We underline again that δm and δ_Z should generally be chosen to remove any divergences (unphysical terms, but finite terms can be kept since these tell us how the effective mass of the particle changes because of the interaction λ with its environment.

4-point correlators and the effective coupling

We will now consider 4-point correlators up to 1-loop level, as this will show how not only masses and residues (amplitude/normalization of the propagator) are renormalized, but also the coupling strength between fields.

For convenience of notation, let us write $Z(J)$ (*) simply as $Z(J) = Z(0)e^{iW(J)}$. The normalized 2-point correlator is then:

$$\frac{\langle T\{\varphi(x_1)\varphi(x_2)\}\rangle}{\langle 0|0\rangle} = Z^{-1}(0)\,\frac{-i\delta}{\delta J(x_1)}\,\frac{-i\delta}{\delta J(x_2)}\,Z(J)\Big|_{J=0} = \frac{\delta}{\delta J(x_1)}\,\frac{\delta}{\delta J(x_2)}\,W(J)\Big|_{J=0}$$

where we used that $\frac{\delta}{\delta J(x)}W(J)\big|_{J=0} = 0$.

This is really a statement that one-point functions $\langle\varphi(x)\rangle$ are zero, which is the case for any interacting theory that is invariant under $\varphi(x) \to -\varphi(x)$. We already saw that this was true in the non-interacting case.

(*) In the free case, we found $Z(J) = Z(0)\exp\left(-\iint d^4x\,d^4(y)\,J(x)\,G_F(x-y)\,J(y)\right)$, but generally $Z(J)$ is defined by adding source terms $J\varphi$ in the Lagrangian (which can be done also in the interacting case)

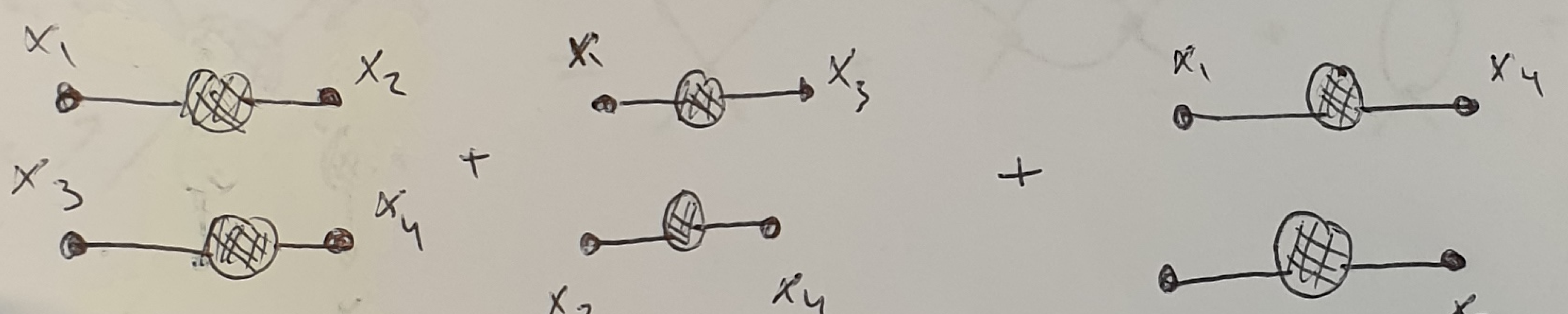
In this notation, the 4-point correlator is then:

$$\frac{\langle T\{\varphi(x_1)\varphi(x_2)\varphi(x_3)\varphi(x_4)\}\rangle}{\langle 0|0\rangle} = \frac{\delta}{\delta J_1(x_1)}\frac{\delta}{\delta J_2(x_2)} W(J)\Big|_{J=0} \frac{\delta}{\delta J(x_3)}\frac{\delta}{\delta J(x_4)} W(J)\Big|_{J=0}$$

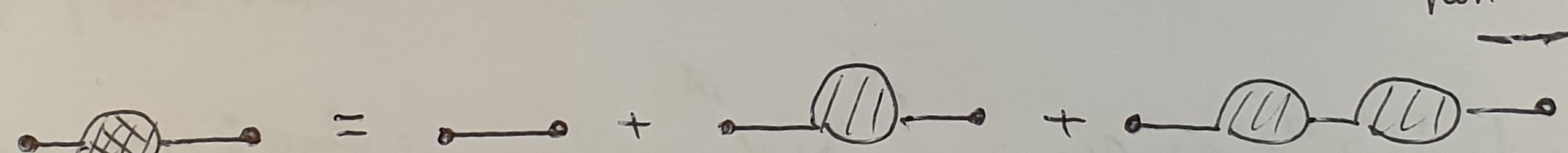
$$+ (2 \leftrightarrow 3) + (2\to 4,\ 3\to 2,\ 4\to 2)$$

$$+ \frac{\delta}{\delta J(x_1)}\frac{\delta}{\delta J(x_2)}\frac{\delta}{\delta J(x_3)}\frac{\delta}{\delta J(x_4)} W(J)\Big|_{J=0}. \qquad (*)$$

The latter term would be zero in the non-interacting (free) case $\lambda = 0$, as we proved with a direct computation previously. Comparing with the expression for the two-point correlator, the first three terms are products of such correlators. As we've seen before, this corresponds to disconnected diagrams:



is the complete non-truncated propagator

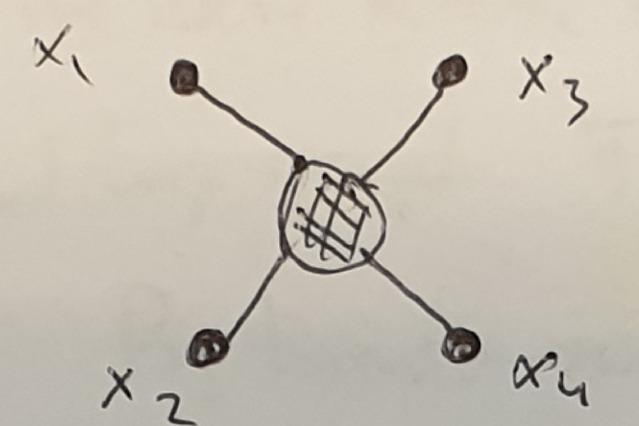
where  and

we defined  previously (sum over all truncated 1PI diagrams with at least one vertex).

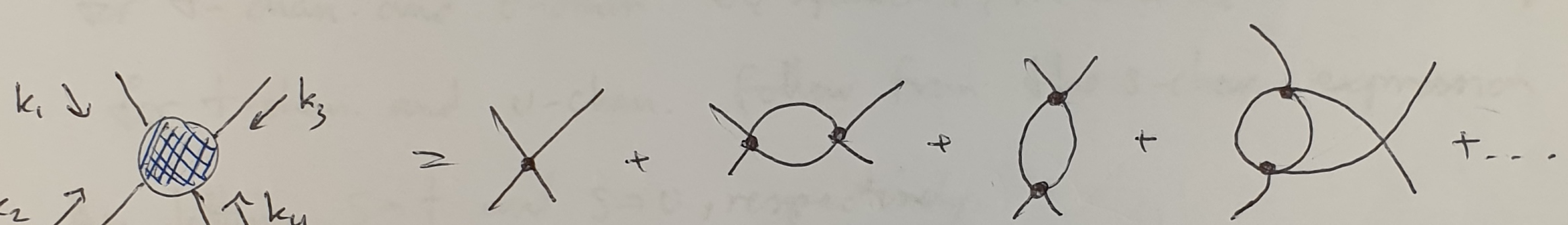
Since the above represent all possible disconnected diagrams (except vacuum bubbles like ∞, but these are divided out by $\langle 0|0\rangle$), it is the fourth term in $(*)$ that contains all connected diagrams. Note that the fourth term does not appear for a 4-point correlator in free theory since the vacuum $|0\rangle$ in the interacting case is different from the free case $|0\rangle_{free}$.

This argument can be extended to all higher point correlaters and $W(J)$ is referred to as the generator of all connected diagrams (more about generators in the next chapter). Thus:

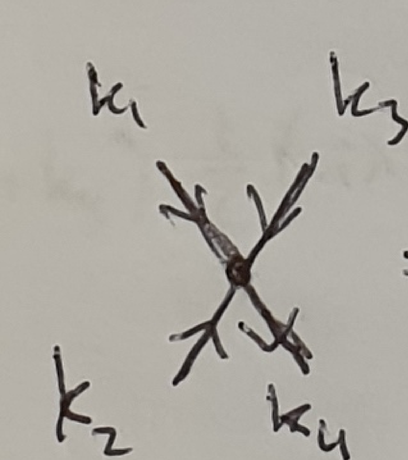
$$\frac{\langle T\{\varphi(x_1)\,\varphi(x_2)\,\varphi(x_3)\,\varphi(x_4)\}\rangle_{connected}}{\langle 0|0\rangle} = \frac{\delta}{\delta J_1}\cdots\frac{\delta}{\delta J_4} W(J)\Big|_{J=0} \qquad (J_i = J(x_i))$$

$=$ 

and the truncated diagram in momentum space (where we lop off propagators connected to the end points) is:

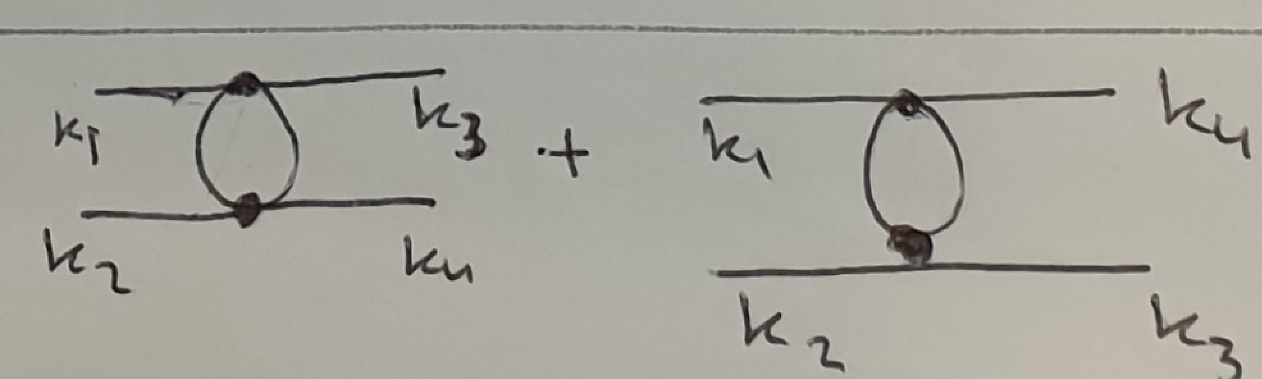


The first term is the single vertex and equal to the vertex Feynman rule we gave previously:



$= (-i\lambda)\,(2\pi)^4\,\delta^4(k_1+k_2+k_3+k_4)$

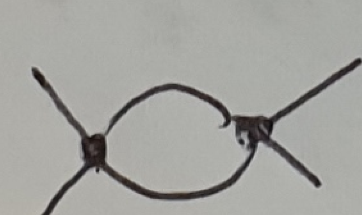
Before evaluating the next three 1-loop diagrams, it is convenient to introduce some new terminology. The Mandelstam variables are defined as:

(*) Could also have drawn the two last as 

$$s \equiv (k_1+k_2)^2 = (k_3+k_4)^2$$
$$t \equiv (k_1+k_3)^2 = (k_2+k_4)^2$$
$$u \equiv (k_1+k_4)^2 = (k_2+k_3)^2$$

In the special case that the incoming momenta are on-shell, $k_i^2 = m^2$ for all i, it follows that $s+t+u = 4m^2$.

In the first loop graph, we have k_1+k_2 entering a vertex, and so it is referred to as the s-channel loop graph. The other two graphs are then t- and u-channel ones. By Lorentz-invariance, the analytic expression for the s-channel graph can only depend on s (has to be L-invariant since the propagator should be, as we've seen previously) and correspondingly for t-chan. and u-chan. By symmetry, we also see that the expression for t-chan. and u-chan. follow from the s-chan. expression simply by letting $s \to t$ and $s \to u$, respectively.

Consider now the s-chan.  graph. Using our derived Feynman rules:

 $= (2\pi)^4 \delta^4(k_1+k_2+k_3+k_4) \frac{1}{2}(-i\lambda)^2 \int \frac{d^4 l}{(2\pi)^4} \frac{i}{l^2-m^2+i\varepsilon} \cdot \frac{i}{(k_1+k_2-l)^2-m^2+i\varepsilon}$

If $k_1+k_2=0$, the poles occur at $l^0 = \pm\left(\sqrt{m^2+\vec{l}^2} - i\varepsilon\right)$ which lie in the 2nd and 4th quadrant: Wick rotation is ok.

If the external momentum k_1+k_2 is finite, we can still do a Wick rotation, but just like for the sunset diagram extra care must be taken when k_1+k_2 exceeds a certain threshold value. To see this, let $q = k_1+k_2$.

We have:

$$\int \frac{d^4 l}{(2\pi)^4} \frac{i}{l^2-m^2+i\varepsilon} \frac{i}{(l-q)^2-m^2+i\varepsilon}$$ Use Feynman parametrization:

$$\frac{1}{ab} = \int_0^1 \frac{dz}{(az+b(1-z))^2}$$ with $a = p^2-m^2$ and $b = (l-q)^2-m^2$

($i\varepsilon$ absorbed into m^2). "Diagonalizing" the momenta in the denominator (without ^{2})

$$az+b(1-z) = l^2-m^2-2lq(1-z)+q^2(1-z)$$

by defining $l' = l-q(1-z)$, so that the denominator becomes (without ^{2})

$$(l')^2-m^2+q^2(1-z)z.$$ Clearly, $d^4l' = d^4 l$ so the integral is now

$$\int_0^1 dz \int \frac{d^4 l'}{(2\pi)^4} \frac{1}{\left[(l')^2-m^2+q^2 z(1-z)\right]^2}.$$ Just like for the sunset diagram the poles lie at:

$(l'^0)^2 = R - i\varepsilon$ with $R \in \mathbb{R}$ and we can always do the Wick rotation.

Using $\frac{1}{B^2} = \int_0^\infty \rho e^{-B\rho} d\rho$ for $\mathrm{Re}\, B > 0$ and introducing dimensional regularization, we get: (after renaming $z \to x$ and doing the Wick rotation to Euclidean momenta)

$$= i\,(2\pi)^4\,\delta^4(k_1+k_2+k_3+k_4)\,\frac{1}{2}\lambda^2\mu^{4-D}\int_0^\infty \rho\,d\rho\int_0^1 dx$$

$$\cdot\int\frac{d^D l_E}{(2\pi)^D}\cdot\exp\left[-\rho\left(x\,l_E^2+(1-x)(k_{1E}+k_{2E}-l_E)^2+m^2\right]\right.$$

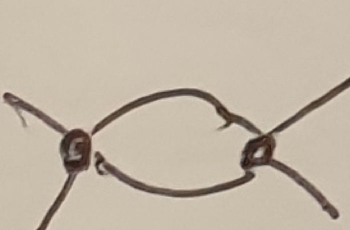
$$= i\,(2\pi)^4\,\delta^4(k_1+k_2+k_3+k_4)\,\frac{\lambda^2\mu^{4-D}}{2\cdot(4\pi)^{D/2}}\,\Gamma(2-D/2)\int_0^1\frac{dx}{[m^2-x(1-x)s]^{2-D/2}}\,.$$

We obtained this by using similar steps as in the sunset diagram case. In doing the Gaussian integral over l_E, we completed the square and set $(k_{1E}+k_{2E})^2 \equiv -s$, where $k_E = (k_{E,0}, k_{E,1}, k_{E,2}, k_{E,3})$, $k_E^2 = k_{E,0}^2 + k_{E,1}^2 + k_{E,2}^2 + k_{E,3}^3$ and $k_{E,0} = -ik^0$, $k_{E,i} = (\vec{k})_i$.

Setting new $D = 4-2\varepsilon$, expanding $\frac{\mu^{4-D}}{(4\pi)^{D/2}}\cdot\Gamma(2-D/2)$ for $\varepsilon \ll 1$ like before and using that

$$\int_0^1 \frac{1}{[f(x)]^\varepsilon}\,dx = \int_0^1 [f(x)]^{-\varepsilon}\,dx \approx \int_0^1 (1-\varepsilon\ln[f(x)])\,dx \quad \text{for } \varepsilon \ll 1,$$

we obtain:



$$= i\,(2\pi)^4\,\delta^4(k_1+k_2+k_3+k_4)\,\frac{\lambda^2}{32\pi^2}\left[\frac{1}{\varepsilon}-\gamma_E+\ln(4\pi)+\ln\left(\frac{\mu^2}{m^2}\right)+Q\left(\frac{s}{m^2}\right)\right]$$

where:

$$Q(z) = -\int_0^1 dx\,\ln[1-x(1-x)z] = 2\cdot\left[1-\sqrt{\frac{4}{z}-1}\cdot\operatorname{atan}\left(\frac{1}{\sqrt{\frac{4}{z}-1}}\right)\right]$$

Similarly (again) to the sunset diagram, $-i\Sigma$ for this diagram requires extra care for s exceeding a threshold value, namely $s > 4m^2$.

This can be seen by noting that the argument of the ln in $Q(z)$ becomes negative for $z > 4$ and Q thus acquires an imaginary part.

This happens in the integration region $\frac{1}{2} - \frac{1}{2}\sqrt{1-\frac{4}{z}} < x < \frac{1}{2} + \frac{1}{2}\sqrt{1-\frac{4}{z}}$.

Selecting the principal branch of ln, which has no imaginary part for $z < 4$, it follows that (look at our previous treatment of $\ln(1-z)$ for sunset)

$$\operatorname{Im}\left[Q(z \pm i\varepsilon)\right] = -\int_{\frac{1}{2}\left(1-\sqrt{1-\frac{4}{z}}\right)}^{\frac{1}{2}\left(1+\sqrt{1-\frac{4}{z}}\right)} dx \cdot (\mp \pi) \cdot \Theta(z-4) \qquad (z \text{ real here})$$

$$= \pm \pi \sqrt{1-\frac{4}{z}} \cdot \Theta(z-4)$$

Thus, $Q(z)$ has a branch cut for $z > 4$ and the $i\varepsilon$ in $m^2 \to m^2 - i\varepsilon$ makes $z \to z + i\varepsilon$ and makes the upper sign correct.