To solve the integral, we make use of Feynman parametrization:

$$\frac{1}{C_1 C_2 \dots C_n} = \int_0^1 dx_1 \dots \int_0^1 dx_n \cdot \frac{(n-1)!\ \delta(x_1 + \dots x_n - 1)}{(x_1 C_1 + \dots x_n C_n)^n} \qquad (X)$$

PROOF

Start by noting that

$$\frac{1}{AB} = \int_0^1 dx \frac{1}{[xA + (1-x)B]^2} = \int_0^1 dx \int_0^1 dy \frac{\delta(x+y-1)}{(xA+yB)^2} \qquad (*)$$

since $\frac{1}{AB} = \frac{1}{A-B}\left(\frac{1}{B} - \frac{1}{A}\right) = \frac{1}{A-B} \int_B^A \frac{dz}{z^2}$. By changing int. variable

to

$$u = \frac{z-B}{A-B} \Rightarrow du = \frac{dz}{A-B}, \text{ so that } z = uA + (1-u)B,$$

we get the desired result. (the second equality in $(*)$ is trivially seen).

Next, by differentiating with respect to B in this formula, we have:

$$\frac{1}{AB^2} = \int_0^1 \int_0^1 dy \; \frac{2y \cdot \delta(x+y-1)}{[xA+yB]^3}$$

$$\frac{1}{AB^3} = \int_0^1 dx \int_0^1 dy \; \frac{3y \cdot \delta(x+y-1)}{[xA+yB]^4}$$

$$\vdots$$

$$\frac{1}{AB^n} = \int_0^1 dx \int_0^1 dy \; \frac{n \cdot y^{n-1} \, \delta(x+y-1)}{[xA+yB]^{n+1}} \qquad (+)$$

We mention this because we will use $(+)$ in our main proof of (X).

First, we have seen that (X) holds for $n=2$. Now, we show that if the formula holds for n, then it holds for $n+1$. This completes the proof by induction.

To see this, take (X) and multiply it by $\frac{1}{C_{n+1}}$. We get:

$$\frac{1}{C_1 C_2 \cdots C_n} \cdot \frac{1}{C_{n+1}} = \int_0^1 dx_1 \cdots \int_0^1 dx_n \; \frac{(n-1)! \, \delta(x_1 + \cdots x_n - 1)}{(x_1 C_1 + \cdots x_n C_n)^n} \cdot \frac{1}{C_{n+1}}$$

use $(+)$ with $(\ldots) = B$ and $C_{n+1} = A$.

$$= \int_0^1 dx_1 \cdots \int_0^1 dx_n \; (n-1)! \, \delta(x_1 + x_2 \cdots x_n - 1) \int_0^1 dx \int_0^1 dy \; \frac{n \cdot y^{n-1} \, \delta(x+y-1)}{[x C_{n+1} + y(x_1 C_1 + \cdots x_n C_n)]^{n+1}}$$

$$= \int_0^1 dx_1 \cdots \int_0^1 dx_n \int_0^1 dy \; \frac{n! \, \delta(yx_1 + \cdots yx_n - y) \cdot y^{n-1}}{[(1-y) C_{n+1} + y(x_1 C_1 + \cdots x_n C_n)]^{n+1}}$$

$$= \int_0^1 dy \int_0^1 dx_1 \cdots \int_0^1 dx_n \cdot \frac{n!\, \delta(yx_1 + \dots yx_n - y)\, y^n}{[(1-y)\, C_{n+1} + y\,(x_1 C_1 + \dots x_n C_n)]^{n+1}}$$ used $\delta(x) = y \cdot \delta(yx)$.

$$= \int_0^1 dy \int_0^y dz_1 \dots \int_0^y dz_n \frac{n!\, \delta(z_1 + \dots z_n - y)}{[(1-y)\, C_{n+1} + (z_1 C_1 + \dots z_n C_n)]^{n+1}}$$ defined $z_i = y \cdot x_i$

Now define $y' \equiv 1-y$, so that $dy' = -dy$ and:

$$= -\int_1^0 dy' \int_0^{1-y'} dz_1 \dots \int_0^{1-y'} dz_n \frac{n!\, \delta(z_1 + \dots z_n + y' - 1)}{[y' C_{n+1} + (z_1 C_1 + \dots z_n C_n)]^{n+1}}$$

Rename $y' \equiv z_{n+1}$:

$$= \int_0^1 dz_1 \dots \int_0^1 dz_{n+1} \frac{n!\, \delta(z_1 + \dots z_{n+1} - 1)}{(z_1 C_1 + \dots z_{n+1} C_{n+1})^{n+1}}$$ qed.

In the last step, we made use of the fact that the interior integral domains can be extended from $\int_0^{1-y'}$ to $\int_0^1$ without changing the result, since y' is always positive and because there is a δ-function within the integral.*

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* To see this, consider first $\int_0^1 dx \int_0^{1-x} dy\, \delta(x+y-1)\, f(x,y)$. The δ-function kicks in at $y = 1-x$, so we can extend $\int_0^{1-x}$ to $\int_0^1$ w.o. changing the result as $x \geq 0$.
Next, look at $\int_0^1 dx \int_0^{1-x} dy_1 \int_0^{1-x} dy_2\, \delta(x + y_1 + y_2 - 1)\, f(x, y_1, y_2)$. $\int_0^{1-x} dy_2$ can be extended to $\int_0^1 dy_2$ for the same reason. If we do that first, and then interchange the two interior integrals to get $\int_0^1 dx \int_0^1 dy_2 \int_0^{1-x} dy_1$, we can use the same rationale on $\int_0^{1-x} dy_1 \to \int_0^1 dy_1$. This can now be generalized to any number of interior integrals.