TFY4305 solutions exercise set 5 2014

Problem 3.5.4

a) Looking at Fig. 1 in the textbook, we see that the restoring force is given

$$F = (\sqrt{h^2 + x^2} - L_0)k \ . \tag{1}$$

The force is in the direction of the spring. We need the component of the force along the wire, which is given by

$$\begin{aligned} F_{\text{wire}} &= F \sin\theta \\ &= F \frac{x}{\sqrt{h^2 + x^2}} \ , \end{aligned} \tag{2}$$

Newton's second law of motion then becomes

$$m\ddot{x} + b\dot{x} + kx\left(1 - \frac{L_0}{\sqrt{h^2 + x^2}}\right) = 0 \ . \tag{3}$$

b) Equilibrium solutions are given by $\ddot{x} = \dot{x} = 0$. This yields

$$kx\left(1 - \frac{L_0}{\sqrt{h^2 + x^2}}\right) = 0 \ . \tag{4}$$

Thus $x = 0$ and $1 - \frac{L_0}{\sqrt{h^2+x^2}} = 0$. This gives $\underline{\underline{x = 0}}$ and $\underline{\underline{x = \pm\sqrt{L_0^2 - h^2}}}$. The nonzero solutions exist only if $L_0 \geq h$.

c) If $m = 0$, the equation of motion reads

$$\dot{x} = \frac{kx}{b}\left(\frac{L_0}{\sqrt{h^2 + x^2}} - 1\right) \ . \tag{5}$$

The function that determines the stability of the fixed points is

$$f(x) = \frac{kx}{b}\left(\frac{L_0}{\sqrt{h^2 + x^2}} - 1\right) \ . \tag{6}$$

The derivative is given by

$$f'(x) = \frac{k}{b}\left(\frac{L_0}{\sqrt{h^2+x^2}} - 1 - \frac{L_0 x^2}{(k^2+x^2)^{\frac{3}{2}}}\right) . \quad (7)$$

This yields

$$f'(0) = \frac{k}{b}\left(\frac{L_0}{h} - 1\right) \quad (8)$$

The origin is stable for $L_0 < h$ and unstable for $L_0 > h$. Moreover

$$f'(\pm\sqrt{L_0^2 - h^2}) = \frac{k}{b}\left(1 - \frac{L_0^2}{h^2}\right) . \quad (9)$$

Thus the fixed points $x = \pm\sqrt{L_0^2 - h^2}$ are stable since $L_0 \geq h$. The stability of the fixed points depends on the ratio $y \equiv L_0/h$. We therefore plot the bifurcation diagram as a function of this dimensionles variable.

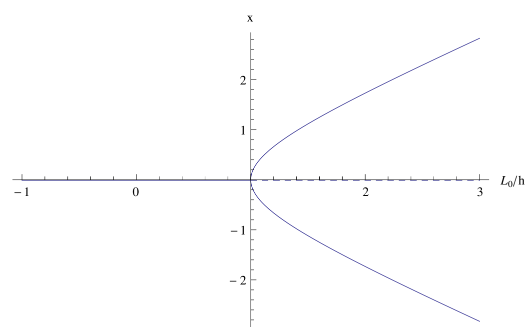

Figure 1: Bifurcation diagram for the bead on a wire. The origin changes stability in $x = 0$.

d) We can rewrite the equation of motion as

$$\frac{m}{k}\frac{d^2x}{dt^2} + \frac{b}{k}\frac{dx}{dt} + x\left(1 - \frac{L_0}{\sqrt{h^2+x^2}}\right) = 0 . \quad (10)$$

Defining a new dimensionless time variable τ via

$$t = \frac{b}{k}\tau . \quad (11)$$

In terms of the new time variable, Newton's equation reads

$$\frac{m}{k}\frac{k^2}{b^2}\frac{d^2x}{d\tau^2} + \frac{dx}{d\tau} + x\left(1 - \frac{L_0}{\sqrt{h^2+x^2}}\right) = 0 . \quad (12)$$

One can therefore ignore the second-order term if

$$\underline{\underline{\frac{mk}{b^2} \ll 1}} . \quad (13)$$

Problem 3.6.5

The forces acting on the bead are gravity and the force from the spring (see Fig. 3.6.7). The component of gravity along the wire is $F = mg\sin\theta$. The force from the spring is

$$F_{\text{spring}} = k\left(\sqrt{x^2+a^2} - L_0\right) . \tag{14}$$

The component of this force along the wire is

$$F_{\text{wire}} = -\frac{x}{\sqrt{x^2+a^2}} F_{\text{spring}} . \tag{15}$$

The sum of F and F_{wire} is zero when the system is in equilibrium. This yields

$$mg\sin\theta - kx\left(1 - \frac{L_0}{\sqrt{x^2+a^2}}\right) = 0 . \tag{16}$$

b) Dividing Eq. (16) by kx, we can write

$$\frac{mg}{kx}\sin\theta = \left(1 - \frac{L_0}{a}\frac{1}{\sqrt{1+(\frac{x}{a})^2}}\right) . \tag{17}$$

Rearranging terms, we find

$$1 - \frac{mg}{ka}\frac{a}{x}\sin\theta = \frac{L_0}{a}\frac{1}{\sqrt{1+(\frac{x}{a})^2}} . \tag{18}$$

This is on the form

$$1 - \frac{h}{u} = \frac{R}{\sqrt{1+u^2}} , \tag{19}$$

if we identify $\underline{\underline{u = x/a}}$, $\underline{\underline{R = L_0/a}}$, and $\underline{\underline{h = \frac{mg}{ka}\sin\theta}}$.

c) Note that the variable u can be positive and negative. Without loss of generality we can restrict h to positive values (negative values correspond to tilting the wire the other way) The function $g(u) = 1 - h/u$ approaches zero as $u \to \pm\infty$. The maximum of the function $h(u) = R/\sqrt{1+u^2}$ is R. For $R < 1$, it is then clear that the functions intersect at a single point and hence there is one fixed point. This is shown in Fig. 2

For $R > 1$, the function $g(u)$ still intersects the function $h(u)$ for positive values of u. However, there may be one or two intersections for negative values of u depending on h. In fact, as Figs. 3 and 4 suggest, a saddle-node bifurcation is taking place at a critical value of h. Since the derivatives of the functions $g(u)$ and $h(u)$ are equal at the bifurcation, h and u satisfy the two equations

$$1 - \frac{h}{u} = \frac{R}{\sqrt{1+u^2}} , \tag{20}$$

$$\frac{h}{u^2} = -\frac{Ru}{(1+u^2)^{\frac{3}{2}}} . \tag{21}$$

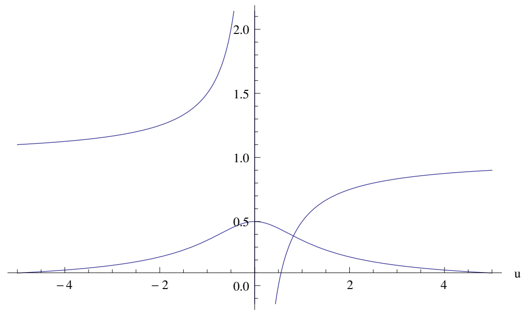

Figure 2: Graphical analysis. $R = \frac{1}{2}$ and $h = \frac{1}{2}$.

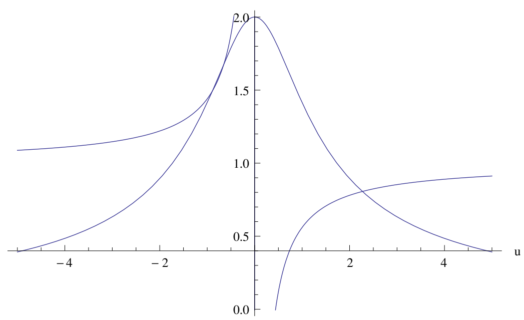

Figure 3: Graphical analysis. $R = 2$ and $h = 0.44$.

d) The right-hand-side of Eq. (19) can be expanded in powers of u. This yields

$$1 - \frac{h}{u} \approx (r+1)(1 - \frac{1}{2}u^2) , \tag{22}$$

where $r = R - 1$. Ignoring the term $\sim ru^2$ which is much smaller than the term $\sim u^2$, we obtain

$$-\frac{h}{u} \approx r - \frac{1}{2}u^2 , \tag{23}$$

or

$$\underline{\underline{\frac{1}{2}u^3 - ru - h = 0}} . \tag{24}$$

e) We first define the functions $f(u) = \frac{1}{2}u^3 - ru$ and $g(u) = h(u)$. The bifurcation takes place when $f(u) = g(u)$ and $f'(u) = g'(u)$. The latter equation gives

$$\frac{3}{2}u^2 - r = 0 . \tag{25}$$

This yields $\underline{\underline{R(u) = 1 + \frac{3}{2}u^2}}$. The equation $g(u) = h(u)$ gives $\underline{\underline{h(u) = \frac{1}{2}u^3 - ru = -u^3}}$

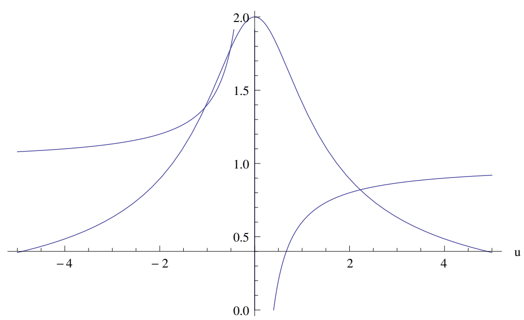

Figure 4: Graphical analysis. $R = 2$ and$h = 0.4$.

f) The exact equations for the bifurcation are given by Eqs. (20) and (21). Solving the latter with respect to $\frac{h}{u}$ and inserting it into the former, one finds

$$1 + \frac{Ru^2}{(1+u^2)^{\frac{3}{2}}} \quad = \quad \frac{R}{\sqrt{1+u^2}} \ . \tag{26}$$

Solving this equation with respect to R gives

$$\underline{\underline{R(u) = (1+u^2)^{\frac{3}{2}}}} \ . \tag{27}$$

Using Eq. (20), we find

$$\underline{\underline{h(u) = -\frac{Ru^3}{(1+u^2)^{\frac{3}{2}}} = -u^3}} \tag{28}$$

Note that the exact Eqs. (27) and (28) reduce to the approximate solutions upon expanding $R(u)$ to first nontrivial order in u for small u.

g) and h) were not a part of the exercise.