM $\alpha = 1$

$$p \text{ not subcritical: } Z(\cdot,t) \notin L^p(\mathbb{R}^N)$$

HOWEVER

$$\lim_{t\to\infty} t^{-1+\gamma+N\theta(1-\frac{1}{p})}\Big\|u(\cdot,t) - \int_0^t M_f(s)Z(\cdot,t-s)\,\mathrm{d}s\Big\|_{L^p(|x|>\nu t^\theta)} = 0, \quad \gamma<1$$

$$\lim_{t\to\infty} \frac{t^{N\theta(1-\frac{1}{p})}}{\log t}\|u(\cdot,t) - M(t)Z(\cdot,t)\|_{L^p(|x|>\nu t^\theta)} = 0, \quad \gamma=1$$

$$\lim_{t\to\infty} t^{N\theta(1-\frac{1}{p})}\Big\|u(\cdot,t) - M_\infty Z(\cdot,t)\Big\|_{L^p(|x|>\nu t^\theta)} = 0, \quad \gamma>1$$

Space-time convolution

Extra assumption: $\|f(\cdot,t)\|_{L^p(\mathbb{R}^N)} \le C(1+t)^{-\gamma}$

INHOMOGENEOUS: COMPACT SETS

$\alpha = 1,\ N > 2\beta$

$$\|t^{\min\{\gamma, N/(2\beta)\}} u(\cdot,t) - \mathfrak{L}\|_{L^p(K)} \to 0 \quad \text{as } t \to \infty$$

$$\|f(\cdot,t)(1+t)^{\gamma} - g\|_{L^1(\mathbb{R}^N)} \to 0, \quad g \in L^1(\mathbb{R}^N) \quad \text{if } \gamma \le N/(2\beta)$$

$$\mathfrak{L} = \begin{cases} c_\beta I_\beta[g], & \gamma < N/(2\beta) \\ c_\beta I_\beta[g] + M_\infty F(0), & \gamma = N/(2\beta) \\ M_\infty F(0), & \gamma > N/(2\beta) \end{cases}$$

$$I_\beta[g](x) = \int_{\mathbb{R}^N} \frac{g(x-y)}{|y|^{N-2\beta}}\,dy$$

$$t^{\frac{N}{2\beta}} \|u(\cdot,t) - M_\infty Z(\cdot,t)\|_{L^p(K)} \to 0,\ \gamma > N/(2\beta)$$

INHOMOGENEOUS PROBLEM: STATIONARY CASE $\alpha = 1,\ N > 2\beta$

$$f(x,t) = g(x) \qquad (\gamma = 0)$$

$$\|u(\cdot,t) - c_\beta I_\beta[g]\|_{L^p(K)} \to 0 \quad \text{as } t \to \infty$$

$$(-\Delta)^\beta (c_\beta I_\beta[g]) = g$$

BUT

$$|x|^{N-2\beta} I_\beta[g](x) \to \|g\|_{L^1(\mathbb{R}^N)} \text{ as } |x| \to \infty$$

⬇

$$I_\beta[g] \notin L^p(\mathbb{R}^N) \text{ if } p \in [1, p_c)$$

INHOMOGENEOUS PROBLEM: PRECEDENTS

$\alpha \in (0,1)$

$$\gamma > 1: \qquad \lim_{t\to\infty} t^{\sigma(p)} \|u(\cdot,t) - M_\infty Y(\cdot,t)\|_{L^p(\mathbb{R}^N)} = 0$$

$$\sigma(p) = 1 - \alpha + N\theta(1 - \frac{1}{p}) \qquad M_\infty := \int_0^\infty \int_{\mathbb{R}^N} f(x,t)\,\mathrm{d}x\mathrm{d}t < \infty$$

$$\begin{cases} p \in [1,\infty], & N < 4\beta \\ p \in [1, p_c), & N > 4\beta \end{cases} \qquad p_c := N/(N - 2\beta)$$

[Kemppainen-Siljander-Zacher-JDE-2017]

LARGE DIMENSIONS: $N > 4\beta$ $\alpha \in (0,1)$

$$Y(x,t) = t^{-\sigma_*(\alpha,\beta)} G(xt^{-\theta}), \qquad \sigma_*(\alpha,\beta) := 1-\alpha+N\theta$$

$$G > 0 \text{ smooth outside the origin,} \qquad \lim_{|\xi|\to 0} |\xi|^{N-4\beta} G(\xi) = \lambda$$

$$|\xi|^{N+2\beta} G(\xi) \le C, \quad |\xi| \ge 1$$

⬇

$$Y(\cdot,t) \in L^p(\mathbb{R}^N) \iff p \in [1, p_*), \qquad p_* := N/(N-4\beta)$$

$$\|Y(\cdot,t)\|_{L^p(\mathbb{R}^N)} = Ct^{-\sigma(\alpha,\beta,p)}, \quad \sigma(\alpha,\beta,p) := \sigma_*(\alpha,\beta) - \frac{N\theta}{p}$$

$$Y \in L^1_{\text{loc}}([0,\infty); L^p(\mathbb{R}^N)) \iff p \in [1, p_c), \qquad p_c := N/(N-2\beta)$$

INHOMOGENEOUS PROBLEM

$\alpha \in (0,1),\ N > 4\beta$

p subcritical ($p \in [1, p_c)$):

$$\lim_{t\to\infty} \frac{1}{\phi(t)} \left\| u(\cdot,t) - \int_0^t M_f(s) Y(\cdot, t-s)\,\mathrm{d}s \right\|_{L^p(\mathbb{R}^N)} = 0$$

$$\phi(t) = \begin{cases} t^{1-\gamma-\sigma(\alpha,\beta,p)}, & \gamma < 1 \\ t^{-\sigma(\alpha,\beta,p)} \log t, & \gamma = 1 \\ t^{-\sigma(\alpha,\beta,p)}, & \gamma > 1 \end{cases} \qquad M_f(t) := \int_{\mathbb{R}^N} f(y,t)\,\mathrm{d}y$$

$\gamma \geq 1$: SIMPLIFICATION

$\alpha \in (0,1),\ N > 4\beta$

$$\gamma = 1: \quad \lim_{t\to\infty} \frac{t^{\sigma(\alpha,\beta,p)}}{\log t} \big\| u(\cdot,t) - M(t)t^{1-\alpha} Y(\cdot,t) \big\|_{L^p(\mathbb{R}^N)} = 0$$

$$M(t) := \int_0^t M_f(s)(t-s)^{\alpha-1}\,\mathrm{d}s = \int_{\mathbb{R}^N} u(x,t)\,\mathrm{d}x$$

$$\gamma > 1: \quad \lim_{t\to\infty} t^{\sigma(\alpha,\beta,p)} \big\| u(\cdot,t) - M_\infty Y(\cdot,t) \big\|_{L^p(\mathbb{R}^N} = 0$$

$$M_\infty = \lim_{t\to\infty} M(t)t^{1-\alpha} = \int_0^\infty \int_{\mathbb{R}^N} f(y,s)\,\mathrm{d}y\mathrm{d}s$$

OUTER REGIONS

$\alpha \in (0,1),\ N > 4\beta$

Extra assumption: $\|f(\cdot,t)\|_{L^p(\mathbb{R}^N)} \le C(1+t)^{-\gamma}$ if $p \ge p_c$

$$\lim_{t\to\infty} \frac{1}{\phi(t)} \left\| u(\cdot,t) - \int_0^t M_f(s) Y(\cdot, t-s)\,\mathrm{d}s \right\|_{L^p(\{|x|>\nu t^\theta\})} = 0$$

$$\phi(t) = \begin{cases} t^{1-\gamma-\sigma(\alpha,\beta,p)}, & \gamma < 1 \\ t^{-\sigma(\alpha,\beta,p)} \log t, & \gamma = 1 \\ t^{-\sigma(\alpha,\beta,p)}, & \gamma > 1 \end{cases} \qquad M_f(t) := \int_{\mathbb{R}^N} f(y,t)\,\mathrm{d}y$$

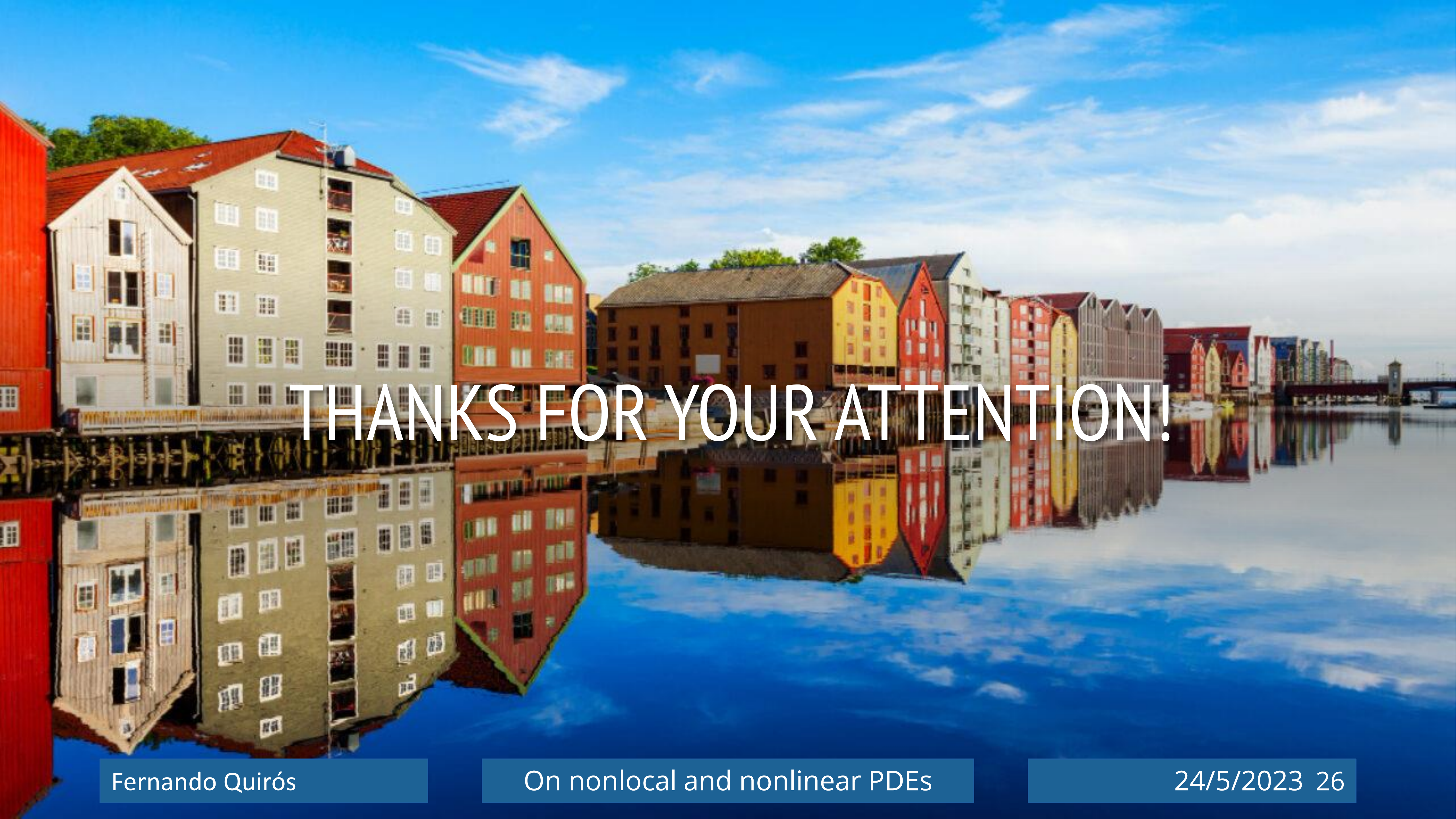
COMPACT SETS: $N > 4\beta$ $\quad\alpha \in (0,1)$

$$\|t^{\min\{\gamma,1+\alpha\}}u(\cdot,t) - \mathfrak{L}\|_{L^p(K)} \to 0 \quad \text{as } t \to \infty$$

$$\|f(\cdot,t)(1+t)^{\gamma} - g\|_{L^1(\mathbb{R}^N)} \to 0, \quad g \in L^1(\mathbb{R}^N) \quad \text{if } \gamma \le 1+\alpha$$

$$\mathfrak{L} = \begin{cases} c_\beta I_\beta[g], & \gamma < 1+\alpha \\ c_\beta I_\beta[g] + \lambda I_{2\beta}[\mathcal{F}], & \gamma = 1+\alpha \\ \lambda I_{2\beta}[\mathcal{F}], & \gamma > 1+\alpha \end{cases}$$

$$I_\beta[h](x) = \int_{\mathbb{R}^N} \frac{h(x-y)}{|y|^{N-2\beta}}\,\mathrm{d}y \qquad \mathcal{F}(x) = \int_0^\infty f(x,s)\,\mathrm{d}s.$$



THANKS FOR YOUR ATTENTION!