TFY4305 solutions exercise set 3 2014

Problem 3.6.2

a) The dynamics is governed by the equation

$$\dot{x} = h + rx - x^2 , \tag{1}$$

where h and r are parameters.

The fixed points are found by solving $h + rx - x^2 = 0$ which yields

$$x_\pm = \frac{r \pm \sqrt{r^2 + 4h}}{2} . \tag{2}$$

i) $h = 0$:
In this case, the fixed points are given by $x = 0$ and $x = r$. Futhermore $f'(x) = r - 2x$ and so $f'(0) = r$ and $f'(r) = -r$. The origin is stable and $x = r$ is unstable for $r < 0$ and vice versa. The bifurcation diagram for the transcritical bifurcation is shown in Fig. 1.

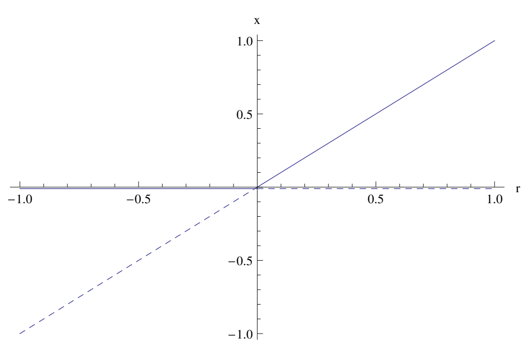

Figure 1: Bifurcation diagram for $h = 0$. The origin changes stability in $r = 0$.

ii) $h > 0$:
The solutions exist for all values of h. Moreover $f'(x_\pm) = \mp\sqrt{r^2 + 4h}$ and so x_+ is always stable while x_- is always unstable. This is shown in Fig. 2.

iii) $h < 0$:
The solutions exist only if $r^2 + 4h \geq 0$ i.e. if $h \geq -r^2/4$. This defines a critical value

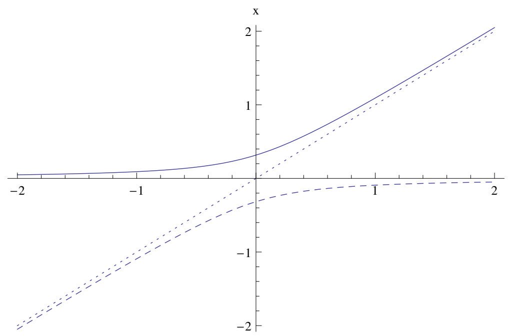

Figure 2: Bifurcation diagram for $h = 0.1$.

$h_c(r) = -\frac{r^2}{4}$ or $r_c(h) = \pm\sqrt{-4h}$. Moreover, $f'(x_\pm) = \mp\sqrt{r^2+4h}$ so the fixed point x_+ is always stable and x_- is always unstable. This is shown in Fig. 3.

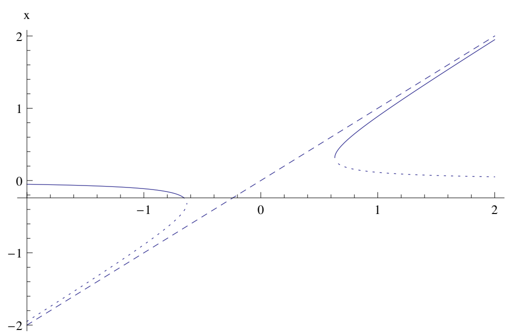

Figure 3: Bifurcation diagram for $h = -0.1$. The upper branch is stable and the lower branch is unstable. The two curves correspond to $r < 0$ and $r > 0$, respectively.

b) In Fig. 4, we show the different regions in the rh-plane separated by the curve $h_c(r) = -r^2/4$.

c) The potential satisfies

$$\frac{dV}{dx} = -h - rx + x^2 \; . \tag{3}$$

Integration yields

$$V(x) = \frac{1}{3}x^3 - \frac{1}{2}rx^2 - hx \; , \tag{4}$$

where we have set the integration constant to zero.

In Fig. 5, we plot the potential $V(x)$ for the different regions in the rh-plane. Solid curve: $r = \frac{1}{2}$ and $h = \frac{1}{4}$ dotted curve: $r = \frac{1}{2}$ and $h = -\frac{1}{16}$, and dashed curve: $r = \frac{1}{2}$ and $h = -\frac{1}{4}$.

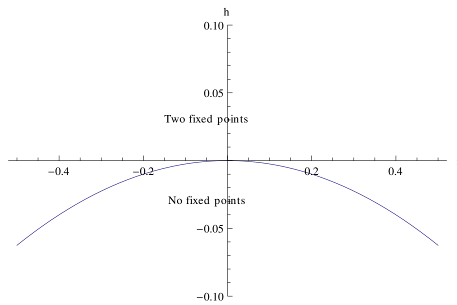

Figure 4: Regions in the rh-plane with different number of fixed points.

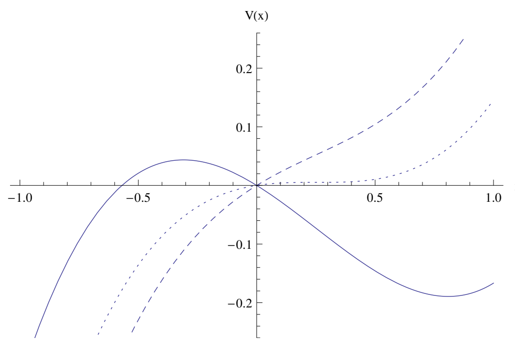

Figure 5: Potential $V(x)$ for different values of r and h. See main text for details.

The local maximum of $V(x)$ corresponds to the unstable fixed point and the local minimum corresponds to the stable fixed point. When there is no local extrema there are no fixed points.

Problem 4.1.2

The dynamics is governed by

$$\dot{\theta} \;=\; 1 + 2\cos\theta \;. \tag{5}$$

The fixed points are found by solving the equation $1 + 2\cos\theta = 0$. This yields the fixed points $\underline{\underline{\theta = \frac{2\pi}{3}}}$ and $\underline{\underline{\theta = \frac{4\pi}{3}}}$. Moreover

$$f'(\theta) \;=\; -2\sin\theta \;, \tag{6}$$

and so $f'(\frac{2\pi}{3}) = -\sqrt{3}$ and $f'(\frac{4\pi}{3}) = \sqrt{3}$. The fixed point $x^* = \frac{2\pi}{3}$ is therefore stable and the fixed point $x =^* \frac{4\pi}{3}$ is unstable. The function $f(\theta)$, the flow and the fixed points are shown in Fig. 6.

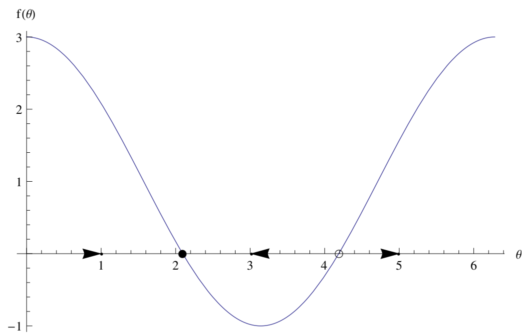

Figure 6: Vector field of $f(\theta) = 1 + 2\cos\theta$.

Problem 4.4.1

The equation is

$$mL^2\ddot{\theta} + b\dot{\theta} + mgL\sin\theta \quad = \quad \Gamma \ . \tag{7}$$

Division by mgL yields

$$\frac{L}{g}\ddot{\theta} + \frac{b}{mgL}\dot{\theta} + \sin\theta \quad = \quad \frac{\Gamma}{mgL} \ . \tag{8}$$

We next introduce a dimensionless time variable τ via

$$\tau \quad = \quad \frac{mgL}{b}t \ . \tag{9}$$

This yields the dimensionless equation

$$\frac{L^3m^2g}{b^2}\frac{d^2\theta}{d\tau^2} + \frac{d\theta}{d\tau} + \sin\theta \quad = \quad \frac{\Gamma}{mgL} \ . \tag{10}$$

The first term can be ignored if the coefficient of $\frac{d^2\theta}{d\tau^2}$ is much smaller than unity, i.e. if

$$\underline{\underline{L^3m^2g \ll b^2}} \ . \tag{11}$$