TFY4305 solutions exercise set 19 2014

Problem 10.7.1

a) We make the ansatz

$$g(x) \approx 1 + c_2 x^2 \ . \tag{1}$$

Inserting this into the equation $g(x) = \alpha g(g(x/\alpha))$ yields

$$\begin{aligned} 1 + c_2 x^2 &= \alpha \left[1 + c_2 \left(1 + c_2 \frac{x^2}{\alpha^2} \right)^2 \right] \ . \\ &= \alpha \left[1 + c_2 \left(1 + c_2 \frac{x^2}{\alpha^2} + c_2^2 \frac{x^4}{\alpha^4} \right) \right] \ , \end{aligned} \tag{2}$$

Multiplying out the right-hand-side, matching powers of x and omitting the x^4-terms, we find

$$1 = \alpha(1 + c_2) \ , \tag{3}$$

$$c_2 = \frac{2c_2^2}{\alpha} \ . \tag{4}$$

Insering the second into the first, we obtain a quadratic equation for α

$$\frac{1}{2}\alpha^2 + \alpha - 1 = 0 \ . \tag{5}$$

The solutions are

$$\alpha = -1 \pm \sqrt{3} \ : \tag{6}$$

The relevant solution is $\underline{\underline{\alpha = -1 - \sqrt{3}}}$. This yields $\underline{\underline{c_2 = -1/2 - \sqrt{3}/2}}$ since $c_2 < 0$ in order for $g(x)$ to have maximum.

b) The ansatz for $g(x)$ is now

$$g(x) \approx 1 + c_2 x^2 + c_4 x^4 \ . \tag{7}$$

This yields

$$g(x) \approx 1 + c_2 x^2 + c_4 x^4 \; . \quad = \tag{8}$$

Inserting this into the equation $g(x) = \alpha g(g(x/\alpha))$, we obtain

$$1 + c_2 x^2 + c_4 x^4 = \alpha \left[1 + c_2 \left(1 + c_2 \frac{x^2}{\alpha^2} + c_4 \frac{x^4}{\alpha^4} \right)^2 + c_4 \left(1 + c_2 \frac{x^2}{\alpha^2} + c_4 \frac{x^4}{\alpha^4} \right)^4 \right] . \tag{9}$$

Multiplying out the right-hand-side, matching powers of x thorugh order x^4, we find

$$\begin{aligned} 1 &= \alpha(1 + c_2 + c_4) \, , && (10) \\ 1 &= \frac{1}{\alpha}[2c_2 + 4c_4] \, , && (11) \\ c_4 &= \frac{1}{\alpha^3}\left[c_2^3 + 2c_2c_4 + 4c_4^2 + 6c_2^2c_4\right] \, . && (12) \end{aligned}$$

This yields

$$\begin{aligned} \alpha &= \frac{1}{1 + c_2 + c_4} \, , && (13) \\ c_2 &= -2 - \frac{1}{2}\alpha + \frac{2}{\alpha} \, , && (14) \\ c_4 &= 1 + \frac{1}{2}\alpha - \frac{1}{\alpha} \, . && (15) \\ & && (16) \end{aligned}$$

Numerical solution of the coupled equations gives

$$\begin{aligned} \alpha &= \underline{\underline{-2.53403}} \, , && (17) \\ c_2 &= \underline{\underline{-1.52224}} \, , && (18) \\ c_4 &= \underline{\underline{0.12761}} \, . && (19) \end{aligned}$$

Problem 10.7.5

a) From example 10.7.1 in the textbook, we have $\underline{\underline{f(x, R_0) = -x^2}}$ and $f(x, R_1) = 1 - x^2$. This yields $f(f(x)) = 1 - (1 - x^2)^2$ and therefore

$$\begin{aligned} \alpha f(f(x/\alpha, R_1)) &= \alpha\left[1 - (1 - x^2/\alpha^2)^2\right] \\ &= \underline{\underline{\frac{2}{\alpha}x^2 - \frac{1}{\alpha^3}x^4}} \, . && (20) \end{aligned}$$

b) Equating $f(x, R_0)$ and Eq. (20) , neglecting terms of order x^4, yields

$$-1 = \frac{2}{\alpha} \, , \tag{21}$$

or $\underline{\underline{\alpha = -2}}$.