TFY4305 solutions exercise set 4 2014

Problem 3.4.5

The dynamics is governed by the equation

$$\dot{x} \quad = \quad r - 3x^2 \; , \tag{1}$$

where r is a parameter. The fixed points are $\underline{\underline{x^* = \pm\sqrt{\frac{r}{3}}}}$, and exist only for $r \geq 0$. Moreover $f'(x) = -6x$ and $f'(x^*) = \mp 6\sqrt{\frac{r}{3}}$. The positive fixed point is therefore always stable and the negative is always unstable. When they coalesce, i.e. when $r = 0$, it is half-stable. The bifurcation at $r = 0$ is a saddle-node bifurcation.

Problem 3.4.6

The dynamics is governed by the equation

$$\dot{x} \quad = \quad rx - \frac{x}{1+x} \; , \tag{2}$$

where r is a parameter. The fixed points are found by solving $rx = x/(1+x)$, which gives $\underline{\underline{x^* = 0}}$ and $\underline{\underline{x^* = \frac{1}{r} - 1}}$. Moreover,

$$f'(x) \quad = \quad r - \frac{1}{(1+x)^2} \; , \tag{3}$$

and therefore $f'(x = 0) = r - 1$. $x = 0$ is thus unstable for $r > 1$ and stable for $r < 1$. Furthermore $f'(x = \frac{1}{r} - 1) = r(1 - r)$. Thus the fixed point $x^* = \frac{1}{r} - 1$ is unstable for $0 < r < 1$ and stable for $r < 0$ and $r > 1$. This implies that the system undergoes a transcritical bifurcation at $r = 1$, where the origin changes stability. The bifurcation diagram is shown in Fig. 1. **Note**: The fixed point $x = \frac{1}{r} - 1$ exists for $r < 0$ as well but it does not undergo any bifurcation.

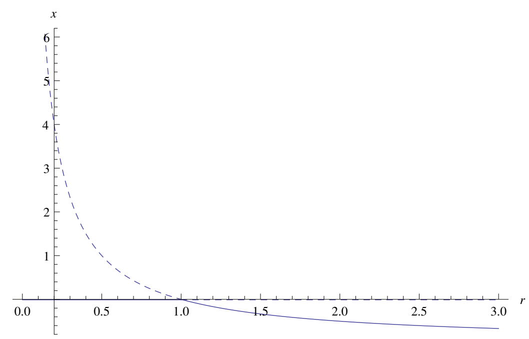

Figure 1: Bifurcation diagram of problem 3.4.6.

Problem 4.3.8

The dynamics is governed by

$$f(\theta) \quad = \quad \frac{\sin 2\theta}{1+\mu \sin\theta} \; . \tag{4}$$

The fixed points are given by the zeros of the function $f(\theta)$ and the solutions are $\theta = 0$, $\theta = \frac{1}{2}\pi$, $\theta = \pi$ and $\theta = \frac{3}{2}\pi$. We must be a little careful, however, when the denominator vanishes at the fixed points. This can take place only for $\mu = \pm 1$. For example, for $\mu = 1$, one can show by L'Hopital's rule that

$$\lim_{\theta \to \frac{3}{2}\pi^{\pm}} f(\theta) = \pm\infty \; , \tag{5}$$

i. e. it is no longer a fixed point.

In Fig. 2, we show the function $f(\theta)$ for $\mu = 0.95$, $\mu = 1.0$, and $\mu = 1.15$. Clearly the fixed point $\theta = \frac{3}{2}\pi$ exists for all values of μ except $\mu = \pm 1$, but it changes stability from stable to unstable. Hence this is a transcritical bifurcation. The other fixed points $\theta = 0$, $\frac{1}{2}\pi$, and π remain unstable, stable, and unstable, respectively. Note that this seems to contradict the rule that fixed points are stable and unstable in an alternating manner. However, this requires a continuous function and $f(\theta)$ is not continuous for at $\theta = \frac{3}{2}\pi$ for $\mu = 1$.

The situation is analogous for $\mu = -1$, where the fixed point $\theta = \pi/2$ change stability.

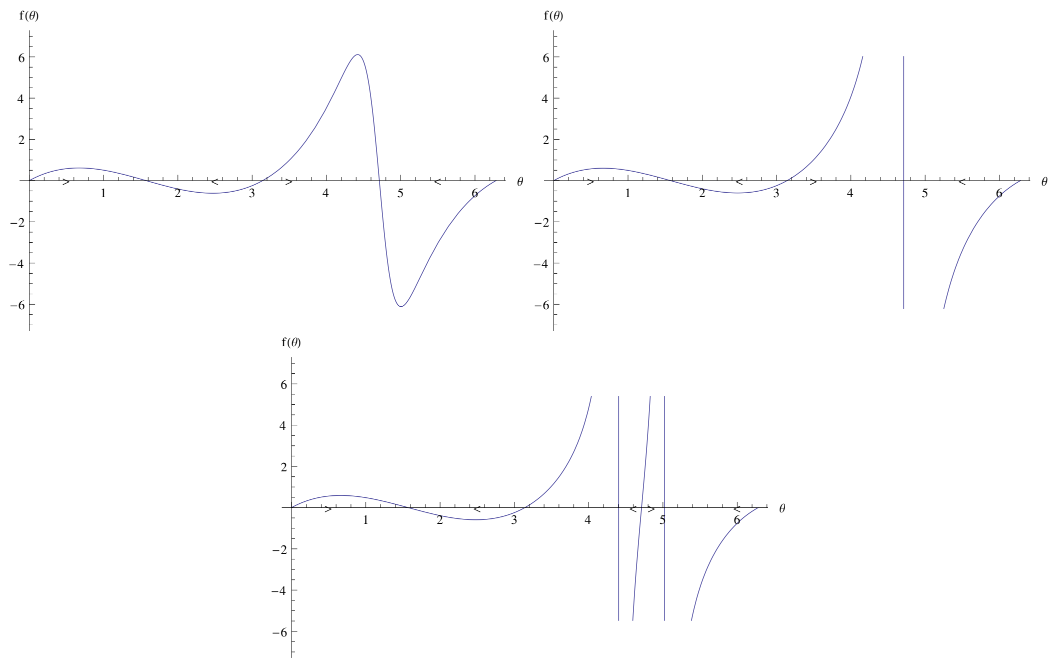

Figure 2: The function $f(\theta)$ for $\mu = 0.95$, $\mu = 1.0$, and $\mu = 1.15$.