TFY4305 solutions exercise set 11 2014

Problem 6.6.5

a) The equation that governs the dynamics is

$$\ddot{x} + f(\dot{x}) + g(x) \quad = \quad 0 \, . \tag{1}$$

If we define $y = \dot{x}$, we can write Eq. (1) as

$$\begin{aligned} \dot{x} &= y \, , & (2) \\ \dot{y} &= -f(y) - g(x) \, . & (3) \end{aligned}$$

Under the transformation $t \to -t$ and $y \to -y$, both sides of the first equation change sign. Since $f(y)$ is an even function and dy/dt is invariant under the above transformation, the second equation is also invariant. Hence the system is invariant under time translation.

b) The Jacobian matrix evaluated at a fixed point (x^*, y^*) us

$$A(x^*, y^*) \quad = \quad \begin{pmatrix} 0 & 1 \\ -g'(x^*) & -f'(y^*) \end{pmatrix} \, , \tag{4}$$

The eigenvalues are

$$\lambda_{1,2} \quad = \quad \frac{-f'(y^*) \pm \sqrt{[f'(y^*)]^2 - 4g'(x^*)}}{2} \, . \tag{5}$$

Any fixed point must have $y^* = 0$. If $f(y)$ is an even function, $f'(y)$ is an odd function and so $f'(0) = 0$. Thus the eigenvalues reduce to

$$\lambda_{1,2} \quad = \quad \pm i\sqrt{g'(x^*)} \, . \tag{6}$$

Depending on the sign of $g'(x^*)$, the eigenvalues are either purely imaginary or purely real with different sign. This corresponds to either a center or a saddle. In the special case

$g'(x^*) = 0$, both eigenvalues are vanishing. Inspecting the matrix $A(x^*, y^*)$ in this case, one finds a single eigenvector

$$v = \begin{pmatrix} 1 \\ 0 \end{pmatrix} . \tag{7}$$

Thus we have a degenerate node.

Problem 6.8.7

The dynamics is governed by the equations

$$\dot{x} = x(4 - y - x^2) , \tag{8}$$
$$\dot{y} = y(x - 1) . \tag{9}$$

The fixed points are $(0, 0)$, $(1, 3)$, and $(\pm 2, 0)$. The Jacobian matrix is given by

$$A(x, y) = \begin{pmatrix} 4 - y - 3x^2 & -x \\ y & x - 1 \end{pmatrix} . \tag{10}$$

Evaluated at the origin, we find

$$A(0, 0) = \begin{pmatrix} 4 & 0 \\ 0 & -1 \end{pmatrix} . \tag{11}$$

Hence the eigenvalues are $\lambda = -1$ and $\lambda = 4$, and so the origin is a saddle. In the same way, one finds that $(-2, 0)$ is a stable node ($\lambda = -9$ and $\lambda = -3$), $(2, 0)$ is a saddle point ($\lambda = -9$ and $\lambda = 1$), and $(1, 3)$ is a stable spiral ($\lambda = (-1 \pm i\sqrt{2})$).

A closed orbit would have to encircle the node or the spiral or both. If an orbit would encircle the node, it would have to cross the x-axis. But the flow on the x-axis is horizontal and so curves would cross which is forbidden. A cycle cannot encircle the spiral since the spiral is joined to the saddle at $(2, 0)$ by a branch of its unstable manifold and cycles cannot cross trajectories. The phase portrait is shown in Fig. 1.

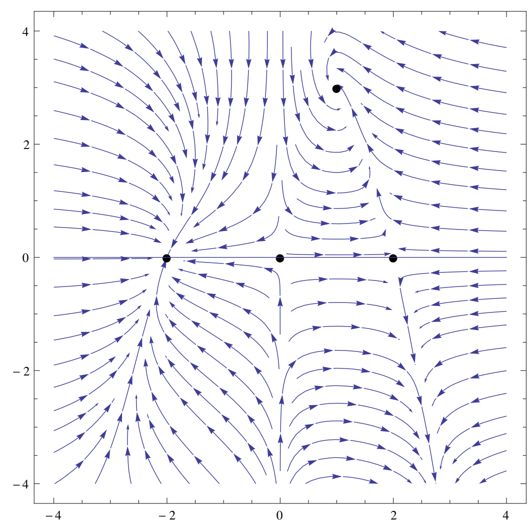

Figure 1: Phase portrait of problem 6.8.7.