TFY4305 solutions exercise set 12 2014

Problem 7.1.8

The dynamics is governed by the second-order equation

$$\ddot{x} + a\dot{x}(x^2\dot{x}^2 - 1) + x = 0 , \tag{1}$$

where $a > 0$ is a constant. The equation can be written as

$$\dot{x} = y , \tag{2}$$
$$\dot{y} = -x - ay(x^2 + y^2 - 1) . \tag{3}$$

a) The only fixed point is the origin. The Jacobian matrix is given by

$$A(x,y) = \begin{pmatrix} 0 & 1 \\ -1 - 2axy & a(1 - x^2 - 3y^2) \end{pmatrix} . \tag{4}$$

Evaluated at the origin, we find

$$A(0,0) = \begin{pmatrix} 0 & 1 \\ -1 & a \end{pmatrix} , \tag{5}$$

and so the characteristic equation becomes $\lambda(\lambda - 1) + 1 = 0$. The eigenvalues are

$$\lambda = \frac{a \pm \sqrt{a^2 - 4}}{2} . \tag{6}$$

Thus $\Delta = 1$ and $\tau = a$. If $a > 2$, the origin is an unstable node and if $a < 2$ it is an unstable spiral. If $a = 2$, it is a degenerate node with eigenvector

$$v = \begin{pmatrix} 1 \\ 1 \end{pmatrix} . \tag{7}$$

The phase portraits for different values of a are shown Fig. 1

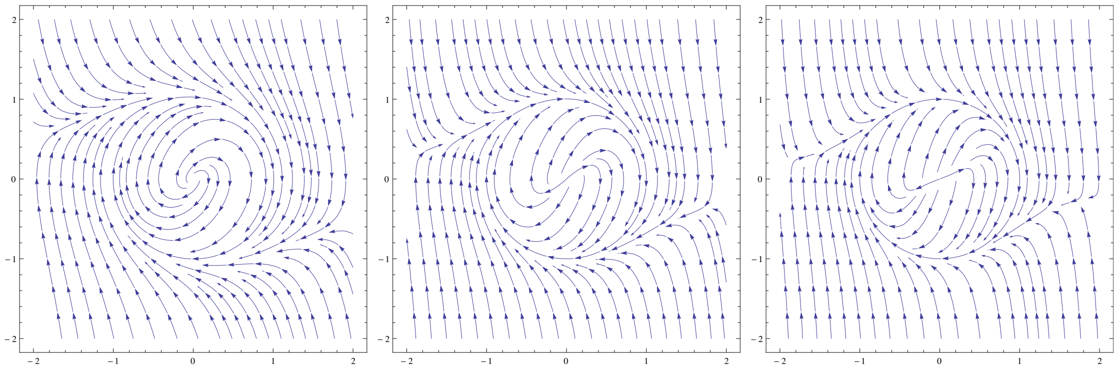

Figure 1: Phase portrait for $a = 1$, $a = 2$, and $a = 3$ in 7.1.8.

b) Going to polar coordinates, we obtain

$$\dot{r} = -ar\sin^2\theta(r^2 - 1) , \quad (8)$$
$$\dot{\theta} = -1 - a\sin\theta\cos\theta(r^2 - 1) . \quad (9)$$

It is easy to see that the equation $r = 1$ is a solution to Eqs. (8). Eq. (9) yields $\dot{\theta} = -1$ and so $\theta = -t + \theta_0$. Thus the limit cycle is a circle with radius one. From the equation for θ, it follows that the period is $T = 2\pi/\omega = 2\pi$.

c) If $r > 1$, then $\dot{r} < 0$ and if $r < 1$, then $\dot{r} > 0$. Thus the limit cycle is stable.

d) Assume there is another periodic trajectory. If there is a point on that trajectory such that $r < 1$, c) shows that $\dot{r} > 0$. If there is a point on that trajectory such that $r > 1$, c) shows $\dot{r} < 0$. Either way we cannot return to a point $r \neq 1$ and therefore there is no other periodic trajectory.

Problem 7.2.5

The dynamics is governed by the equations

$$\dot{x} = f(x, y) , \quad (10)$$
$$\dot{y} = g(x, y) . \quad (11)$$

a) Assume that this is a gradient system. Then $\dot{\mathbf{x}} = -\nabla V(x, y)$. This implies

$$\dot{x} = -\frac{\partial V}{\partial x} , \quad (12)$$
$$\dot{y} = -\frac{\partial V}{\partial y} . \quad (13)$$

This yields $f = -\partial V/\partial x$ and $g = -\partial V/\partial y$. Taking the partial derivatives of these equations and using that partial derivatives commute (smooth vector field), we obtain $\partial f/\partial y = -\partial^2 V/\partial x\partial y = \partial g/\partial x$.

b) We define the function $V(x, y)$ by

$$V(x,y) = -\int_0^x f(\bar{x},y)d\bar{x} - \int_0^y g(0,\bar{y})d\bar{y} \; . \tag{14}$$

This yields

$$\frac{\partial V}{\partial x} = -f(x,y) \tag{15}$$

by the fundamental theorem of calculus. Moreover,

$$\begin{aligned}
\frac{\partial V}{\partial y} &= -\frac{\partial}{\partial y}\int_0^x f(\bar{x},y)d\bar{x} - g(0,y) \\
&= -\int_0^x \frac{\partial f(\bar{x},y)}{\partial y}d\bar{x} - g(0,y) \\
&= -\int_0^x \frac{\partial g(\bar{x},y)}{\partial \bar{x}}d\bar{x} - g(0,y) \\
&= -g(x,y) \; ,
\end{aligned} \tag{16}$$

which shows that it is a gradient system as long as f and g are smooth so everything is well defined.