TFY4305 solutions exercise set 16 2014

Problem 8.2.11

a) The damped Duffing equation reads

$$\ddot{x} + \mu\dot{x} + x - x^3 = 0 \; . \tag{1}$$

This can be written as

$$\begin{aligned} \dot{x} &= y \; , & (2) \\ \dot{y} &= -\mu y - x + x^3 \; . & (3) \end{aligned}$$

The fixed points are given by $(0, 0)$ and $(\pm 1, 0)$. The Jacobian matrix reads

$$A(x, y) = \begin{pmatrix} 0 & 1 \\ -1 + 3x^2 & -\mu \end{pmatrix} \; . \tag{4}$$

This yields

$$A(0, 0) = \begin{pmatrix} 0 & 1 \\ -1 & -\mu \end{pmatrix} \; , \tag{5}$$

and the eigenvalues are

$$\lambda = \frac{-\mu \pm \sqrt{\mu^2 - 4}}{2} \; . \tag{6}$$

This shows that the real part goes through a zero as μ goes through zero. The imaginary part is nonzero. Thus the origin goes from a stable spiral to an unstable spiral as μ decreases through zero.

b) The phase portraits for $\mu = 1$, $\mu = 0$, and $\mu = -1$ are shown in Fig. 1. The other fixed points $(\pm 1, 0)$ are clearly visible. These are saddles.

The fixed point loses its stability for $\mu = 0$, but there are no limit cycles before or after the bifurcation. We notice that there is a band of closed curves for $\mu = 0$. Thus there is no

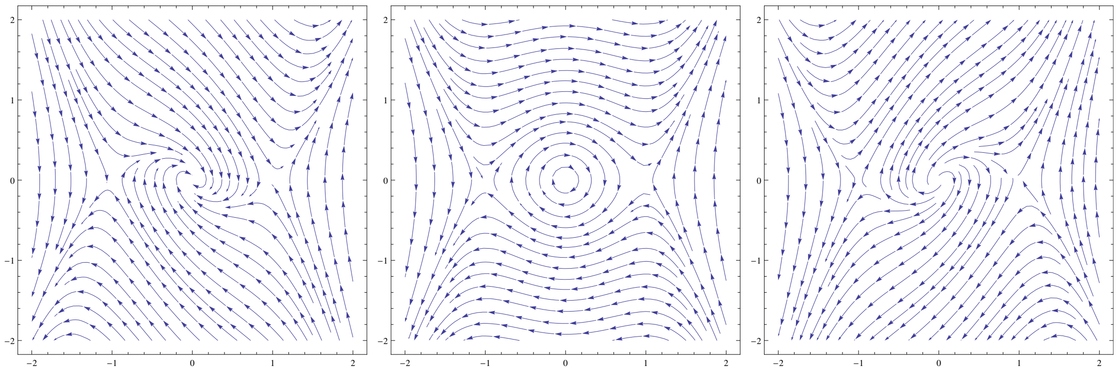

Figure 1: Phase portrait for $\mu = 1$, $\mu = 0$, and $\mu = -1$ for problem 8.2.11.

limit cycle since the periodic solutions are not isolated. Since μ corresponds to a conservative system (no damping) the Hopf bifurcation is of degenerate type. The conserved quantity for $\mu = 0$, is easily found from Eq. (1) by multiplying with $\dot{x}$:

$$\dot{x}(\ddot{x} + x - x^3) = 0 \; . \tag{7}$$

Upon integration, this yields

$$\frac{1}{2}(\dot{x}^2 + \frac{1}{2}x^2 - \frac{1}{4}x^4) \;\; = \;\; c \; , \tag{8}$$

where c is an integration constant.

Problem 8.4.2

The equations that govern the dynamics are

$$\begin{aligned} \dot{r} &= r(\mu - \sin r) \; , && (9) \\ \dot{\theta} &= 1 \; , && (10) \end{aligned}$$

where μ is a parameter. Note that the problem is one-dimensional since the equations for $\dot{\theta}$ and $\dot{r}$ are decoupled. The derivative of function $f(r) = r(\mu - \sin r)$ then determines the fixed points of $f(r) = 0$, which corresponds to circles. This yields

$$f'(r) \;\; = \;\; \mu - \sin r - r\cos r. \tag{11}$$

$r = 0$ is a solution to $f(r) = 0$ and $f'(0) = \mu$ and so the sign of μ determines the stability of the origin. We see that for $\mu > 1$, $\dot{r} = 0$ has no other solutions. This is shown in Fig. 2 (upper left). For $\mu = 1$, a half-stable limit cycle is born. This has radius $r = \frac{1}{2}\pi, \frac{5}{2}\pi$ etc. This is shown in Fig. 2 (upper right). For $\mu < 1$, this splits into a stable and an unstable limit cycle (lower left). The stable limit cycles moves towards the origin as μ decreases and

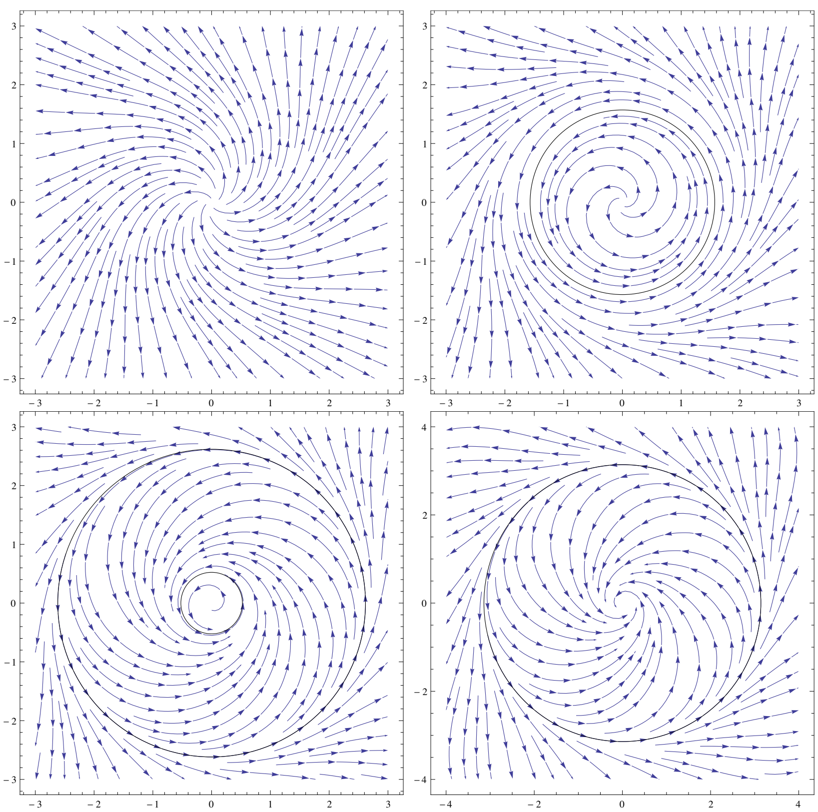

Figure 2: Phase portrait for $\mu = 2$, $\mu = 1$, $\mu = \frac{1}{2}$, $\mu = 0$, and $\mu = -1$ in problem 8.4.2.

coalesce with $r = 0$ for $\mu = 0$ (lower right). A similar pattern is found for negative μ. Using $\mu - \sin r^* = 0$, the stability is determined by $f'(r^*) = -r^* \cos r^*$.

In Fig. 3, we show the intersection of $g(x) = \sin x$ and $h(x) = \mu$ for $\mu = 1$, $\mu = \frac{1}{2}$, and $\mu = -1$ giving the radii of the closed orbits.

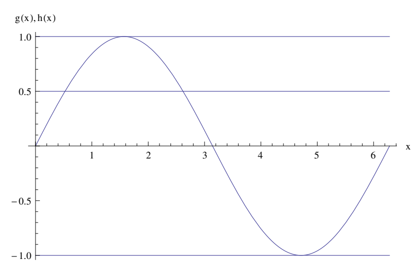

Figure 3: Intersection of $g(x) = \sin x$ and $h(x) = \mu$ for $\mu = 1$, $\mu = \frac{1}{2}$, and $\mu = -1$.