NTNU
Faculty of Natural Sciences and Technology
Department of Physics

Exam TFY 4305 Nonlinear dynamics
Fall 2014

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kl. 09.00-13.00

Examination support material:
Approved calculator
Rottmann: Matematisk Formelsamling
Rottmann: Matematische Formelsammlung
Barnett & Cronin: Mathematical Formulae

The problem set is four pages. Useful formulas can be found on page 4. Read carefully. Ask if in doubt. Viel Glück! Veel success! Good luck! Buena suerte! Bonne chance!

Problem 1

In this problem, we consider the population of herring in the North Sea and the effects of fishing. Let $N(t)$ be the population of herring at time t. The model is

$$\dot{N} = rN\left(1-\frac{N}{K}\right) - H\frac{N}{A+N} , \tag{1}$$

where $r > 0$, $A > 0$, $H > 0$ and $K > 0$ are parameters. The first term in (1) is the term in the logistic growth model and the second term incorporates the effects of fishing.

a) Show that (1) can be written in dimensionless form

$$\frac{dx}{d\tau} = x(1-x) - h\frac{x}{a+x} , \qquad (2)$$

and find the dimensionless parameters x, τ, a, and h.

b) $x^* = 0$ is a fixed point. Find the stability of $x^* = 0$ as a function of a and h.

c) Find the other fixed points of the system For which values of a and h are these solutions biologically meaningful? Determine the stability of the fixed points as functions of a and h.

d) Show that a bifurcation takes place at $h = a$ and classify it. Hint: Taylor expansion of (2) about $x = 0$.

e) There is another bifurcation at $h = \frac{1}{4}(a+1)^2$ for $a < a_c$, where a_c is a critical value. Classify the bifurcation and find the critical value a_c.

f) Sketch the bifurcation diagram in the ah-plane and indicate the number of fixed points in the different regions.

Problem 2

Consider the following two-dimensional system

$$\begin{aligned} \dot{x} &= y + ax(1 - 2b - x^2 - y^2) , && (3) \\ \dot{y} &= -x + ay(1 - x^2 - y^2) , && (4) \end{aligned}$$

where a and b are real parameters satisfying $0 < a \leq 1$ and $0 \leq b < \frac{1}{2}$.

a) Rewrite the system in polar coordinates (r, θ).

b) Show that the system has at least one limit cycle and find an expression for the period T. Hint: The origin is the only fixed point of (3)–(4).

c) Classify the fixed point $(x, y) = (0, 0)$ as a function of a and b.

Problem 3

Consider the discrete map

$$x_{n+1} = x_n^2 + cx_n \,, \tag{5}$$

where c is a real parameter.

a) Find the fixed points of the map (5). Determine the stability of the fixed points as functions of c.

b) Find the period-2 cycle. For which values of c does it exist?

c) Classify the bifurcations in a) and b) and sketch the bifurcation diagram.

Problem 4

The questions below are independent of each other.

a) We are going to consider a so-called **fat fractal**. It is constructed as follows: S_0 is a line segment of length 1. S_1 is found by removing the middle quarter of S_0. S_1 then consists of two line segments. S_2 is found by removing a line segment of length $\frac{1}{16}$ in the middle of each of the segments of S_1. The construction is shown in Fig. 1. We repeat this ad infinitum and the fat

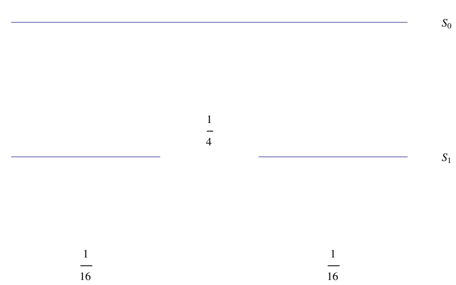

Figure 1: Construction of the fat fractal C'.

fractal is the set $C' = S_\infty$. Find the measure of C'. Is C' countable?

b) In Fig. 2 we have plotted a phase portrait of a two-dimensional system. Find the index of the fixed points located at $(0,0)$ and $(\pi, 0)$.

c) A two-dimensional system is given by

$$\dot{x} = y + \mu x \,, \tag{6}$$

$$\dot{y} = -x + \mu y - x^2 y \,, \tag{7}$$

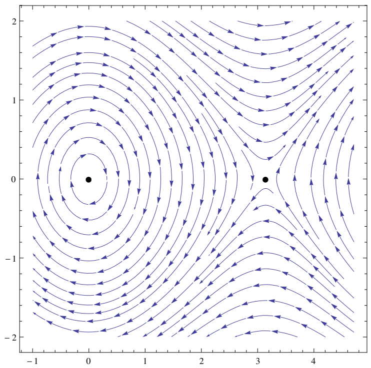

Figure 2: Vector field with fixed points located at $(0, 0)$ and $(\pi, 0)$.

where μ is a real-valued parameter. The left panel in Fig. 3 shows a plot of the vector field (6)–(7) for $\mu = -0.1$. The right panel shows the vector field for $\mu = 0.1$. The system goes through a bifuration at a critical value μ_c. Find μ_c and classify the bifurcation.

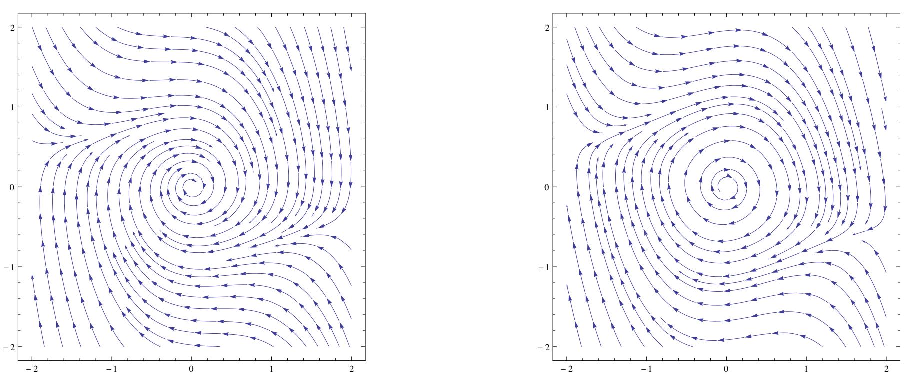

Figure 3: Vector field (6)–(7) for $\mu = -0.1$ (left) and $\mu = 0.1$ (right).

Useful formulas:

$$\dot{\theta} = \frac{x\dot{y} - y\dot{x}}{r^2}\,, \tag{8}$$

$$(x^2 + cx)^2 + c(x^2 + cx) - x = [x(x + c - 1)]\left[x^2 + (c + 1)x + c + 1\right]. \tag{9}$$