TFY4305 solutions exercise set 8 2014

Problem 6.3.9

The dynamics is governed by the set of equations

$$\dot{x} = y^3 - 4x \ , \tag{1}$$
$$\dot{y} = y^3 - y - 3x \ . \tag{2}$$

a) Taking the difference between $\dot{x} = 0$ and $\dot{y} = 0$, we find $y = x$. Inserted in either equation, we find $y^3 - 4y = 0$ and so $y = 0$ or $y = \pm 2$. The fixed points are therefore $\underline{\underline{(0,0)}}$, $\underline{\underline{(-2,-2)}}$, and $\underline{\underline{(2,2)}}$. The Jacobian matrix is

$$A(x,y) = \begin{pmatrix} -4 & 3y^2 \\ -3 & 3y^2 - 1 \end{pmatrix} . \tag{3}$$

Evaluated at the origin, we find

$$A(0,0) = \begin{pmatrix} -4 & 0 \\ -3 & -1 \end{pmatrix} , \tag{4}$$

and so $\lambda_1 = -4$ and $\lambda_2 = -1$. The origin is therefore a $\underline{\underline{\text{stable node}}}$. The corresponding eigenvectors are $(1,1)$ and (0,1).

Evaluated at $\pm(2,2)$, we find

$$A(\pm(2,2)) = \begin{pmatrix} -4 & 12 \\ -3 & 11 \end{pmatrix} , \tag{5}$$

and so $\lambda_1 = 8$ and $\lambda_2 = -1$. Hence $\pm(2,2)$ are $\underline{\underline{\text{saddle points}}}$. The eigenvectors are $(1,1)$ and $(4,1)$.

b) From Eqs. (1) and (2), we obtain

$$\frac{dy}{dx} = \frac{y^3 - y - 3x}{y^3 - 4x} \ . \tag{6}$$

For a point on the axis $y = x$, this yields

$$\frac{dy}{dx} = 1 \ . \tag{7}$$

Hence, the tangent to the trajectory $y(x)$ is along the line $y = x$ and so we will stay on the line.

c) Eqs. (1) and (2) give

$$\frac{d(x-y)}{dt} = -(x-y) \ . \tag{8}$$

Integration of this equation yields

$$x(t) - y(t) = (x_0 - y_0)e^{-(t-t_0)} \ , \tag{9}$$

where $x_0 = x(t_0)$ and $y_0 = y(t_0)$. This shows that

$$\underline{\underline{\lim_{t\to\infty} |x(t) - y(t)| = 0}} \ . \tag{10}$$

d) The phase portrait is shown in Fig. 1. close to the origin we clearly see the stable

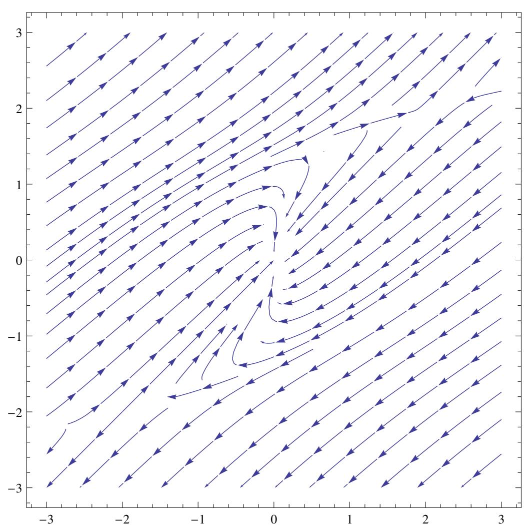

Figure 1: Phase portrait of problem 6.3.9.

eigendirection given by the line $x = 0$, which is the corresponding eigendirection for the eigenvector of $A(0, 0)$. Likewise, the unstable direction close the points $(\pm(2, 2)$ is given by the line $y = x$.

Problem 6.3.10

The dynamics is governed by the set of equations

$$\dot{x} = xy \,, \tag{11}$$
$$\dot{y} = x^2 - y \,. \tag{12}$$

a) The Jacobian matrix is

$$A(x,y) = \begin{pmatrix} y & x \\ 2x & -1 \end{pmatrix} . \tag{13}$$

Evaluated at the origin, we find

$$A(0,0) = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} , \tag{14}$$

and so $\lambda_{1,2} = -1, 0$. Hence $\Delta = 0$ and linearization predicts that the origin is a nonisolated fixed point.

b) $\dot{x} = 0$ gives either $x = 0$ or $y = 0$. Inserting this into $\dot{y} = 0$ then implies $y = 0$ or $x = 0$. Hence the origin is an isolated fixed point.

c) We note that $\dot{x} = 0$ and $\dot{y} = -y$ along the y-axis. This implies that one flows towards the origin along the y-axis exponentially fast. Assume that you start in the positive quadrant, (x_0, y_0). Can you ever cross one of the axes? You cannot cross the y-axis since the flow here is vertical (recall uniqueness!). Similarly you cannot cross the x-axis since $\dot{y} = x^2 > 0$ here. This implies that $\dot{x} > 0$ for all t and you will flow away from the fixed point. If you start in the 4th quadrant, $\dot{x}$ is negative and $\dot{y}$ is positive. Eventually you will cross the x-axis and from there you will flow away from $x = 0$. Since the equations are symmetric under $t \to -t$ and $x \to -x$, the same arguments are valid, except arrows must be reversed. Hence if we start in the 2nd or 3rd quardrant, we eventually have $\dot{x} < 0$ and we move away $x = 0$. The phase portrait is shown in Fig. 2.

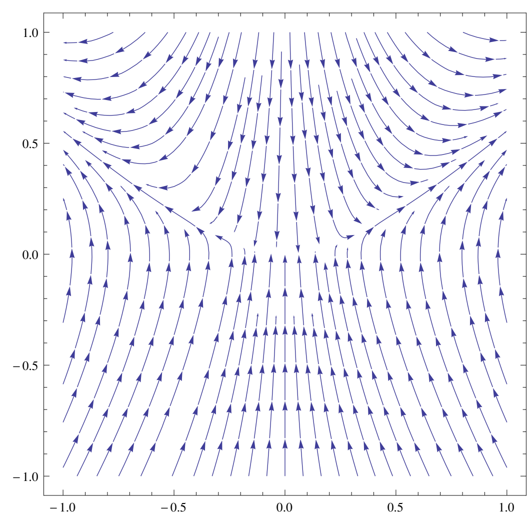

Figure 2: Phase portrait of problem 6.3.10.