representation (three-dim.) of SO(3). This is valid for a real vector, but what about spinors?. We have that:

$$\begin{pmatrix} \alpha' \\ \beta' \end{pmatrix} = U(\vec{\theta}) \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \quad \text{with} \quad U(\vec{\theta}) = e^{-i\vec{\theta}\cdot\vec{\sigma}/2}.$$

Here, $\vec{\theta}$ points along the axis of rotation and $|\vec{\theta}|$ is the angle of rotation in the right-hand sense.

Note that for a matrix A, we understand $e^{A} = 1 + A + \frac{1}{2}A^2 \ldots$

The set of all such matrices constitute the group SU(2):

$U(\vec{\theta})$ is a unitary matrix with determinant +1.

SHORT INTRO TO GROUP THEORY : SU(2) vs. SO(3)

These two groups are of particular importance since they are related to both rotations of spins (half-integer as well as integer spins) and to internal symmetries between particles and in Lagrangians describing relevant quantum fields.

Let's introduce some terminology:

<u>Isomorphism</u> A one-to-one correspondence between elements G of a group $\mathcal{G}$ and G' of a group $\mathcal{G}'$ such that if $G_i G_j = G_k$, then $G_i' G_j' = G_k'$.

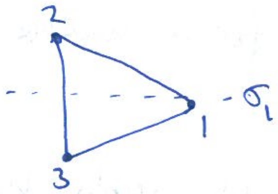
<u>Homomorphism</u> Let f be a mapping that maps G of $\mathcal{G}$ onto G' of $\mathcal{G}'$: $G' = f(G)$. If $f(G_i G_j) = f(G_i) f(G_j)$ holds for any two elements, f is a homomorphic mapping.

Note that homomorphism signifies a n-to-one correspondence between the elements of the group, in general. A homomorphism with $n=1$ is an isomorphism.

(EXAMPLE) Return to the equilateral triangle. The allowed symmetry operations constitute a group C_{3v}:

$C_{3v} = \{E, C_3, C_3^{-1}, \sigma_1, \sigma_2, \sigma_3\}$. C_3 = rotation counterclock $\frac{\pi}{3}$ around center

σ_j = mirror reflection around symmetry axis



Multiplication table (entries $G_j \circ G_i$)

$G_j \backslash G_i$	E	C_3	C_3^{-1}	σ_1	σ_2	σ_3
E	E	C_3	C_3^{-1}	σ_1	σ_2	σ_3
C_3	C_3	C_3^{-1}	E	σ_3	σ_1	σ_2
C_3^{-1}	C_3^{-1}	E	C_3	σ_2	σ_3	σ_1
σ_1	σ_1	σ_2	σ_3	E	C_3	C_3^{-1}
σ_2	σ_2	σ_3	σ_1	C_3^{-1}	E	C_3
σ_3	σ_3	σ_1	σ_2	C_3	C_3^{-1}	E

Compare this with group $C_2 = \{E, C\}$, where C = rotation $\pi \Rightarrow C^2 = E$. Let the mapping be defined:

$E, C_3, C_3^{-1} \xrightarrow{f} E \qquad \sigma_1, \sigma_2, \sigma_3 \xrightarrow{f} C$.

This fulfills the homomorphism criterion, as seen from the multiplication tables:

C_2	E	C
E	E	C
C	C	E

As seen, there are three elements in C_{3v} corresponding to one element in C_2 : homomorphism (<u>not</u> isomorphism).

The above is related to <u>representations</u> of a group:

Let $\mathcal{G}$ be a group with elements G_i and associate a square matrix with each element, $\hat{D}(G_i)$. If the matrices satisfy $\hat{D}(G_j)\hat{D}(G_i) = \hat{D}(G_k)$ for the corresponding relation $G_j G_i = G_k$, then the set of matrices $\hat{D}(G_i)$ is a representation of $\mathcal{G}$.

$\Longrightarrow$ The mapping $\hat{D} : G_i \to \hat{D}(G_i)$ is then homomorphic as the same matrix may be associated with several elements.

<u>Faithful representation</u> : one-to-one correspondence between G_i and $\hat{D}(G_i)$.

Note that a group can have representations in several dimensions.

EXAMPLE : 1D rep. of C_{3v} :

$$\hat{D}(E) = 1, \quad \hat{D}(C_3) = 1, \quad \hat{D}(C_3^{-1}) = 1$$

$$\hat{D}(\sigma_1) = -1, \quad \hat{D}(\sigma_2) = -1, \quad \hat{D}(\sigma_3) = -1.$$

Now consider the rotation group. Rotation matrix :

$$\hat{R} = \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{pmatrix} \begin{pmatrix} \cos\gamma & -\sin\gamma & 0 \\ \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$\hat{R}$ is a 3×3 orthogonal matrix with $\det(\hat{R}) = +1$, since only proper rotations (not inversions) are considered.

Thus, the set of rotation matrices forms the group $SO(3)$.

The range of the parameters is important when discussing the relation to $SU(2)$, as we will now see.

A possible representation of $SO(3)$ in two-dim. is given by:

$$\hat{D}^{(1/2)}[R(\alpha,\beta,\gamma)] = \begin{bmatrix} e^{-i(\alpha+\gamma)/2}\cos\frac{\beta}{2} & -e^{-i(\alpha-\gamma)/2}\sin\frac{\beta}{2} \\ e^{i(\alpha-\gamma)/2}\sin\frac{\beta}{2} & e^{i(\alpha+\gamma)/2}\cos(\frac{\beta}{2}) \end{bmatrix}$$

We see that $\hat{D}^{(1/2)}$ is a special, unitary 2×2 matrix.

Hence, these matrices form the group $SU(2)$ <u>if</u> the parameter range of (α,β,γ) covers all possible $SU(2)$ matrices.

<u>Now</u>: $\hat{R}(0,0,0)$ and $\hat{R}(0,0,2\pi)$ are both corresponding to the same identity operation. However, $\hat{D}(0,0,0) = -\hat{D}(0,0,2\pi)$.

In effect, both $\hat{D}^{(1/2)}(R)$ and $-\hat{D}^{(1/2)}(R)$ correspond to $\hat{R}$.

$SU(2)$ would then be a double-valued representation of $SO(3)$ $\Rightarrow$ not allowed for a true representation.

However, for $\hat{D}^{(1/2)}$ to constitute $SU(2)$ we need e.g. $0 \le \gamma < 4\pi$.

By restricting the range to $0 \le \gamma < 2\pi$, $\hat{D}^{(1/2)}$ can represent $SO(3)$ – but then it is no longer $SU(2)$.

Instead, the factor group $SU(2)/Z_2$ with $Z_2 = \{E, \bar{E}\}$, $\bar{E} = -E$ is isomorphic to $SO(3)$: this removes the double-valuedness (equivalent to restricting the parameters).

We may turn the argument around: since there are two elements in $SU(2)$ corresponding to $SO(3)$, the groups are homomorphic $\Rightarrow$ $SO(3)$ is a representation of $SU(2)$ (not faithful, however).

General properties:

- The representations of $SU(2)$ may be classified by a number $j = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$ and the dimensionality is $2j+1$.
- The spin j rep. of $SU(2)$ is $\{\hat{D}^{(j)}\}$ where $\hat{D}^{(j)}$ are rotation matrices for angular momentum $j = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$

j = half-integer: the rep. is faithful to $SU(2)$ (e.g. $j = 1/2$ like above).

j = integer: the rep. is also a representation for $SO(3)$, and faithful as long as $j \neq 0$. (e.g. $j = 1$).