CHAPTER 7 - QUANTUM ELECTRODYNAMICS

We introduce here:

- The Dirac equation and its solution
- Feynman rules for QED
- Some pragmatic calculational tools.

This theory is a major cornerstone in particle physics and serves as foundation for much of the material that we will look at later.

THE DIRAC EQUATION

Non-rel. QM : Schrödinger equation.

Rel. QM : - Klein-Gordon eq. for spin-0.
- Dirac eq. for spin-$\frac{1}{2}$
- Proca eq. for spin-1.

With Feynman rules in hand, the underlying field equation takes a backseat.* However, the notation of Feynman rules for spin $\frac{1}{2}$ requires some familiarity with the Dirac eq.

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* As we saw for ABC-theory, where we didn't have to mention any field eq. (K-G).

Schrödinger equation motivated:

Start with $\vec{p}^2/2m + V = E$ and apply:

$\vec{p} \to \frac{\hbar}{i}\nabla$, $E \to i\hbar\frac{\partial}{\partial t}$: $-\frac{\hbar^2}{2m}\nabla^2\Psi + V\Psi = i\hbar\frac{\partial\Psi}{\partial t}$.

Same principle for K-G eq.:

Start with $E^2 - \vec{p}^2c^2 = m^2c^4$ $(p^\mu p_\mu - m^2c^2 = 0)$.

Consider a free particle to begin with $(V=0)$

Now, perform: $p_\mu \to i\hbar\partial_\mu$ where $\partial_\mu \equiv \frac{\partial}{\partial x^\mu}$

In effect, this means $(\frac{E}{c}, -\vec{p}) \to i\hbar(\frac{1}{c}\frac{\partial}{\partial t}, \vec{\nabla})$.

$$\Rightarrow \quad -\hbar^2\partial^\mu\partial_\mu\varphi - m^2c^2\varphi = 0 \quad \text{or}$$

$$-\frac{1}{c^2}\frac{\partial^2\varphi}{\partial t^2} + \nabla^2\varphi = \left(\frac{mc}{\hbar}\right)^2\varphi.$$

In the presence of an EM potential, do the usual replacements $E \to E - e\phi$, $\vec{p} \to \vec{p} - \frac{e\vec{A}}{c}$ (for charge $-e$) so that we obtain:

$$(i\hbar\partial_t - e\varphi)^2\Psi = c^2\left(-i\hbar\nabla - \frac{e\vec{A}}{c}\right)^2\Psi + m^2c^4\Psi.$$

Note that S-E is $O(t)$, while K-G is $O(t^2)$.

Ψ is a scalar field, but any solution to the Dirac (or any equation) is also a solution of K-G (no reference to spin in K-G).

 116 The opposite is not true.

Now, Dirac's strategy was to factor the $E-\vec{p}$ relation.
If $\vec{p}=0$, this is easy:

$$(p^0)^2 - m^2c^2 = 0 = (p^0+mc)(p^0-mc).$$

$$\Rightarrow (p^0-mc)=0 \quad \text{or} \quad (p^0+mc)=0.$$ (Both guarantee $p^\mu p_\mu - m^2c^2=0$)

For non-zero $\vec{p}$, we need something like:

$$(p^\mu p_\mu - m^2c^2) = (\beta^\kappa p_\kappa + mc)(\gamma^\lambda p_\lambda - mc).$$

where β^κ and γ^λ are undetermined coefficients. We obtain:

$$\beta^\kappa \gamma^\lambda p_\kappa p_\lambda - mc(\beta^\kappa - \gamma^\kappa)p_\kappa - m^2c^2,$$

so $\beta^\kappa = \gamma^\kappa$ to cancel linear terms. Now, we must find γ^κ such that $p^\mu p_\mu = \gamma^\kappa\gamma^\lambda p_\kappa p_\lambda$. (*)

<u>Problem</u>: This is impossible for scalars γ^κ.

<u>Solution</u>: But it works if they are matrices!

Writing out (*), we get:

$$(p^0)^2 - (p^1)^2 - (p^2)^2 - (p^3)^2 = (\gamma^0)^2(p^0)^2 + (\gamma^1)^2(p^1)^2 + (\gamma^2)^2(p^2)^2 + (\gamma^3)^2(p^3)^2$$
$$+ (\gamma^0\gamma^1+\gamma^1\gamma^0)p_0p_1 + \ldots (\gamma^2\gamma^3+\gamma^3\gamma^2)p_2p_3$$

Note that one may produce the spin term in the Pauli equation (S.E. with spin) from Dirac equation; and also recover P.E.S. fully in the low-energy limit.

Thus, we need a set of matrices that get rid off the cross-terms:

$$(\gamma^0)^2 = 1\ ,\ (\gamma^1)^2 = (\gamma^2)^2 = (\gamma^3)^2 = -1 \quad \text{and}$$

$$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 0 \quad \text{for } \mu \neq \nu$$

We see that scalars couldn't do the trick since they commute. This may be written compactly as:

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} \quad \text{where } g^{\mu\nu} = \begin{pmatrix} 1 & & & 0 \\ & -1 & & \\ & & -1 & \\ 0 & & & -1 \end{pmatrix} \text{ is}$$

the Minkowski metric and:

$\{A, B\} \equiv AB + BA$ is the anti-commutator.

There are a number of essentially equivalent sets of γ-matrices that do the trick; the smallest ones are 4×4:

$$\gamma^0 = \begin{pmatrix} \underline{1} & \underline{0} \\ \underline{0} & -\underline{1} \end{pmatrix} \ ;\ \gamma^i = \begin{pmatrix} \underline{0} & \underline{\sigma}^i \\ -\underline{\sigma}^i & \underline{0} \end{pmatrix},$$

where $\underline{\ldots}$ means a 2×2 matrix. Now, the Dirac equation factorizes:

Think of K-G as an equation describing the length of a vector $|\vec{v}|^2$ and Dirac equation describes $\vec{v}$; you can always get K-G from Dirac but less degrees of freedom then.

ALTERNATIVE MOTIVATION DIRAC EQ.

We saw that $p^\mu p_\mu - m^2c^2 = (\gamma^\kappa p_\kappa + mc)(\gamma^\lambda p_\lambda - mc) = 0$ with appropriate γ-matrices. We then took one of these factors, say $\gamma^\mu p_\mu - mc$, and set it to zero. Assigning operators and acting on a wavefunction we got:

$$i\hbar\gamma^\mu \partial_\mu \psi - mc\psi = 0.$$

<u>But</u>: $\gamma^\mu p_\mu - mc$ is not zero! (check diagonal elements)

$AB = 0$ does not imply $A = 0$ or $B = 0$ when A, B are matrices.

Alternative derivation (same result):

Consider K-G equation: $\left(\nabla^2 - \frac{1}{c^2}\partial_t^2\right)\psi = \left(\frac{mc}{\hbar}\right)^2\psi$.

Try to factorize this (in operator form already):

$$\left(\nabla^2 - \frac{1}{c^2}\partial_t^2\right)\psi = \left(A\partial_x + B\partial_y + C\partial_z + \frac{i}{c}D\partial_t\right)\left(A\partial_x + B\partial_y + C\partial_z + \frac{i}{c}D\partial_t\right)\psi$$

$$= \left(\frac{mc}{\hbar}\right)^2\psi \equiv \kappa^2\psi.$$

This is solved by:

$$\left(A\partial_x + B\partial_y + C\partial_z + \frac{i}{c} D\partial_t\right)\Psi = \pm k\Psi$$

The terms A, B, C, D are identified as γ-matrices to get rid off cross-terms:

$$A, B, C = i\gamma_k \;, \quad D = \gamma^0$$

$$\Rightarrow \quad i\hbar\gamma^\mu\partial_\mu \pm mc\Psi = 0$$

$$(p^\mu p_\mu - m^2c^2) = (\gamma^\mu p_\mu + mc)(\gamma^\lambda p_\lambda - mc) = 0$$

The conventional choice is now to consider:

$$\gamma^\mu p_\mu - mc = 0 \qquad \text{(doesn't matter which one).}$$

Letting $p_\mu \to i\hbar\partial_\mu$, we obtain the final form of the Dirac equation:

$$\boxed{i\hbar\gamma^\mu\partial_\mu\psi - mc\psi = 0}$$

$\psi = \begin{pmatrix}\psi_1\\ \psi_2\\ \psi_3\\ \psi_4\end{pmatrix}$ is a Dirac spinor (<u>not</u> a 4-vector). Hence, it does not transform via a Lorentz transformation when changing inertial system.

SOLUTIONS TO THE DIRAC EQUATION

Consider first the simple case where ψ is independent on $\vec{r}$: $\frac{\partial\psi}{\partial x_j} = 0$, $(j = x,y,z)$. This should be a case with $\vec{p} = 0$ since $p_\mu \to i\hbar\partial_\mu$. We obtain:

$$\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}\partial\psi_A/\partial t\\ \partial\psi_B/\partial t\end{pmatrix} = -i\frac{mc^2}{\hbar}\begin{pmatrix}\psi_A\\ \psi_B\end{pmatrix}$$

Bring to more conventional form by multiplying with γ^0:

$$\gamma^0(\gamma^0 p_0 - \vec{\gamma}\cdot\vec{p} - mc)\psi \to i\partial_t\psi - \gamma^0\vec{\gamma}\cdot\vec{p}\psi - m\gamma^0\psi = 0$$

$\Rightarrow i\partial_t\psi = H_D\psi$ with $H_D = \vec{\alpha}\cdot\vec{p} + \beta m$, $\vec{\alpha} = \gamma^0\vec{\gamma}$ and $\beta = \gamma^0$. Add $V(r)$ now.

with $\varphi_A = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}$ and $\varphi_B = \begin{pmatrix} \varphi_3 \\ \varphi_4 \end{pmatrix}$. Solutions are:

$$\varphi_A(t) = e^{-i(mc^2/\hbar)t}\,\varphi_A(0) \quad , \quad \varphi_B(t) = e^{+i(mc^2/\hbar)t}\,\varphi_B(0)$$

For a particle at rest $E = mc^2$, so φ_A carries the usual factor $e^{-iEt/\hbar}$. The negative ~~energy~~ sign in φ_B $(e^{iEt/\hbar})$ represents antiparticles with positive energy.

For instance: φ_A may represent an electron, in which case φ_B represents a positron.

The p and $\bar{p}$ parts are each 2×1 spinors since they have $s = \frac{1}{2}$. For $\vec{p} = 0$, we then have four independent solutions:

$$\psi^{(1)} = e^{-i(mc^2/\hbar)t}\begin{pmatrix}1\\0\\0\\0\end{pmatrix} \quad , \quad \psi^{(2)} = e^{-i(mc^2/\hbar)t}\begin{pmatrix}0\\1\\0\\0\end{pmatrix}$$

(electron $\uparrow$) (electron $\downarrow$)

$$\psi^{(3)} = e^{+i(mc^2/\hbar)t}\begin{pmatrix}0\\0\\1\\0\end{pmatrix} \quad , \quad \psi^{(4)} = e^{+i(mc^2/\hbar)t}\begin{pmatrix}0\\0\\0\\1\end{pmatrix}$$

(positron $\uparrow$) (positron $\downarrow$)

Let's now look for solutions with $\vec{p} \neq 0$ in a plane-wave form:

$\Psi(\vec{r},t) = a e^{-i/\hbar (Et - \vec{p}\cdot\vec{r})} u(E,\vec{p})$ or in 4-vector notation $\Psi(x) = a e^{-(i/\hbar) x\cdot p} u(p)$. Here, a is a normalization constant and we must determine $u(p)$.

We obtain:

$$\gamma^\mu p_\mu a e^{-(i/\hbar)x\cdot p} u - mca e^{-(i/\hbar)x\cdot p} u = 0$$

$$\Rightarrow (\gamma^\mu p_\mu - mc) u = 0.$$

Simpler than original equation: no derivatives.

Now, we have:

$$\gamma^\mu p_\mu = \frac{E}{c}\begin{pmatrix} \underline{1} & \underline{0} \\ \underline{0} & -\underline{1} \end{pmatrix} - \vec{p}\cdot\begin{pmatrix} 0 & \vec{\underline{\sigma}} \\ -\vec{\underline{\sigma}} & \underline{0} \end{pmatrix},$$ so that:

$$(\gamma^\mu p_\mu - mc) u = \begin{pmatrix} \left(\frac{E}{c} - mc\right) u_A - \vec{p}\cdot\vec{\underline{\sigma}} u_B \\ \vec{p}\cdot\vec{\underline{\sigma}} u_A - \left(\frac{E}{c} + mc\right) u_B \end{pmatrix}$$

$$\Rightarrow u_A = \frac{c}{E - mc^2}(\vec{p}\cdot\vec{\underline{\sigma}}) u_B \quad \& \quad u_B = \frac{c}{E + mc^2}(\vec{p}\cdot\vec{\underline{\sigma}}) u_A$$

Combining these equations, we find:

$$u_A = \frac{c^2 (\vec{p}\cdot\vec{\underline{\sigma}})^2}{E^2 - m^2c^4} u_A.$$

This is consistent, since $(\vec{p}\cdot\vec{\underline{\sigma}})^2 = \vec{p}^{\,2}\,\underline{1}$ which gives:

$$E^2 - m^2c^4 = \vec{p}^{\,2}c^2 \qquad \left(E = \pm\sqrt{m^2c^4 + \vec{p}^{\,2}c^2}\right)$$

The wave functions are then determined up to a normalization constant:

i) $u_A = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, then $u_B = \frac{c}{E + mc^2} \begin{pmatrix} p_z \\ p_x + ip_y \end{pmatrix}$

ii) $u_A = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, then $u_B = \frac{c}{E + mc^2} \begin{pmatrix} p_x - ip_y \\ -p_z \end{pmatrix}$

iii) $u_B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, then $u_A = \frac{c}{E - mc^2} \begin{pmatrix} p_z \\ p_x + ip_y \end{pmatrix}$

iv) $u_B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, then $u_A = \frac{c}{E - mc^2} \begin{pmatrix} p_x - ip_y \\ -p_z \end{pmatrix}$.

For (i) and (ii), one must use $E = +\sqrt{\ldots}$ to avoid diverging u_B when $\vec{p} \to 0$: these are particle solutions.

Opposite for (iii) and (iv).

One tends to normalize these eigenspinors so that $u^\dagger u = 2|E|/c$, $\dagger$ means conjugate transpose:

$$u = \begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{pmatrix} \Longrightarrow u^\dagger = (\alpha^*, \beta^*, \gamma^*, \delta^*).$$

Note that the choices (i) - (iv) above do not correspond immediately to spin $\uparrow, \downarrow$ electrons / positrons.

Neither of them are eigenstates of the spin operator

$$\vec{S} = \frac{\hbar}{2} \vec{\Sigma} = \frac{\hbar}{2} \begin{pmatrix} \vec{\underline{\sigma}} & \underline{0} \\ \underline{0} & \vec{\underline{\sigma}} \end{pmatrix}$$ in general.

Only for the special case of $p_x = p_y = 0$ will (i)–(iv) be eigenspinors of S_z (and $\vec{S}^2$).

We noted that $E = -\sqrt{\ldots}$ corresponds to antiparticles with positive energy (since free particles must have positive energy). It is customary to relabel the antipart. states with "v" and flip the sign of energy and momentum. The four independent solutions are then:

$$u^{(1)} = N \begin{bmatrix} 1 \\ 0 \\ \frac{c p_z}{E + mc^2} \\ \frac{c(p_x + i p_y)}{E + mc^2} \end{bmatrix}, \quad u^{(2)} = N \begin{bmatrix} 0 \\ 1 \\ \frac{c(p_x - i p_y)}{E + mc^2} \\ \frac{-c p_z}{E + mc^2} \end{bmatrix}$$

$$v^{(1)} = N \begin{bmatrix} \frac{c(p_x - i p_y)}{E + mc^2} \\ \frac{-c p_z}{E + mc^2} \\ 0 \\ 1 \end{bmatrix}, \quad v^{(2)} = -N \begin{bmatrix} \frac{c p_z}{E + mc^2} \\ \frac{c(p_x + i p_y)}{E + mc^2} \\ 1 \\ 0 \end{bmatrix},$$

where $N = \sqrt{(|E| + mc^2)/c}$ and $E = \sqrt{m^2c^4 + \vec{p}^2 c^2}$ everywhere.

$u^{(1)}, u^{(2)}$: two spin states for an electron with energy E and momentum $\vec{p}$.

$v^{(1)}, v^{(2)}$: –"– for a positron –"–

Notice that the u's satisfy $(\gamma^\mu p_\mu - mc)\, u = 0$, whereas the v's satisfy $(\gamma^\mu p_\mu + mc)\, v = 0$ since we have reversed the sign of E and $\vec{p}$ by convention.

Plane-wave solutions particularly useful since they describe particles with specified E and $\vec{p}$, which can be controlled in an experiment.

BILINEAR COVARIANTS

A Dirac spinor transforms under a change of inertial system as follows:

$$\psi \to \psi' = S\psi$$ ($'$-system moving with speed v in the x-direction)

where

$$S = \begin{pmatrix} a_+ \underline{1} & a_- \underline{\sigma_1} \\ a_- \underline{\sigma_1} & a_+ \underline{1} \end{pmatrix}$$ with $a_\pm = \pm\sqrt{\frac{1}{2}(\gamma \pm 1)}$ and $\gamma = \dfrac{1}{\sqrt{1-\frac{v^2}{c^2}}}$.

Proof not given here (see e.g. Bjorken & Drell textbook)

It is important to note $\psi^\dagger \psi$ is <u>not</u> a scalar.

This may be seen since $(\psi^+\psi)' \neq \psi^+\psi$:

$$(\psi^+\psi)' = (\psi^+)'\psi' = \psi^+ S^+ S \psi \quad \text{and}$$

$$S^+ S = \gamma \begin{pmatrix} \underline{1} & -\frac{v}{c}\underline{\sigma_1} \\ -\frac{v}{c}\underline{\sigma_1} & \underline{1} \end{pmatrix} \neq 1.$$

Same as for a 4-vector: only the proper metric [relative sign between the components $((x^0)^2-(x^1)^2-(x^2)^2-(x^3)^2)$] makes it transform like a scalar. In this case, it is $\bar{\psi}\psi = \psi^+\gamma^0\psi = |\psi_1|^2 + |\psi_2|^2 - |\psi_3|^2 - |\psi_4|^2$ which is a relativistic invariant. Here, we introduced the adjoint spinor: $\underline{\bar{\psi} \equiv \psi^+\gamma^0}$.

Using this notation and considering how ψ transforms under parity, one finds the following transformation behavior under parity $\psi \to \psi' = \gamma^0\psi$, from which one may derive:

- $\bar{\psi}\psi$ = scalar
- $\bar{\psi}\gamma^5\psi$ = pseudoscalar $\quad \left(\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} \underline{0} & \underline{1} \\ \underline{1} & \underline{0} \end{pmatrix}\right)$
- $\bar{\psi}\gamma^\mu\psi$ = vector
- $\bar{\psi}\gamma^\mu\gamma^5\psi$ = pseudovector

THE PHOTON

Classical electrodynamics described by Maxwell's equations

i) $\nabla \cdot \vec{E} = 4\pi\rho$ iii) $\nabla \cdot \vec{B} = 0$

ii) $\nabla \times \vec{E} + \frac{1}{c}\frac{\partial \vec{B}}{\partial t} = 0$ iv) $\nabla \times \vec{B} - \frac{1}{c}\frac{\partial \vec{E}}{\partial t} = \frac{4\pi}{c}\vec{J}$

where ρ is the charge density and $\vec{J}$ is the current density

In relativistic theory notation, one introduces the antisymmetric field tensor $F_{\mu\nu}$:

$$F_{\mu\nu} = \begin{bmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{bmatrix}$$

and the 4-vector $J^\mu = (c\rho, \vec{J})$. Equations i) and iv) now read: $\partial_\mu F^{\mu\nu} = \frac{4\pi}{c} J^\nu$.

From $F^{\mu\nu} = -F^{\nu\mu}$, one finds then that $\underline{\partial_\mu J^\mu = 0}$

This can be written as $\nabla \cdot \vec{J} = -\frac{\partial \rho}{\partial t}$: charge continuity equation.

QFT = classical field theory + commutation relations for operators + attach operators on the field.
The latter incorporates the quantum aspect.

The homogeneous equations i) and iii) may be re-expressed as:

$$\left.\begin{aligned}\vec{B} &= \nabla \times \vec{A} \\ \vec{E} &= -\nabla V - \frac{1}{c}\frac{\partial \vec{A}}{\partial t}\end{aligned}\right\} \begin{aligned} & F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu \\ & \text{with } A^\mu = (V, \vec{A}).\end{aligned}$$

The fields are the physical observables ($\vec{E}$ & $\vec{B}$), whereas the potentials V & $\vec{A}$ (or just A^μ) are not uniquely determined:

$A_\mu' = A_\mu + \partial_\mu \lambda$ gives the same fields since $\partial^\mu A^{\nu\prime} - \partial^\nu A^{\mu\prime} = \partial^\mu A^\nu - \partial^\nu A^\mu$. Here, λ is any differentiable function of position and time.

$\Rightarrow$ Gauge transformation.

We can choose a particular gauge if we like:

$\partial_\mu A^\mu = 0$ is the Lorentz [redacted] condition, which means we restrict λ to satisfy $\Box \lambda = 0$.

$\Box \equiv \partial^\mu \partial_\mu$. In this gauge, the inhomogeneous Maxwell's equations simplify to $\Box A^\mu = \frac{4\pi}{c} J^\mu$.

For empty space with $J^\mu = 0$, we may choose a gauge with $A^0 = 0$ such that the Lorentz condition reads $\nabla \cdot \vec{A} = 0 \Rightarrow$ Coulomb gauge.

Note: by selecting $A^0 = 0$, we restrict ourselves to one particular inertial system, thus breaking Lorentz covariance. Alternatively, we must perform a gauge transf. along with every Lorentz transf. in order to have the Coulomb gauge in the new inertial system.

In QED, A^μ is the "wavefunction" of the photon.

For a free photon: $\Box A^\mu = 0$ (Klein-Gordon for $m = 0$)

Plane-wave solutions: $A^\mu(x) = a e^{-(i/\hbar) p \cdot x} \varepsilon^\mu(p)$

$\varepsilon^\mu \equiv$ polarization vector characterizing the spin of the photon.

Inserted into $\Box A^\mu = 0$ [which is only valid under the the [redacted]], we find $p^\mu p_\mu = 0$

$\Rightarrow m = 0$ and $E = |p|c$.

Lorentz condition

Proca eq. = K-G eq. + Lorentz-condition.

Now, the Lorentz condition dictates that $p^\mu \varepsilon_\mu = 0$. In the Coulomb gauge, we have:

$$\varepsilon^0 = 0, \quad \vec{\varepsilon} \cdot \vec{p} = 0.$$

Therefore, the polarization vector $\vec{\varepsilon}$ is $\perp$ to $\vec{p}$: a free photon is transversely polarized in the Coulomb gauge. For a given $\vec{p}$, there are two linearly independent 3-vectors $\perp \vec{p}$: e.g. $\varepsilon_{(1)} = (1,0,0)$ and $\varepsilon_{(2)} = (0,1,0)$ for $\vec{p} \sim (0,0,1)$.

Photon has $s=1$, shouldn't there be three available spin states? No:

Massive particle of spin s : $2s+1$ different spin orientations.
Massless particle of spin $s \neq 0$: 2 different spin orientations (regardless of s) *
(—"— spin $s=0$: 1 spin orientation.)

With this gauge, we explicitly eliminated the $m_s = 0$ solution. However, the same physics would transpire if we didn't specify the gauge: in that case, longitudinal "ghost" photons decoupled from everything else would appear.

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* No rest frame for photon, 3-dim. rot-group exchanged with a 2D Poincaré group.

FEYNMAN RULES FOR QED

Wavefunctions for free electrons and positrons of momentum $p = (\frac{E}{c}, \vec{p})$ and $E = \sqrt{m^2c^4 + \vec{p}^2c^2}$ have this form:

Electrons	Positrons
$\psi(x) = a\, e^{-ipx/\hbar}\, u^{(s)}(p)$	$\psi(x) = a\, e^{ipx/\hbar}\, v^{(s)}(p)$

$s = 1,2$ denotes two spin states. The spinors satisfy

$$(\gamma^\mu p_\mu - mc)u = 0 \qquad\qquad (\gamma^\mu p_\mu + mc)v = 0.$$

Their adjoints $\bar{u} = u^+\gamma^0$, $\bar{v} = v^+\gamma^0$ satisfy

$$\bar{u}(\gamma^\mu p_\mu - mc) = 0 \qquad\qquad \bar{v}(\gamma^\mu p_\mu + mc) = 0.$$

They are orthogonal and normalized:

$$\bar{u}^{(1)}u^{(2)} = 0, \quad \bar{u}u = 2mc \qquad\qquad \bar{v}^{(1)}v^{(2)} = 0, \quad \bar{v}v = -2mc.$$

Also, they are complete in the sense that:

$$\sum_{s=1,2} u^{(s)}\bar{u}^{(s)} = (\gamma^\mu p_\mu + mc) \qquad\qquad \sum_{s=1,2} v^{(s)}\bar{v}^{(s)} = (\gamma^\mu p_\mu - mc)$$

This is important as we normally average over electron and positron spin, and need the complete set.

<u>Note</u>: all of the above is equally valid for μ^- and μ^+, τ^- and τ^+, or changing even to quarks and antiquarks (all have spin $\frac{1}{2}$).

Photons

$A^\mu(x) = a\, e^{-ip\cdot x/\hbar}\, \varepsilon^\mu_{(s)}$ with $s=1,2$ being the two spin polarization states. The polarization vectors satisfy $\varepsilon^\mu p_\mu = 0$ (Lorentz condition) and are orthogonal + normalized:

$\varepsilon^{\mu\, *}_{(1)} \varepsilon_{\mu(2)} = 0$ & $\varepsilon^{\mu *} \varepsilon_\mu = 1$.

<u>Coulomb gauge</u>: $\varepsilon^0 = 0$, $\vec{\varepsilon}\cdot\vec{p} = 0$ and one has

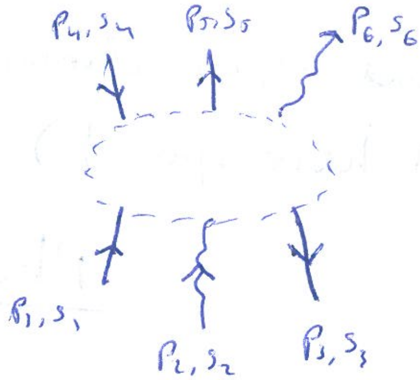
$$\sum_{s=1,2} (\varepsilon_{(s)})_i (\varepsilon^*_{(s)})_j = \delta_{ij} - \hat{p}_i \hat{p}_j$$

for the 3-vectors. Explicit pair: $\varepsilon_{(1)} = (1,0,0)$, $\varepsilon_{(2)} = (0,1,0)$.

To calculate $\mathcal{M}$ for a given diagram:

1) <u>Notation</u>. Label inc. and outgoing momenta and spin with $p_1, p_2, \dots p_n$ and $s_1, s_2, \dots s_n$. Internal momenta q_n. Arrows on external lines indicate if it's an

e^- or e^+, while arrows on internal lines are assigned so that the direction of flow is preserved (one arrow in, one arrow out). External photons have arrows that point forward.



2) External lines

Electrons : Incoming  u

Outgoing  $\bar{u}$

Positrons : Incoming  $\bar{v}$

Outgoing  v

Photons : Incoming  ε^{μ}

Outgoing  $(\varepsilon^{\mu})^*$

3) Vertex factors

Each vertex contributes $ig_e\gamma^{\mu}$, where $g_e = \sqrt{4\pi\alpha}$ is a dim. less coupling constant.

For quarks where $|q| \neq e$, one should use

$$g = -q\sqrt{4\pi/\hbar c}.$$

4) <u>Propagators</u>. Internal lines contribute as follows:

Electrons & positrons $\dfrac{i(\gamma^\mu q_\mu + mc)}{q^2 - m^2c^2}$

Photons: $-\dfrac{i g_{\mu\nu}}{q^2}$.

5) <u>Conservation of E and $\vec{p}$</u>. For each vertex, write $(2\pi)^4 \delta^4(k_1+k_2+k_3)$. Positive k for arrow into vertex and negative k for arrow out of vertex, <u>except</u> for external positrons.

6) <u>Integrate over internal momenta</u>. For each internal q, write $\int \dfrac{d^4q}{(2\pi)^4}$.

7) <u>Cancel the δ-function</u>. Remove $(2\pi)^4\delta^4(p_1+p_2+\ldots-p_n)$ and what remains is $-i\mathcal{M}$.

<u>Total amplitude</u>: sum the amplitudes for each contributing diagram taking into account antisymmetrization.

8) <u>Antisymmetrization</u>. Include a relative minus sign between diagrams that differ only by interchanging two inc. (or outg.) electrons or positrons <u>or</u> of an incoming electron with an outgoing positron (or vice versa): Pauli principle for fermionic wavefunction.

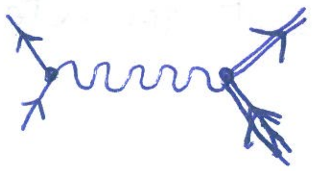
Fermion loops handled later.

EXAMPLES

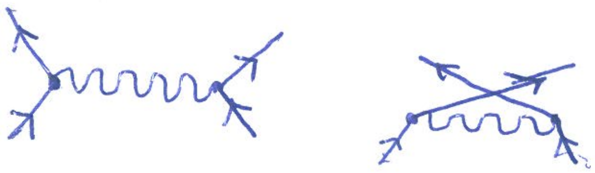
The most important processes in QED are shown below.

Second-order

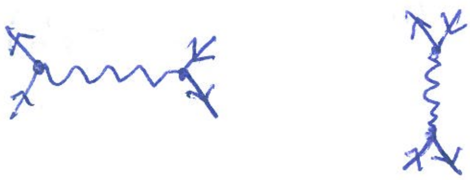
Elastic



$e-\mu$ scattering ($e+\mu \to e+\mu$)
(Mott scatt. if $M \gg m$ → Rutherford scatt. if $v \ll c$)



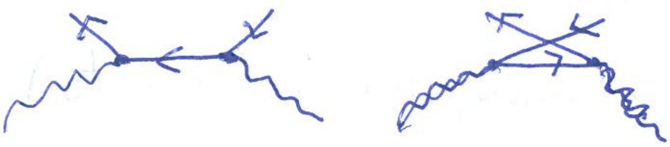
e–e scattering ($e^- + e^- \to e^- + e^-$)
(Møller scattering)



$e^- - e^+$ scattering ($e^- + e^+ \to e^- + e^+$)
(Bhabha scatt.)

Inelastic

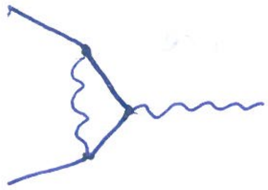
Pair annihilation ($e^- + e^+ \to \gamma + \gamma$)



Pair production ($\gamma + \gamma \to e^- + e^+$)

Compton scattering ($\gamma + e^- \to \gamma + e^-$)

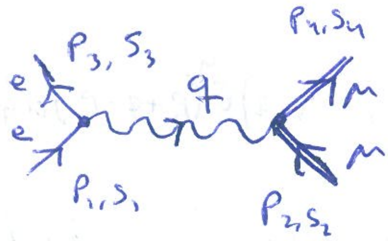
Third-order



Anomalous magnetic moment of electron.

Let's work out $\mathcal{M}$ in detail for some of these processes and illustrate the rules "in action".

EXAMPLE e-μ scattering



$$(2\pi)^4 \int [\bar{u}^{(s_3)}(p_3)\, (ig_e \gamma^\mu)\, u^{(s_1)}(p_1)] \frac{-ig_{\mu\nu}}{q^2} \times$$
$$[\bar{u}^{(s_4)}(p_4)\, ig_e \gamma^\nu u^{(s_2)}(p_2)]\, \delta^4(p_1 - p_3 - q) \times$$
$$\delta^4(p_2 + q - p_4)\, d^4 q$$

After q-integration and dropping δ-function:

$$\mathcal{M} = -\frac{g_e^2}{(p_1 - p_3)^2} [\bar{u}^{(s_3)}(p_3)\gamma^\mu u^{(s_1)}(p_1)][\bar{u}^{(s_4)}(p_4)\gamma_\mu u^{(s_2)}(p_2)].$$

Note that $\mathcal{M}$ is a <u>number</u>.

EXAMPLE e-e scattering

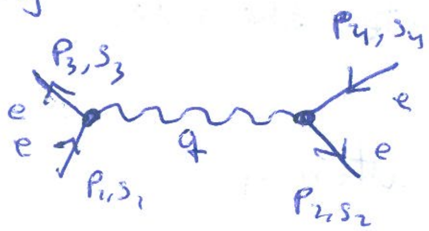
Two contributing diagrams related via $(p_3, s_3) \leftrightarrow (p_4, s_4)$.
Relative minus sign due to antisymmetrization:

$$\mathcal{M} = \frac{-g_e^2}{(p_1 - p_3)^2} [\bar{u}(3)\gamma^\mu u(1)][\bar{u}(4)\gamma_\mu u(2)]$$
$$+ \frac{g_e^2}{(p_1 - p_4)^2} [\bar{u}(4)\gamma^\mu u(1)][\bar{u}(3)\gamma_\mu u(2)]$$

Used the result from $e-\mu$ scattering. There are <u>two</u> diagrams rather than <u>one</u> since the e^- are identical and can be rearranged in two ways.

$\boxed{\text{EXAMPLE}}$ $e^- - e^+$ scattering

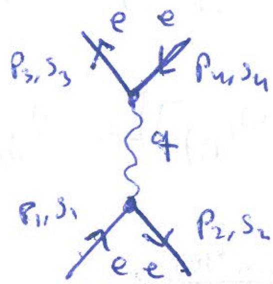
First contribution



$$(2\pi)^4 \int [\bar{u}(3)(ig_e\gamma^\mu)u(1)] \frac{-ig_{\mu\nu}}{q^2} [\bar{v}(2)(ig_e\gamma^\nu)v(4)] \delta^4(p_1-p_3-q)\,\delta^4(p_2+q-p_4)\, d^4q$$

Note that the order is always adjoint spinor/gamma matrix/spinor when moving along a fermion line. Moving backwards (along the arrow) of antiparticle line → forward in time.

Second contribution



Note that the fundamental vertex is always

$$(2\pi)^4 \int [\bar{u}(3)(ig_e\gamma^\mu)v(4)] \frac{-ig_{\mu\nu}}{q^2} [\bar{v}(2)\, ig_e\gamma^\nu u(1)] \delta^4(q-p_3-p_4)\,\delta^4(p_1+p_2-q)\, d^4q$$

By interchanging inc. e^+ and outg. e^-, we get

Therefore, the amplitudes add with a relative minus-sign.

CASIMIR'S TRICK & TRACE THEOREMS

If all spins and polarizations are known a priori in the experiment, appropriate spinors and pol. vectors are inserted into the expression for $\mathcal{M}$.

<u>More often</u>: spins are not known, but random. Then, the cross section is given by the average over all initial spin configurations and the sum over all final spin configurations.

$$\langle |\mathcal{M}|^2 \rangle \equiv \text{average over initial spins and sum over final spins.}$$

Let's introduce some convenient notation:

$$\not{a} = a^\mu \gamma_\mu \qquad \not{a}^* = \gamma^\mu a_\mu^*$$

$$\bar{\Gamma} = \gamma^0 \Gamma^\dagger \gamma^0 \quad \text{for a matrix } \Gamma.$$

Consider now e-μ scattering. We have found:

$$|\mathcal{M}|^2 = \frac{g_e^4}{(p_1-p_3)^4} [\bar{u}(3)\gamma^\mu u(1)][\bar{u}(4)\gamma_\mu u(2)][\bar{u}(3)\gamma^\nu u(1)]^*[\bar{u}(4)\gamma_\nu u(2)]^*$$

The structure has the form:

$$G \equiv [\bar{u}(a)\Gamma_1 u(b)][\bar{u}(a)\Gamma_2 u(b)]^*$$

where $\Gamma_{1,2}$ are 4×4 matrices. For a scalar (1×1 matrix), $k = +$, so we have:

$$[\bar{u}(a)\Gamma_2 u(b)]^k = [u^+(a)\gamma^0\Gamma_2 u(b)]^+ = u^+(b)\Gamma_2^+\gamma^{0+}u(a)$$

$$= \bar{u}(b)\bar{\Gamma}_2 u(a) \qquad (\text{since } \gamma^{0+} = \gamma^0 \text{ and } (\gamma^0)^2 = 1)$$

We now sum over spin orientations of particle b:

$$\sum_{b\text{ spins}} G = \bar{u}(a)\Gamma_1\left\{\sum_{s_b=1,2} u^{(s_b)}(p_b)\bar{u}^{(s_b)}(p_b)\right\}\bar{\Gamma}_2 u(a)$$

$$= \bar{u}(a)\Gamma_1(\not{p}_b + m_b c)\bar{\Gamma}_2 u(a) = \bar{u}(a)Qu(a)$$

where $Q \equiv \Gamma_1(\not{p}_b + m_b c)\bar{\Gamma}_2$.

Now, for a:

$$\sum_{a\text{ spins}}\sum_{b\text{ spins}} G = \sum_{s_a=1,2}\bar{u}^{(s_a)}(p_a)Qu^{(s_a)}(p_a)$$

Write out matrix multiplication:

$$\sum_{s_a=1,2}\bar{u}_i^{(s_a)}Q_{ij}u_j^{(s_a)} = Q_{ij}\left\{\sum_{s_a=1,2}u^{(s_a)}\bar{u}^{(s_a)}\right\}_{ji}$$

$$= Q_{ij}(\not{p}_a + m_a c)_{ji} = \mathrm{Tr}\{Q(\not{p}_a + m_a c)\}$$

where $\mathrm{Tr}(A) \equiv \sum_i A_{ii}$. We used that u and $\bar{u}$ are 4×1 and 1×4 spinors, respectively.

In total, we have:

$$\sum_{\text{all spins}} [\bar{u}(a)\Gamma_1 u(b)][\bar{u}(a)\Gamma_2 u(b)]^* = \mathrm{Tr}\{\Gamma_1(\not{p}_b + m_b c)\bar{\Gamma}_2(\not{p}_a + m_a c)\}$$

There are no spinors left after summing over the spins: only matrix multiplication and a trace at the end.

This is known as Casimir's trick.

Note: if u is replaced with v, the corresponding mass changes sign.

For e-μ scattering, we then have $\Gamma_2 = \gamma^\nu$ and $\bar{\Gamma}_2 = \gamma^0\gamma^{\nu\dagger}\gamma^0 = \gamma^\nu$. We find:

$$\langle |\mathcal{M}|^2 \rangle = \frac{g_e^4}{4(p_1-p_3)^4} \mathrm{Tr}\{[\gamma^\mu(\not{p}_1 + mc)\gamma^\nu(\not{p}_3 + mc)]\}$$
$$\times \mathrm{Tr}\{\gamma_\mu(\not{p}_2 + Mc)\gamma_\nu(\not{p}_4 + Mc\}$$

where $m = m_e$, $M = m_\mu$ and $\frac{1}{4}$ is included to average over initial spins (2 part. with 2 config. = 4).

To evaluate the traces appearing in $\langle |\mathcal{M}|^2 \rangle$, there is a set of trace theorems that may be derived (see book) from some fundamental properties of $\mathrm{Tr}\{\ldots\}$.

We have that

1. $\mathrm{Tr}\{A+B\} = \mathrm{Tr}\{A\} + \mathrm{Tr}\{B\}$
2. $\mathrm{Tr}\{\alpha A\} = \alpha \mathrm{Tr}\{A\}$
3. $\mathrm{Tr}\{AB\} = \mathrm{Tr}\{BA\}$
4. $g_{\mu\nu} g^{\mu\nu} = 4$
5. $\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2g^{\mu\nu}$
6. $\not{a}\not{b} + \not{b}\not{a} = 2a\cdot b$

From this follows e.g. $\mathrm{Tr}\{\gamma^\mu \gamma^\nu\} = 4g^{\mu\nu}$ and

$\mathrm{Tr}\{\not{a}\not{b}\} = 4\cdot a \cdot b$. (Exercise for students: prove odd number of γ has $\mathrm{Tr}=0$ Can use that $\mathrm{Tr}\{\gamma^5\}=0$ and $\{\gamma^5, \gamma^\mu\}=0$)

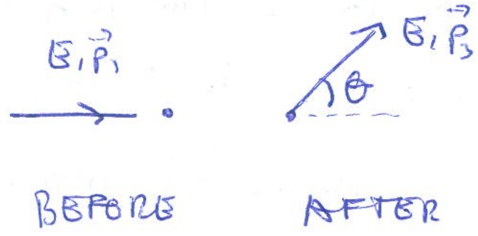
CROSS SECTIONS & LIFETIMES

We can now explicitly evaluate $\frac{d\sigma}{d\Omega}$ for e.g. Mott and Rutherford scattering:

e scatters off a much heavier muon $M \gg m = m_e$.
Find $\frac{d\sigma}{d\Omega}$ in the lab frame (M at rest) when neglecting the recoil of M.

Using the rules for cross-section related to $\mathcal{M}$ given previously, one finds (prob. 6.8 in book)

$$\frac{d\sigma}{d\Omega} = \left(\frac{\hbar}{8\pi Mc}\right)^2 \langle |\mathcal{M}|^2 \rangle$$



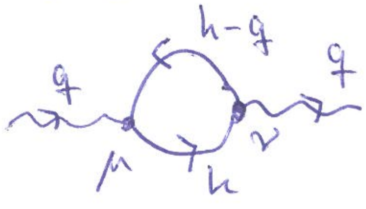
$\boxed{\text{INTERLUDE}}$: treatment of bubble diagrams (fermion loops)

The general rule is:

- Include a factor (-1), take the Tr, and follow the fermion lines ~~backwards~~ along the arrows (~~just like~~ opposite for external lines).

The Tr corresponds to including all possible spin orientations of the fermion as it connects with itself in the end (just like integration over momenta).

Example



$$\Rightarrow -\mathrm{Tr}\int \frac{d^4k}{(2\pi)^4}\frac{\{\gamma_\mu(\not k-\not q+m)\gamma_\nu(\not k+m)\}}{[(k-q)^2-m^2c^2][k^2-m^2c^2]}$$

But is the direction important? Could we also go in the opposite direction with the arrows?

The answer is that for a loop with 2 (even) number of fermions, we can. The proof is:

$$\mathrm{Tr}\{\gamma_\mu \not{p}\, \gamma_\nu \not{q}\} = p^\lambda q^\sigma\, \mathrm{Tr}\{\gamma_\mu \gamma_\lambda \gamma_\nu \gamma_\sigma\}$$

$$= 4 p^\lambda q^\sigma \left(g_{\mu\lambda} g_{\nu\sigma} - g_{\mu\nu} g_{\lambda\sigma} + g_{\mu\sigma} g_{\lambda\nu}\right)$$

$$= 4\left(p_\mu q_\nu - g_{\mu\nu} p^\lambda q_\lambda + q_\mu p_\nu\right) \qquad (*)$$

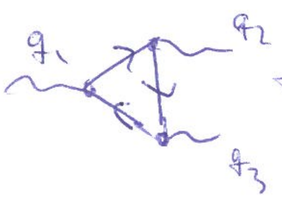
Now the Tr if we go in the opposite direction is:

$\mathrm{Tr}\{\gamma_\mu \not{q}\, \gamma_\nu \not{p}\}$. But we see that $(*)$ is invariant when exchanging $p \leftrightarrow q$. Therefore, the two traces are the same. □

In fact, this is a special case of the more general statement:

$$\mathrm{Tr}\{\gamma_{\mu_1} \gamma_{\mu_2} \cdots \gamma_{\mu_n}\} = \mathrm{Tr}\{\gamma_{\mu_n} \cdots \gamma_{\mu_2} \gamma_{\mu_1}\}.$$

The order should matter for loop with 3 fermions.



topologically different than


?

(flipped it)

We then have:

$p_1 = \left(\frac{E}{c}, \vec{p}_1\right)$, $p_2 = (Mc, 0)$, $p_3 = \left(\frac{E}{c}, \vec{p}_3\right)$, $p_4 = (Mc, 0)$.

$(|\vec{p}_1| = |\vec{p}_3|)$. The averaged amplitude is given by:

$$\langle |\mathcal{M}|^2 \rangle = \frac{8 g_e^4}{(p_1 - p_3)^4} \left[(p_1 p_2)(p_3 p_4) + (p_1 p_4)(p_2 p_3) - (p_1 p_3) M^2 c^2 - (p_2 p_4) m^2 c^2 + 2 (m M c^2)^2 \right]$$

by evaluating the trace according to the proper rules.

Inserting the above 4-momenta:

$$\langle |\mathcal{M}|^2 \rangle = \left(\frac{g_e^2 M c}{\vec{p}^2 \sin^2 \frac{\theta}{2}} \right)^2 \left[(mc)^2 + \vec{p}^2 \cos^2 \frac{\theta}{2} \right].$$

$\Rightarrow$ Mott formula: $\left(\frac{d\sigma}{d\Omega}\right)_{lab} = \left(\frac{\alpha \hbar}{2 \vec{p}^2 \sin^2 \frac{\theta}{2}} \right)^2 \left[(mc)^2 + \vec{p}^2 \cos^2 \frac{\theta}{2} \right].$

Non-relativistic limit $\vec{p}^2 << (mc)^2$ gives Rutherford formula.

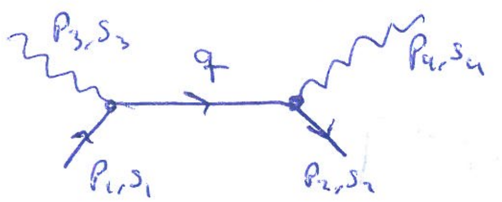
$\frac{d\sigma}{d\Omega} = \left(\frac{e^2}{2mv^2 \sin^2 \frac{\theta}{2}} \right)^2$. Concerning decays: there are <u>no decays</u> in pure QED. No mechanism can convert μ^- to e^- or similar, fermion lines cannot simply terminate in a diagram.

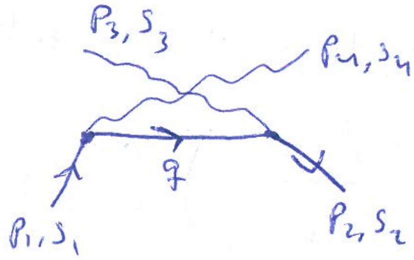
On the other hand, we may have annihilation such as $e^+ + e^- \to \gamma + \gamma$, which is really a scattering event.

Let us analyze this process in the positronium rest frame (CM-frame for e^+e^- pair).

We assume that the bound-state is in the singlet spin configuration and at rest to begin with.

Two contributing diagrams:





The corresponding amplitudes:

$$\mathcal{M}_1 = \frac{g_e^2}{(p_1 - p_3)^2 - m^2c^2} \bar{v}(2) \not{\epsilon}_4 (\not{p}_1 - \not{p}_3 + mc) \not{\epsilon}_3 u(1)$$

$$\mathcal{M}_2 = \frac{g_e^2}{(p_1 - p_4)^2 - m^2c^2} \bar{v}(2) \not{\epsilon}_3 (\not{p}_1 - \not{p}_4 + mc) \not{\epsilon}_4 u(1)$$

Have omitted * on polarization vectors for simpler notation, will reinstate later. Total amplitude $\mathcal{M} = \mathcal{M}_1 + \mathcal{M}_2$ and,

$$p_1 = mc(1,0,0,0) \qquad p_2 = mc(1,0,0,0)$$

$$p_3 = mc(1,0,0,1) \qquad p_4 = mc(1,0,0,-1)$$

It follows that:

$$(p_1 - p_3)^2 - m^2c^2 = (p_1 - p_4)^2 - m^2c^2 = -2(mc)^2.$$

Now use the rule $\not{p}_1 \not{\varepsilon}_3 + \not{\varepsilon}_3 \not{p}_1 = 2\, p_1 \cdot \varepsilon_3$.

Coulomb gauge : $\varepsilon^0 = 0$ and we have $\vec{p}_1 = 0 \Rightarrow \not{p}_1 \not{\varepsilon}_3 = -\not{\varepsilon}_3 \not{p}_1$.

Also, $\not{p}_3 \not{\varepsilon}_3 = -\not{\varepsilon}_3 \not{p}_3$ since $p_3 \cdot \varepsilon_3 = 0$ due to Lorentz condition. Insert this into $\mathcal{M}$ and use that

$(\not{p}_1 - mc)\, u(1) = 0$ [since it solves the Dirac eq.]

$$\Rightarrow (\not{p}_1 - \not{p}_3 + mc) \not{\varepsilon}_3 u(1) = \not{\varepsilon}_3 \not{p}_3 u(1).$$

We then have:

$$\mathcal{M} = -\frac{g_e^2}{2(mc)^2} \bar{v}(2) \left[\not{\varepsilon}_4 \not{\varepsilon}_3 \not{p}_3 + \not{\varepsilon}_3 \not{\varepsilon}_4 \not{p}_4 \right] u(1).$$

Using the properties of γ-matrices and contraction rules, we find that the above can be written as

$$\mathcal{M} = \frac{g_e^2}{(mc)} \bar{v}(2) \left[\vec{\varepsilon}_3 \cdot \vec{\varepsilon}_4 \gamma^0 + i (\vec{\varepsilon}_3 \times \vec{\varepsilon}_4) \cdot \vec{\Sigma} \gamma^5 \right] u(1)$$

where $\vec{\Sigma} = \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix}$ (see p.243 Griffiths)

Now, take into account the spin-singlet symmetry:

$$(\uparrow\downarrow - \downarrow\uparrow)/\sqrt{2}.$$

Amplitude for singlet state:

$$\mathcal{M}_{singlet} = (\mathcal{M}_{\uparrow\downarrow} - \mathcal{M}_{\downarrow\uparrow})/\sqrt{2}.$$

To find $\mathcal{M}_{\uparrow\downarrow}$, use spin-up spinors for the electron and spin-down for positron:

$$u(1) = \sqrt{2mc}\begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \quad \bar{v}(2) = \sqrt{2mc}\,(0\ 0\ 1\ 0)$$

We then find: $\bar{v}(2)\gamma^0 u(1) = 0$ and $\bar{v}(2)\vec{\Sigma}\gamma^3 u(1) = -i2mc\,\hat{z}$.

$$\Rightarrow \mathcal{M}_{\uparrow\downarrow} = -2ig_e^2\,(\vec{\varepsilon}_3 \times \vec{\varepsilon}_4)_z$$

Same procedure for opposite spin:

$$\mathcal{M}_{\downarrow\uparrow} = 2ig_e^2\,(\vec{\varepsilon}_3 \times \vec{\varepsilon}_4)_z = -\mathcal{M}_{\uparrow\downarrow}$$

Total amplitude is then:

$$\mathcal{M}_{singlet} = -2\sqrt{2}\,ig_e^2\,(\vec{\varepsilon}_3 \times \vec{\varepsilon}_4)_z$$

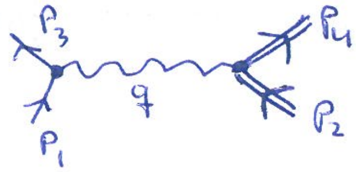
for annihilation of a stationary e^+e^- pair in a spin-singlet configuration into two photons which emerge in the directions $\pm\hat{z}$.

Note that the amplitude for this process in a triplet configuration $(\uparrow\downarrow + \downarrow\uparrow)/\sqrt{2}$ gives 0. Reason: conservation of ~~parity~~ charge conj. in EM processes only allow 2γ, not 3γ from e^+e^- in the singlet and opposite (3γ, not 2γ) for triplets

RENORMALIZATION

We consider now specifically how higher-order diagrams are taken into account by renormalization.

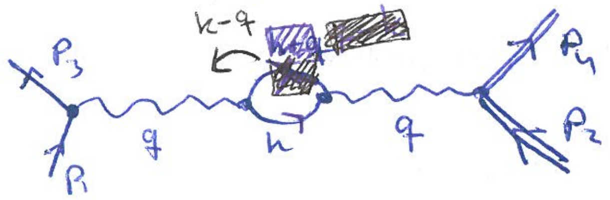
e–μ scattering:



$$\mathcal{M} = -g_e^2 \left[\bar{u}(p_3)\gamma^\mu u(p_1)\right] \frac{g_{\mu\nu}}{q^2} \left[\bar{u}(p_4)\gamma^\nu u(p_2)\right]$$

$(q = p_1 - p_3)$

There are fourth-order corrections, such as the vacuum-polarization diagram:



The temporary $e^- e^+$ pair modifies the effective electric charge of the electron (or alternatively put, the effective coupling constant). One may derive the amplitude by using the extra Feynman rule:

- For each fermion loop (closed), include an overall factor -1 and take the trace.

This gives:

$$\mathcal{M} = \frac{-i g_e^4}{q^4}\left[\bar{u}(p_3)\gamma^\mu u(p_1)\right]\left\{\int \frac{d^4k}{(2\pi)^4} \frac{\mathrm{Tr}\{\gamma_\mu(\not{k}+mc)\gamma_\nu(\not{k}-\not{q}+mc)\}}{(k^2-m^2c^2)\left[(k-q)^2-m^2c^2\right]}\right\}\left[\bar{u}(p_4)\gamma^\nu u(p_2)\right] \quad (*)$$

To include this contribution, we may view it as a modification of the photon propagator:

$$\frac{g_{\mu\nu}}{q^2} \to \frac{g_{\mu\nu}}{q^2} - \frac{i}{q^4} I_{\mu\nu} \quad \text{where } I_{\mu\nu} = -g_e^2 \int d^4k \ldots$$

Problem: this integral is divergent $\sim \ln(k)$.

Strategy: absorb infinities into renormalized masses and coupling constants.

General form of $I_{\mu\nu}$ after integration (only q remaining as 4-vector): $g_{\mu\nu}(\ldots) + q_\mu q_\nu (\ldots)$

We thus write $I_{\mu\nu} = -i g_{\mu\nu} I(q) + q_\mu q_\nu J(q)$.

The 2nd term makes no contribution to the amplitude: q_μ contracts with γ^μ in (*) and gives

$$[\bar{u}(3)\, \not{q}\, u(p_1)] = \bar{u}(p_3)(\not{p}_1 - \not{p}_3)u(p_1) = 0$$

as seen from the basic Dirac equations for u and $\bar{u}$.

The 1st term may be rewritten in the following way:

(Prob. 7.39 Griffiths)

$$I(q) = \frac{g_e}{12\pi^2}\left[\int_{m^2}^{\infty}\frac{dx}{x} - 6\int_0^1 z(1-z)\ln\left(1-\frac{q^2}{m^2c^2}z(1-z)\right)dz\right]$$

The log-divergence is now contained in the first term.

Temporarily introduce a cut-off Λ and define:

$$f(x) \equiv 6\int_0^1 z(1-z)\ln\left[1+x\,z(1-z)\right]dz \qquad \left(x \equiv -\frac{q^2}{m^2c^2}\right).$$

$$\Rightarrow \quad I(q) = \ln\left(\frac{\Lambda^2}{m^2}\right)\cdot\frac{g_e^2}{12\pi^2} - \frac{g_e^2}{12\pi^2}f\left(\frac{-q^2}{m^2c^2}\right)$$

$$f(x) \simeq \begin{cases} x/5 & x \ll 1 \\ \ln x & x \gg 1 \end{cases}$$

The amplitude for e-μ scatt., including vacuum-polarization, now reads:

$$\mathcal{M} = -g_e^2\left[\bar{u}(p_3)\gamma^\mu u(p_1)\right]\frac{g_{\mu\nu}}{q^2}\left[1-\frac{g_e^2}{12\pi^2}\left(\ln\left(\frac{\Lambda^2}{m^2}\right)-f\left(-\frac{q^2}{m^2c^2}\right)\right)\right]$$

$$\times\left[\bar{u}(p_4)\gamma^\nu u(p_2)\right]$$

Here's the critical step: introduce the renormalized coupling constant $g_R \equiv g_e\sqrt{1-\frac{g_e^2}{12\pi^2}\ln\left(\frac{\Lambda^2}{m^2}\right)}$

$$\Rightarrow \quad \mathcal{M} = -g_R^2\left[\bar{u}(p_3)\gamma^\mu u(p_1)\right]\frac{g_{\mu\nu}}{q^2}\left[1+\frac{g_R^2}{12\pi^2}f\left(\frac{-q^2}{m^2c^2}\right)\right]\left[\bar{u}(p_4)\gamma^\nu u(p_2)\right]$$

(up to order g_e^4, made an approximation $\frac{g_e^4}{g_R^2} = g_R^2$ for the last term)

Identical form as tree-level diagram, but with two differences:

① New effective coupling constant $g_e \to g_R$.
It depends (spuriously) on the cut-off Λ, but that's ok: g_R is experimentally measured and determined and corresponds to the physical electric charge (which is finite).
Main point of renormalization: the bare quantity is not the physical quantity (e.g. charge), higher-order corrections to bare quantity gives the effective quantity (physically measurable).

② There is a finite correction term $(1 + \ldots)$, which depends on q. May also be absorbed into the effective coupling constant $\Rightarrow$ "running" coupling constant.

$$g_R(q^2) = g_R(0) \sqrt{1 + \frac{g_R^2(0)}{12\pi^2} f\left(\frac{-q^2}{m^2c^2}\right)} \quad : q\text{-dependent.}$$

High momentum $q \to$ closer approach, meaning the effective charge depends on distance to other particles. Natural consequence of vacuum polarization and screening.

For non-rel. situations, this is a very small effect.

QCD and electrodynamics of quarks – CHAPTER 8

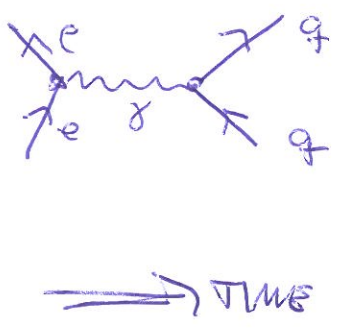
All that we said so far about electrons applies equally well to quarks after substituting $(-e) \to \begin{cases} \frac{2}{3}e \\ -\frac{1}{3}e \end{cases}$.

The "problem" which complicates our observation of how quarks behave is that they never appear freely: we must infer information indirectly via hadrons.

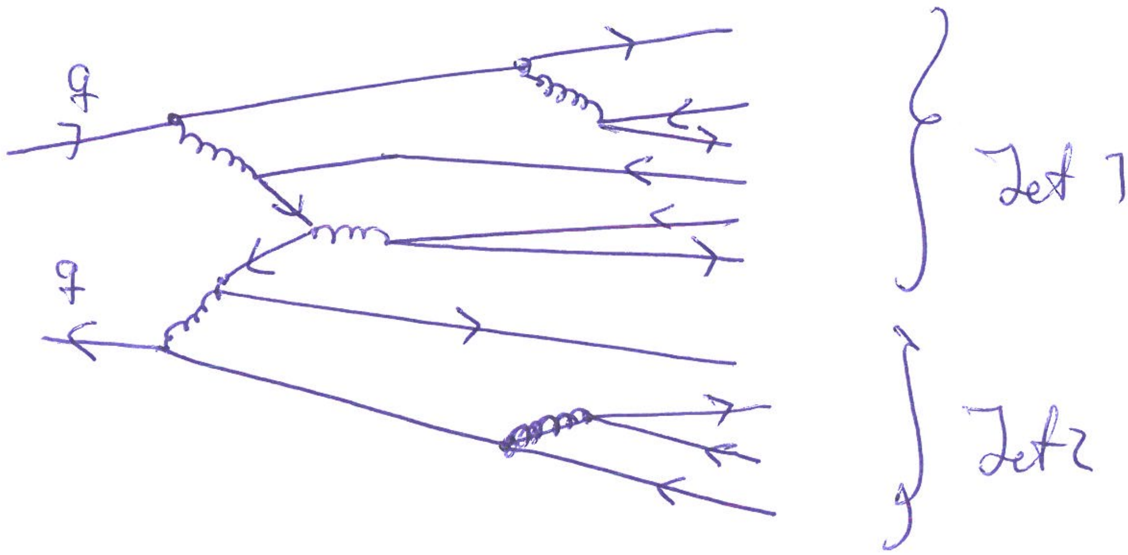
We ~~start~~ here by considering two central examples of hadron-production via $e\bar{e}$ and ep scattering. Then, we develop the framework of QCD: the theory of how colored particles interact. We discuss Feynman rules, color factors, pair annihilation, and finally asymptotic freedom.

Hadron production via e^+e^- collisions

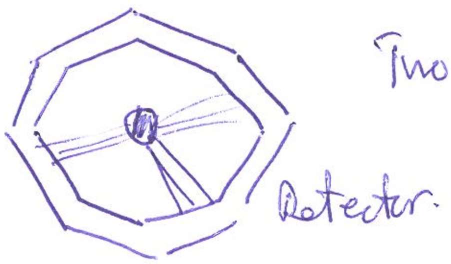
Consider $e^+ + e^- \to q + \bar{q}$



Briefly, the quarks escape as "free" particles, but when their separation reaches 10^{-15} m the strong int. is so great that it triggers a cascade of quark-antiquark pairs $\Rightarrow$ jet formation of hadrons.



What is observed is then $e^+ + e^- \to$ hadrons. The fingerprint of quarks in this is that the hadrons typically emerge in two back-to-back jets along the direction of q and $\bar{q}$, respectively. Sometimes one even sees three-jet events, which is indicative of an emitted gluon carrying a portion of the energy.



Two- and three-jet events.

The observation of three-jet events is generally regarded as the most direct evidence for the existence of gluons.

Note that the first stage in the hadronization and jet formation ($e^+ + e^- \to \gamma \to q + \bar{q}$) is ordinary QED, so we get: (evaluated in CM)

$$\langle |\mathcal{M}|^2 \rangle = Q^2 g_e^4 \left[1 + \left(\frac{mc^2}{E}\right)^2 + \left(\frac{Mc^2}{E}\right)^2 + \left[1 - \left(\frac{mc^2}{E}\right)^2\right] \times \left[1 - \left(\frac{Mc^2}{E}\right)^2\right] \cos^2\theta \right]$$

which leads to a total scattering cross section:

$$\sigma = \frac{\pi Q^2}{3} \left(\frac{\hbar c \alpha}{E}\right)^2 \sqrt{\frac{1 - (Mc^2/E)^2}{1 - (mc^2/E)^2}} \left[1 + \frac{1}{2}\left(\frac{Mc^2}{E}\right)^2\right]\left[1 + \frac{1}{2}\left(\frac{mc^2}{E}\right)^2\right]$$

Here Q is the quark charge in units of e ($\frac{2}{3}$ for u, c, t etc.) and M is the mass of the quark.

σ is imaginary for $E < Mc^2$. Why? Because the process is kinematically forbidden then: not enough energy to even produce rest masses of quarks.

For high energies $E > Mc^2 \gg mc^2$ we get

$$\sigma = \frac{\pi}{3} \left(\frac{\hbar Q c \alpha}{E}\right)^2.$$

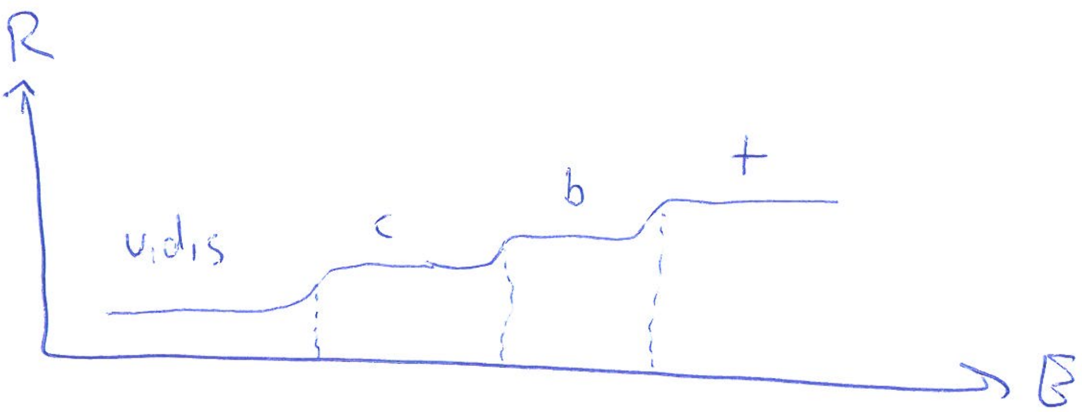
There exists a number of thresholds for the energy as we increase it: first one is able to produce muons, light quarks, then (1300 MeV) the c quark, τ at 1777 MeV, and b at 4500 MeV, and finally t[redacted] quark.

A prediction for experiments would then be to consider the ratio
$$R = \frac{\sigma(e^+e^- \to \text{hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)}$$

Above each threshold, we then have according to σ:

$$R(E) = 3 \sum_i Q_i^2$$

where the sum is over all quark flavors with threshold below E. The factor 3 is due to the three possible colors for each flavor. According to this, we should have some "staircase" graph with a step up every time we exceed a threshold

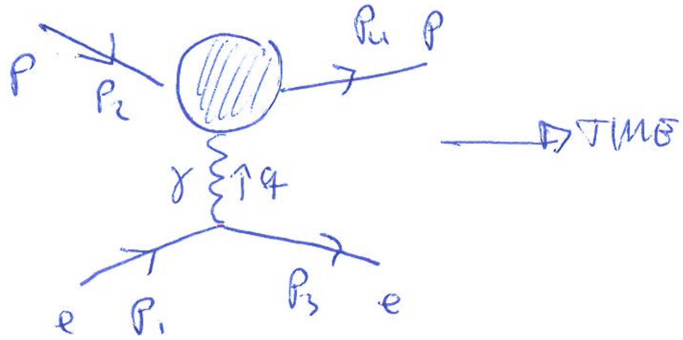


Comparison between experiment and theory is pretty good (except right near the actual thresholds).
One thing missed by our approximations is that the $q\bar{q}$'s are not really free particles as we know, so that the hadronization process cannot exactly be split into two artificial steps $e^+e^- \to q\bar{q}$ and then $q\bar{q} \to$ hadrons.
For instance, we could produce a bound state ($\phi = s\bar{s}$ or $\psi = c\bar{c}$) where the quarks are strongly interacting and our procedure fails. Such events can show up as resonant peaks in σ.

It is worth to note that the color factor of 3 is crucial for the agreement between exp. and theory
$\Rightarrow$ evidence of the color quantum number.

Elastic e-p scattering

Let us now see how we can probe the internal structure of the proton. If p had no internal structure, we could treat it as a point-particle and copy our QED treatment of $e-\mu$ scattering. But since p is not a simple point charge, we need a more flexible formalism to account for this. To lowest order in QED, we may have:



The blob represents the unknown exact structure of the photon-proton vertex interaction.

What we do know, however, is that the e-e vertex and γ propagator are the same, so that we can write generally:

$$\langle |\mathcal{M}|^2 \rangle = \frac{g_e^4}{q^4} L^{\mu\nu}_{\text{electron}} K_{\mu\nu,\,\text{proton}}$$

where $K_{\mu\nu}$ is so far unknown.

So $K_{\mu\nu}$ is a second-rank tensor and it can only depend on three things: p_2, p_4, and q. But since $q = p_4 - p_2$, only two of these are independent.

We choose q and p_2 (let's now drop the subscript so that $p \equiv p_2$ is the initial proton momentum).

With this in mind, the most general form of $K_{\mu\nu}$ is:

$$K^{\mu\nu} = -K_1 g^{\mu\nu} + \frac{K_2}{(Mc)^2} p^\mu p^\nu + \frac{K_4}{(Mc)^2} q^\mu q^\nu$$

$$+ \frac{K^5}{(Mc)^2} (p^\mu q^\nu + p^\nu q^\mu)$$

All K_i's have in this way the same dimension and ~~depend~~ are functions of the only scalar variable in the problem: q^2. Note that $p^2 = M^2c^2$ and

$p_4^2 = M^2c^2 = (q+p)^2 \Rightarrow q \cdot p = -q^2/2.$

These functions are not independent. For instance, one may show (prob. 8.14) that

$$q_\mu K^{\mu\nu} = 0$$

which leads to

$$K_4 = \left(\frac{M_c}{q^2}\right)^2 K_1 + \frac{1}{4} K_2 \text{ and } K_5 = \frac{1}{2} K_2 .$$

We have then reduced $K^{\mu\nu}$ to be expressed with two so-called <u>form factors</u> $K_1(q^2)$ and $K_2(q^2)$:

$$K^{\mu\nu} = K_1\left(-g^{\mu\nu} + \frac{q^\mu q^\nu}{q^2}\right) + \frac{K_2}{(M_c)^2}\left(p^\mu + \frac{1}{2}q^\mu\right)\left(p^\nu + \frac{1}{2}q^\nu\right)$$

With this in hand, we may now evaluate in the standard way

$$\langle |M|^2 \rangle = \left(\frac{2g_e^2}{q^2}\right)^2 \left[K_1\left[(P_1 P_3) - 2(mc)^2\right] + K_2\left[\frac{(P_1 P)(P_3 P)}{(Mc)^2} + \frac{q^2}{4}\right]\right]$$

Working in the laboratory frame with the proton at rest, one obtains (see book for details)

$$\frac{d\sigma}{d\Omega} = \left(\frac{\alpha\hbar}{4ME\sin^2\theta/2}\right)^2 \frac{E'}{E}\left(2K_1 \sin^2\frac{\theta}{2} + K_2 \cos^2\frac{\theta}{2}\right).$$

where E = incident electron energy, E' = outgoing -"-

and θ = scattered angle of electron.

The point is then that by experimentally counting the number of electrons scattered in a given direction and for a range of energies, one determines $K_1(q^2)$ and $K_2(q^2)$.

This gives us a way to model the internal structure of the proton, but a complete theory must be able to <u>calculate</u> what K_1 and K_2 are.

In the simplest model (proton as a point charge) one gets $K_1 = -q^2$ and $K_2 = (2Mc)^2$. This is fine at low energies (e never gets close enough to see the inside of the proton), but it fails at high energies dramatically $\Longrightarrow$ proton has rich internal structure.

Feynman rules for QCD

<u>QCD</u>: interaction between colored particles via gluons.

Strong coupling constant: $g_s = \sqrt{4\pi\alpha_s}$.

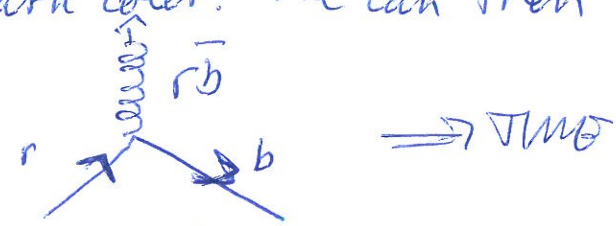
α_s sets the strength of force:

Think of g_s as the fundamental unit of color.

Specification of quark state requires both Dirac spinor $u^{(s)}(p)$ and a three-element column vector c providing color:

$$c = \begin{pmatrix}1\\0\\0\end{pmatrix} \text{ for red, } c = \begin{pmatrix}0\\1\\0\end{pmatrix} \text{ for blue, } c = \begin{pmatrix}0\\0\\1\end{pmatrix} \text{ for green.}$$

Let c_i $(i = 1,2,3)$ run over quark color. We can then have processes like:



Red quark turns into blue quark and gluon carries off missing color. We might expect nine types of gluons then (3×3 combinations of color and anticolor): $r\bar{r}, r\bar{b}, r\bar{g}, b\bar{r}, \ldots$ These states constitute a color octet and a color singlet, which is a more compatible representation with regard to the $SU(3)$ symmetry that QCD is based on (more later):

$$|1\rangle = \frac{r\bar{b} + b\bar{r}}{\sqrt{2}}$$

$$|2\rangle = -i\,(r\bar{b} - b\bar{r})/\sqrt{2}$$

$$|3\rangle = (r\bar{r} - b\bar{b})/\sqrt{2}$$

$$|4\rangle = (r\bar{g} + g\bar{r})/\sqrt{2}$$

$$|5\rangle = -i(r\bar{g} - g\bar{r})/\sqrt{2}$$

$$|6\rangle = (b\bar{g} + g\bar{b})/\sqrt{2}$$

$$|7\rangle = -i\,(b\bar{g} - g\bar{b})/\sqrt{2}$$

$$|8\rangle = (r\bar{r} + b\bar{b} - 2g\bar{g})/\sqrt{6}$$

and the singlet $|9\rangle = (r\bar{r} + b\bar{b} + g\bar{g})/\sqrt{3}$.

Can get e.g. a $r\bar{b}$ gluon by $\dfrac{|1\rangle + i|2\rangle}{\sqrt{2}}$

Note that $|9\rangle$ is invariant under SU(3) transformations [that's why it is called a singlet analogously to a $S_z = 0, S = 0$ spin-singlet state which is invariant under SU(2) spin rotations].

We must now make an important modification (minor) compared to what we have stated previously:

$\boxed{\text{All naturally occurring particles are color singlets}}$

~~colorless~~

Color singlet $\neq$ colorless. For instance, $|3\rangle$ and $|8\rangle$ are colorless in the sense that they have equally much of color/anticolor, but they are not singlets.

Analogy from spin: $S_z = 0$ does not imply $S = 0$ ($\frac{\uparrow\downarrow + \downarrow\uparrow}{\sqrt{2}}$ $m=0$ triplet state has $S = 1$). But $S = 0$ implies $S_z = 0$.

Think of $|9\rangle$ as $\vec{r}^{\,2} = x^2 + y^2 + z^2$ invariant under rotation.

Thus: octet gluons do not appear as free particles, because they are not singlets. But if $|9\rangle$ existed, it should appear not only as mediator but also as free particle. Since gluons are massless, the existence of a singlet gluon would imply a long-ranged strong force which we don't see $\Rightarrow$ only 8 gluons exist.

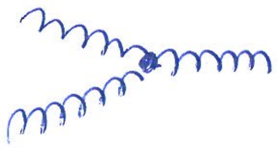
<u>Quantitative description</u>: gluons massless with spin 1, polarization vector ε^{μ} orthogonal to momentum p_{μ}.

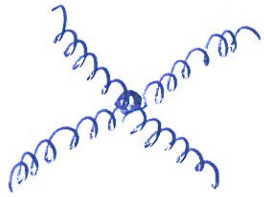
Let's as before use Coulomb gauge: $\varepsilon^0 = 0$, $\vec{\varepsilon} \cdot \vec{p} = 0$.
Note that selecting a particular gauge spoils Lorentz covariance: doing a Lorentz transformation now demands that we also do a belonging gauge transformation to restore the Coulomb gauge). The gauge-choice does, as always, not affect the <u>physics</u>.

To describe the color of the gluon, we use an eight-element vector a:

$$a = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \text{ for } |1\rangle, \quad a = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \text{ for } |7\rangle \text{ etc.}$$

Gluons couple to each other since they carry color (unlike photons that do not carry charge) :





. Before stating the Feynman rules for QCD, we need some notation.

<u>Gell-Mann matrices</u> [they are to SU(3) what Pauli-matrices are to SU(2)]

$$\lambda^1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda^2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\lambda^4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \lambda^5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \quad \lambda^6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\lambda^7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad \lambda^8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

Also need the commutators of λ-matrices which define the structure constants of SU(3) ($f^{\alpha\beta\gamma}$):

$$[\lambda^\alpha, \lambda^\beta] = 2i f^{\alpha\beta\gamma} \lambda^\gamma$$ (summation over γ from 1 to 8)

Antisymmetric : $f^{\alpha\beta\gamma} = -f^{\alpha\gamma\beta} = -f^{\beta\alpha\gamma}$.

Now we can write down Feynman rules.

1. External lines

. Quark with momentum p, spin s, color c.

Incoming  $u^{(s)}(p)\,c$

Outgoing  $\bar{u}^{(s)}(p)\,c^{+}$

For antiquark:

Incoming  $\bar{v}^{(s)}(p)\,c^{+}$

Outgoing  $v^{(s)}(p)\,c$.

where c represents the color of the corresponding quark.

For gluon with momentum p, polarization ε, color a, write:

Incoming 

$\varepsilon_\mu(p)\,a^\alpha$

Outgoing 

$\varepsilon^*_\mu(p)\,(a^\alpha)^*$

2. Propagators

Quarks & antiquarks 

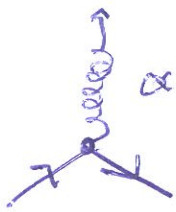
: $\dfrac{i(\not{q}+mc)}{q^2-m^2c^2}$

Gluons 

: $-\dfrac{i g_{\mu\nu}\delta^{\alpha\beta}}{q^2}$

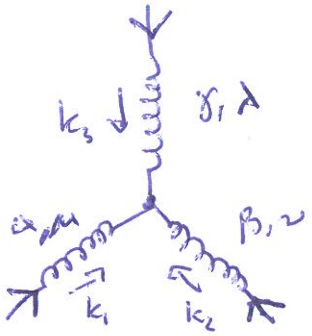
3. Vertices. Each vertex introduces a factor

Quark-gluon



: $-\frac{ig_s}{2}\lambda^\alpha \gamma^\mu$

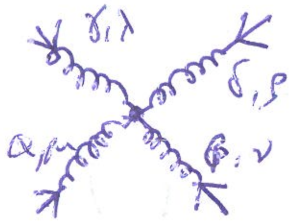
Three-gluon



: $-g_s f^{\alpha\beta\gamma}\left[g_{\mu\nu}(k_1-k_2)_\lambda + g_{\nu\lambda}(k_2-k_3)_\mu + g_{\lambda\mu}(k_3-k_1)_\nu\right]$

Note gluon momenta point <u>into</u> vertex.

Four-gluon



(sum over η implied)

$$-ig_s^2\left[f^{\alpha\beta\eta}f^{\gamma\delta\eta}(g_{\mu\lambda}g_{\nu\rho}-g_{\mu\rho}g_{\nu\lambda}) + f^{\alpha\delta\eta}f^{\beta\gamma\eta} \times (g_{\mu\nu}g_{\lambda\rho}-g_{\mu\lambda}g_{\nu\rho}) + f^{\alpha\gamma\eta}f^{\delta\beta\eta}(g_{\mu\rho}g_{\nu\lambda}-g_{\mu\nu}g_{\lambda\rho})\right]$$

Apart from this, same as in QED. Time for some examples.

Color factors

Let's consider the interaction between two quarks to lowest-order QCD. The focus will be on saying something on the effective potential between quarks (analogue to QED "Coulomb potential").

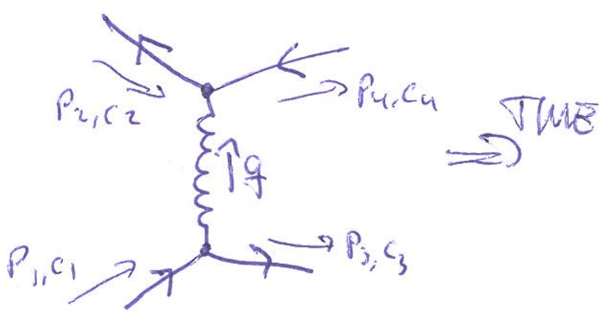
Remember that we are effectively doing a perturbation theory calculation: small α_s. Cannot hope to find asymptotic freedom at lowest order; we only get the short-range potential behavior.

We will find a central result:

> Quarks attract most strongly when they are in a color singlet configuration

Quark & antiquark

Consider $q\bar{q}$ scattering with different flavors (e.g. $u+\bar{d} \to u+\bar{d}$). Lowest-order diagram:



Apply the rules:

$$\mathcal{M} = i\,[\bar{u}(3)\,c_3^\dagger]\left[-\frac{ig_s}{2}\lambda^\alpha\gamma^\mu\right][u(1)c_1]\left[\frac{-ig_{\mu\nu}\,\delta^{\alpha\beta}}{q^2}\right]$$

$$\times\,[\bar{v}(2)\,c_2^\dagger]\left[-\frac{ig_s}{2}\lambda^\beta\gamma^\nu\right][v(4)c_4]$$

$$= -\frac{g_s^2}{4q^2}[\bar{u}(3)\gamma^\mu u(1)][\bar{v}(2)\gamma_\mu v(4)](c_3^\dagger\lambda^\alpha c_1)(c_2^\dagger\lambda^\alpha c_4)$$

Structure is identical to $e-\bar{e}$ scattering except:

1) $g_e \to g_s$
2) Additional color factor $f = \frac{1}{4}(c_3^\dagger\lambda^\alpha c_1)(c_2^\dagger\lambda^\alpha c_4)$.

Thus, we can state that the potential describing $q\bar{q}$ interactions is the same as in QED for opposite charges when replacing α with $f\alpha_s$:

$$V_{q\bar{q}}(r) = -\frac{f\alpha_s\hbar c}{r}.$$

f depends on the color state of the interacting quarks: we can make a singlet or one of the color octets. The latter all yield the same f. Let's compute this one first.

We choose $r\bar{b}$ to be concrete. Color is conserved overall, and thus inc q = out q = red whereas inc $\bar{q}$ = out $\bar{q}$ = antiblue. Thus:

$$c_1 = c_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad c_2 = c_4 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$\Rightarrow f = \frac{1}{4}\left[(1\,0\,0)\lambda^\alpha \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right]\left[(0\,1\,0)\lambda^\alpha \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\right] = \frac{1}{4}\lambda_{11}^\alpha \lambda_{22}^\alpha .$$

Only λ^3 and λ^8 have non-zero entries in (11) and (22) positions $\Rightarrow$ $\underline{f = -\frac{1}{6}}$.

Same procedure for the singlet state $\frac{1}{\sqrt{3}}(r\bar{r} + b\bar{b} + g\bar{g})$ gives $f = \frac{4}{3}$. We have obtained:

$$V_{q\bar{q}}(r) = \begin{cases} -\frac{4}{3}\,\frac{\alpha_s \hbar c}{r} & \text{color singlet} \\ \frac{1}{6}\,\frac{\alpha_s \hbar c}{r} & \text{color octet} \end{cases}$$

It is clear that the force is attractive for the singlet, which helps to explain why $q\bar{q}$ bindings (mesons) occur as color singlets but not octets (which would have produced colored mesons).

Quark & quark

Consider again ~~d~~different flavors (e.g. $u + d \to u + d$).

Lowest-order amplitude:

$$\mathcal{M} = \frac{-g_s^2}{4} \frac{1}{q^2} \left[\bar{u}(3)\gamma^\mu u(1)\right]\left[\bar{u}(4)\gamma_\mu u(2)\right] \left(c_3^\dagger \lambda^\alpha c_1\right)\left(c_4^\dagger \lambda^\alpha c_2\right)$$

Same as for e-μ scattering except $g_e \to g_s$ and the additional color factor $f = \frac{1}{4}\left(c_3^\dagger \lambda^\alpha c_1\right)\left(c_4^\dagger \lambda^\alpha c_2\right)$.

The effective potential is then similar to that of like charges in electrodynamics:

$$V_{qq}(r) = f \frac{\alpha_s \hbar c}{r}.$$

The color configuration for two quarks cannot be singlet. Instead, we have

Triplet (antisymmetric) : $\begin{cases} (rb - br)/\sqrt{2} \\ (bg - gb)/\sqrt{2} \\ (gr - rg)/\sqrt{2} \end{cases}$

Sextet (symmetric) : $\begin{cases} rr,\ bb,\ gg \\ (rb + br)/\sqrt{2} \\ (bg + gb)/\sqrt{2} \\ (gr + rg)/\sqrt{2} \end{cases}$

Doing the calculation as before, we find:

$$V_{qq}(r) = -\frac{2}{3}\,\frac{\alpha_s \hbar c}{r} \quad \text{(triplet)}$$

$$V_{qq}(r) = \frac{1}{3}\,\frac{\alpha_s \hbar c}{r} \quad \text{(sextet)}$$

Note different signs, but we know that neither occurs in nature (in spite of attractive interaction for triplets). We must take into account that the short-range attraction does not prove that binding occurs: we'd have to know about the long-range behavior for the same color configuration.

It does, however, have important consequences for the binding of three quarks where you can show that complete mutual attraction between three quarks is obtained only in a singlet configuration (see book for more details)

For a detailed technical calculation of quark-gluon scattering, see book for example. We will close this chapter by considering asymptotic freedom.

Asymptotic freedom

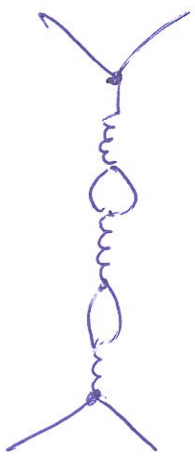
In the last QED-chapter, we found that the loop diagram  caused the effective charge of the electron to be a function of the momentum transfer q:

$$\alpha(q) = \alpha(0)\left[1 + \frac{\alpha(0)}{3\pi}\ln\left(\frac{|q|^2}{(mc)^2}\right)\right] \qquad (*)$$

for $|q|^2 = -q^2 \gg (mc)^2$. The physics in this is that as the charges get closer to each other (larger $|q|^2$), the coupling strength increases due to vacuum polarization. The screening is less effective at shorter distances.

We recall that this diagram also introduces a divergent term that we "soak up" in an effective charge (which is what is measured).

Now, the above equation holds to order $[\alpha(0)]^2$. Higher-order:

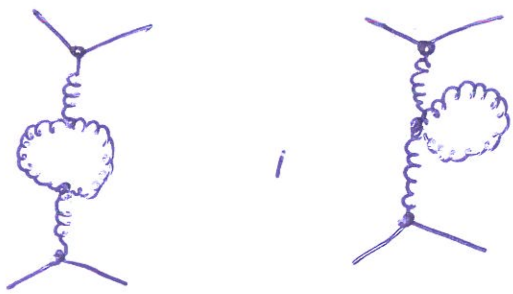
 ,  , ... can be summed explicitly and gives the result:

$$\alpha(q) = \frac{\alpha(0)}{1 - [\alpha(0)/3\pi]\ln\left(\frac{|q|^2}{(mc)^2}\right)} .$$

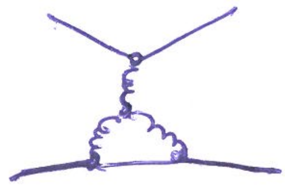
We can in fact understand why it looks like this when taking into account all higher-order loop diagrams since we get a series $1 + x + x^2 + \dots = \frac{1}{1-x}$ where x represents a bubble.

Much of the same thing happens in QCD where $q-\bar{q}$ bubbles  screen quark color and gives (modulo color factors f) the same as Eq. (*).

But QCD has a twist: there are now also virtual gluon bubbles (due to gluon-gluon coupling):



and also diagrams of the form



Now, the gluon contribution works in the opposite direction as the $q-\bar{q}$ bubbles: produces antiscreening or "camouflage."

The formula for the running coupling constant in QCD reads: $\longrightarrow$

$$\alpha_s(|q|^2) = \frac{\alpha_s(\mu^2)}{1 + [\alpha_s(\mu^2)/12\pi](11n - 2f)\ln\left(\frac{|q|^2}{\mu^2}\right)}, \quad |q|^2 >> \mu^2.$$

where n is the number of colors and f is the number of flavors ($n=3$, $f=6$ in SM). Thus, since $11n > 2f$ the coupling constant α_s will decrease with increasing $|q|^2$: at short distances, the "strong force" becomes quite weak.

This is the asymptotic freedom $\Rightarrow$ licence to use perturbation theory (Feynman diagrams) for interquark potentials.

But what is μ^2? In electrodynamics it is natural to define "the charge" of a particle at long-range (fully screened) value. We could, however, use the effective charge at any $|q|^2$ as the reference value as long as $\alpha(|q|^2)$ is small there.

<u>In QCD</u>: α_s large when $q^2 \to 0$, so we cannot use that as reference point.

Instead, we thus use $\alpha_s(\mu^2) < 1$ as the "bare" strength of the coupling constant which we base the perturbation expansion on.

Note that $\alpha_s(|q|^2)$ varies substantially over available energy ranges experimentally whereas $\alpha(|q|^2)$ varies much less.

CHAPTER 10 – WEAK INTERACTIONS

We will here establish Feynman rules for charged ($W^{\pm}$) and neutral (Z^0) weak interactions related to leptons and quarks, and treat some important processes in detail. Finally, we gather EM and weak vertices under the same umbrella: Glashow–Weinberg–Salam electroweak theory.

CHARGED LEPTONIC WEAK INTERACTIONS

Mediators in weak interactions: $W^{\pm}$ (charged) and Z^0 (neutral).

$M_W = 82 \pm 2$ GeV/c^2, $M_Z = 92 \pm 2$ GeV/c^2.

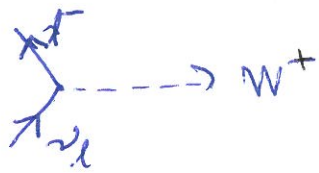
Massful spin-1 bosons: three polarization states ($m_s = -1, 0, 1$).

Propagator: $\dfrac{-i\left[g_{\mu\nu} - q_\mu q_\nu/(Mc)^2\right]}{q^2 - M^2c^2}$

If $q^2 \ll (Mc)^2$ (often the case), then $\rightarrow \dfrac{i g_{\mu\nu}}{(Mc)^2}$ $\left(q^2 \ll (Mc)^2\right)$.

Fundamental leptonic vertex:

Reverse process also possible:



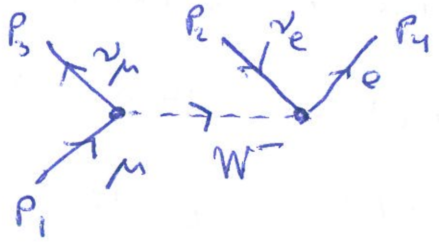
Feynman rules are the same as in QED except:

- Modified propagator (massive)
- Weak vertex factor $-\frac{i g_W}{2\sqrt{2}} \gamma^\mu (1-\gamma^5)$.

The factor $(1-\gamma^5)$ leads to parity violation: γ^μ alone gives a vector coupling (QED), $\gamma^\mu \gamma^5$ gives an axial vector coupling (see previous chapter).

Decay of the muon

$\mu \to e + \nu_\mu + \bar{\nu}_e$



Amplitude: $\mathcal{M} = \frac{g_W^2}{8 (M_W c)^2} [\bar{u}(3) \gamma^\mu (1-\gamma^5) u(1)] [\bar{u}(4) \gamma_\mu (1-\gamma^5) v(2)]$

Using Casimir's trick, we find that:

$$\langle |\mathcal{M}|^2 \rangle = 2 \left(\frac{g_W}{M_W c} \right)^4 (p_1 \cdot p_2)(p_3 \cdot p_4).$$

Analyzing in the muon rest frame: $p_1 = (m_\mu c, \vec{0})$.

we get $p_1 p_2 = m_\mu E_2$. Moreover, $p_1 = p_2 + p_3 + p_4$:

$$(P_3+P_4)^2 = m_e^2c^2 + 2P_3P_4 = (P_1-P_2)^2 = m_\mu^2c^2 - 2P_1P_2$$

$$\Rightarrow \quad P_3\cdot P_4 = \frac{(m_\mu^2-m_e^2)c^2}{2} - m_\mu E_2$$

Have set $m_\nu = 0$, and since $m_\mu \gg m_e$, we can set $m_e = 0$ as an approximation:

$$\langle|\mathcal{M}|^2\rangle = \left(\frac{g_w}{M_w c}\right)^4 m_\mu^2 E_2 (m_\mu c^2 - 2E_2)$$

We use the Golden rule to calculate the decay rate.

$$d\Gamma = \frac{\langle|\mathcal{M}|^2\rangle}{2\hbar m_\mu}\left(\prod_{i=2,3,4} \frac{c\, d^3p_i}{(2\pi)^3 2E_i}\right)(2\pi)^4 \delta^4(P_1-P_2-P_3-P_4)$$

Here, $E_i = |\vec{p}_i|c$. Do the $\vec{p}_3$-integral:

$$d\Gamma = \frac{\langle|\mathcal{M}|^2\rangle c^3}{16(2\pi)^5\hbar m_\mu}\frac{(d^3p_2)(d^3p_4)}{E_2E_3E_4}\,\delta\left(m_\mu c - \frac{E_2}{c} - \frac{E_3}{c} - \frac{E_4}{c}\right)$$

where $E_3 = |\vec{p}_2+\vec{p}_4|c$. Now do the $\vec{p}_2$-integral.

Let $\vec{p}_4 \parallel \hat{z}$ so that:

$$\left(\frac{E_3}{c}\right)^2 = |\vec{p}_2+\vec{p}_4|^2 = \frac{1}{c^2}(E_2^2 + E_4^2 + 2E_2E_4\cos\theta)$$

Also, $d^3p_2 = \left(\frac{E_2}{c}\right)^2 \frac{dE_2}{c}\sin\theta\, d\theta\, d\varphi$.

No φ-dependence $\rightarrow \int d\varphi = 2\pi$.

To do the θ-integration, let $x \equiv \frac{B_3}{c}$ so that

$$dx = -\frac{B_2 B_4 \sin\theta\, d\theta}{c B_3}.$$ We then have:

$$\int_0^\pi \frac{\sin\theta\, d\theta}{B_3}\, \delta\left(m_\mu c - \frac{B_2}{c} - \frac{B_3}{c} - \frac{B_4}{c}\right)$$

$$= \frac{c}{B_2 B_4} \int_{x_-}^{x_+} \delta\left(m_\mu c - x - \frac{B_2}{c} - \frac{B_4}{c}\right) dx$$

$$= \begin{cases} \dfrac{c}{B_2 B_4} & \text{if } x_- < m_\mu c - \frac{B_2}{c} - \frac{B_4}{c} < x_+ \\ 0 & \text{otherwise} \end{cases}$$

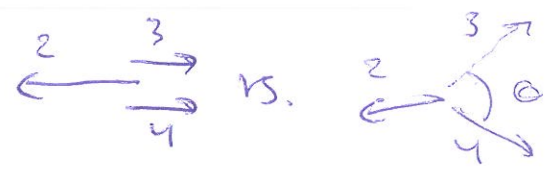
Here, $x_\pm = \frac{1}{c} |B_2 \pm B_4|$. Can write more conveniently by adding $(B_2 + B_4)$ and dividing on 2:

$$\frac{1}{2}\left[|B_2 - B_4| + B_2 + B_4\right] < \frac{1}{2} m_\mu c^2 < B_2 + B_4$$

Equivalent to three inequalities: $\begin{cases} B_2 < \frac{1}{2} m_\mu c^2 \\ B_4 < \frac{1}{2} m_\mu c^2 \\ (B_2 + B_4) > \frac{1}{2} m_\mu c^2. \end{cases}$

This makes sense physically: max. energy for a particle if it emerges <u>opposite</u> to the two others. Conservation of momentum dictates that the particle then picks up <u>half</u> the available energy. Therefore, a pair of particles (e.g. 2+4) must always have at least $\frac{1}{2} m_\mu c^2$.

If there is an angle between the particles, some get more momentum and one gets less.



$$\Gamma = \frac{1}{4\pi^3 \hbar} \left(\frac{g_W}{2M_W c}\right)^4 (m_e c^2)^5 \left[\frac{1}{15}(2a^4 - 9a^2 - 8)\sqrt{a^2-1} + a \ln\left(a + \sqrt{a^2-1}\right)\right]$$

with $a \equiv \frac{m_n - m_p}{m_e}$. Note that we cannot here neglect the electron mass, since the rest energy of the electron is comparable to the released energy: $(m_n - m_p - m_e)c^2$. This is not the case for μ-decay: $(m_\mu - m_e)c^2$.

Putting in numbers, we obtain:

$\tau = \frac{1}{\Gamma} = 1316$ s. Experimental value - 898 ± 16 sec.

Correct $\underbrace{\text{of}}_{\text{order}}$ magnitude, but still deviation. Weak decays range from 15 min to 10^{-13} sec, so it's not too bad.

Reason for discrepancy:

Assumed p and n are point particles (neglecting internal structure) and assume interaction with W in the same way as leptons.

At the same time, we know the Mott formula works very well for e-p scattering, where p is also treated as a point particle.

The crucial question is: what is the net coupling strength of the proton and neutron?

In electrodynamics, all internal complications don't matter because electric charge is conserved. However, we don't know that the same should be valid for a weak coupling: a gluon splitting to a $q\bar{q}$ pair might make a finite contribution to the effective weak coupling (vertex). To account for this in the $n \to p + W$ vertex, one writes

$(1-\gamma^5) \to (c_V - c_A\gamma^5)$, c_V: correction to $\underbrace{\text{"weak charge"}}_{\text{vector}}$

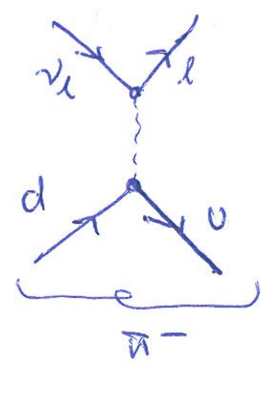
c_A: correction to axial vector "weak charge".

Experimentally: $c_V \simeq 1.00$, $c_A \simeq 1.26$.

Corrected lifetime due to this: <u>914 sec</u>

Decay of the pion

$\pi^- \to l^- + \bar{\nu}_l$ is actually a scattering event of bound quarks:



Analog to positronium decay $(e^+ + e^- \to \gamma + \gamma)$. Analyzing this decay in the framework of weak interactions, we may write:

The blob represents the (unknown) coupling of π^- to W^-.

Let $\tilde{F}^\mu$ describe . Amplitude:

$$\mathcal{M} = \bar{u}(3)\left(\frac{-ig_w}{2\sqrt{2}}\gamma^\mu(1-\gamma^5)\right)v(2)\,\frac{ig_{\mu\nu}}{(M_W c)^2}\tilde{F}^\mu$$

$$= \frac{g_w^2}{8(M_W c)^2}\left[\bar{u}(3)\gamma_\mu(1-\gamma^5)v(2)\right]F^\mu$$

Absorbed a constant into $\tilde{F}^\mu \to F^\mu$. We know it must be a 4-vector to contract γ_μ. Since π^- has spin 0, F^μ can only depend on p^μ: $F^\mu = f_\pi p^\mu$ where f_π is a scalar.

Sum over outgoing spins:

$$\langle|\mathcal{M}|^2\rangle = \left[\frac{f_\pi}{8}\left(\frac{g_w}{M_W c}\right)^2\right]^2 p_\mu p_\nu \operatorname{Tr}\left\{\gamma^\mu(1-\gamma^5)\not{p}_2\gamma^\nu(1-\gamma^5)(\not{p}_3 + m_l c)\right\}$$

$$= \frac{1}{8}\left[f_\pi\left(\frac{g_w}{M_W c}\right)^2\right]^2\left[2(p\cdot p_2)(p\cdot p_3) - p^2(p_2\cdot p_3)\right]$$

Since $p = p_2 + p_3$, $2 p_2 p_3 = (m_\pi^2 - m_\ell^2) c^2$.

$$\Rightarrow \langle |\mathcal{M}|^2 \rangle = \left(\frac{g_w}{2M_w}\right)^4 f_\pi^2 m_\ell^2 (m_\pi^2 - m_\ell^2)$$

This is as far as we get without specifying what f_π is. However, we can calculate branching ratios:

$$\frac{\Gamma(\pi^- \to e^- + \bar{\nu}_e)}{\Gamma(\pi^- \to \mu^- + \bar{\nu}_\mu)} = \frac{m_e^2 (m_\pi^2 - m_e^2)^2}{m_\mu^2 (m_\pi^2 - m_\mu^2)^2} = 1.28 \cdot 10^{-4}.$$

Experimental measurement: $1.23 \pm 0.02 \times 10^{-4}$.

Is this reasonable? π seems to prefer decay into μ in spite of the fact that $m_\mu \gg m_e$.

Mere phase space for decay into lighter particle (mass decrease as large as possible).

Dynamical explanation: If $m_e = 0$, then $\pi^- \to e^- + \bar{\nu}_e$ would be completely forbidden. π^- has $s = 0$, so e and $\bar{\nu}_e$ have to emerge with opposite spins (= equal helicity).

Since $\bar{\nu}_e$ is considered as always right-handed, e should also be that here.

This is slightly modified when coupling to composite particles like the proton, but that's due to strong interaction contamination. The coupling to Z^0 is:

$$\frac{-ig_Z \gamma^\mu (c_V^f - c_A^f \gamma^5)}{2} \qquad (Z^0 \text{ vertex factor})$$

g_Z: neutral coupling constant, $c_{V,A}^f$: coefficients depending on f.

These numbers are determined by the weak mixing angle (Weinberg angle) θ_W. We have that:

$$g_W = \frac{g_e}{\sin\theta_W}$$

$$g_Z = \frac{g_e}{\sin\theta_W \cos\theta_W}$$

f	c_V	c_A
ν_e, ν_μ, ν_τ	$\frac{1}{2}$	$\frac{1}{2}$
e^-, μ^-, τ^-	$-\frac{1}{2} + 2\sin^2\theta_W$	$-\frac{1}{2}$
u, c, t	$\frac{1}{2} - \frac{4}{3}\sin^2\theta_W$	$\frac{1}{2}$
d, s, b	$-\frac{1}{2} + \frac{2}{3}\sin^2\theta_W$	$-\frac{1}{2}$

This will be motivated later on (electroweak theory).

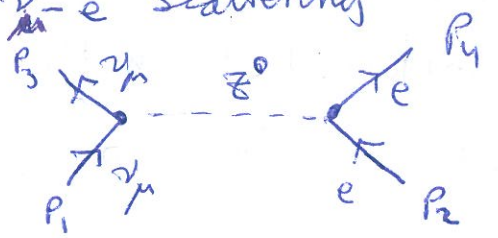
There is no way in the SM to calculate θ_W, but its value may be inferred from experiments: $\theta_W = 28.7^\circ$.

We already know the propagator:

$$\frac{-i(g_{\mu\nu} - q_\mu q_\nu / M_Z^2 c^2)}{q^2 - M_Z^2 c^2} \qquad \text{and} \qquad M_W = M_Z \cos\theta_W.$$

EXAMPLE — Elastic $\nu_\mu - e$ Scattering

$$\nu_\mu + e \rightarrow \nu_\mu + e$$

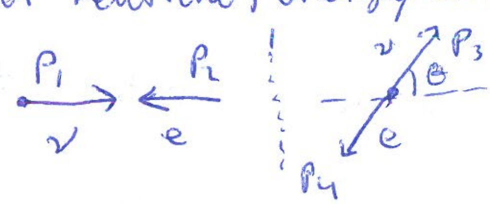


$$\mathcal{M} = \frac{g_Z^2}{8(M_Z c)^2}\left[\bar{u}(3)\gamma^\mu(1-\gamma^5)u(1)\right]\left[\bar{u}(4)\gamma_\mu(c_V - c_A\gamma^5)u(2)\right]$$

where $\{c_V, c_A\}$ are the neutral weak couplings for electrons. Go to the CM frame and assume very high energy scattering so that we may neglect to electron mass (rest energy). Then:

$$\langle|\mathcal{M}|^2\rangle = 2\left(\frac{g_Z E}{M_Z c^2}\right)^4\left[(c_V+c_A)^2 + (c_V-c_A)^2\cos^4\frac{\theta}{2}\right]$$

where E is the electron (or neutrino) energy and θ is the scattering angle



Diff. scatt. cross section for this situation was worked out in chapter 6:

$$\frac{d\sigma}{d\Omega} = 2\left(\frac{\hbar c}{\pi}\right)^2\left(\frac{g_Z}{4M_Z c}\right)^4 E^2\left[(c_V+c_A)^2 + (c_V-c_A)^2\cos^4\frac{\theta}{2}\right]$$

$$\Longrightarrow \sigma = \frac{2}{3\pi}(\hbar c)^2\left(\frac{g_Z}{2M_Z c^2}\right)^4 E^2\left(c_V^2 + c_A^2 + c_V c_A\right)$$

Note: most (not all) neutral processes are "masked" by competing EM ones. For instance, $e^+ + e^- \to \mu^+ + \mu^-$ can occur both via exchange of a virtual Z^0 or γ.

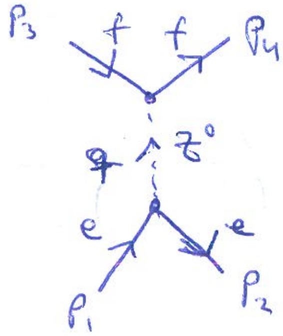
Moreover: there is a weak contamination in every EM process since Z^0 couples to everything that γ does (and more). Even if the effect is small, its smoking gun signature (when observable) is parity violation.

To access weak interactions alone, one can use neutrinos (with no EM coupling) or sufficiently high energies so that $q \sim M_Z c$ and hence the denominator of the Z^0 propagator is small $\to$ large interaction.

EXAMPLE: $e^- e^+$ - scattering near the Z^0 pole.

$e^+ + e^- \to f + \bar{f}$ where f is a quark or lepton.

Assume $m_f \ll M_Z$, but we use the exact form of the Z^0 propagator since we're interested in $q \sim M_Z c$.



By the way, note that Z^0 is its own antiparticle while W^+ and W^- are each others antiparticles.

Then, we get:

$$\mathcal{M} = -\frac{g_Z^2}{4[q^2-(M_Z c)^2]}\left[\bar{u}(4)\gamma^\mu(c_V^f - c_A^f\gamma^5)v(3)\right]$$

$$\times\left(g_{\mu\nu} - \frac{q_\mu q_\nu}{(M_Z c)^2}\right)\left[\bar{v}(2)\gamma^\nu(c_V^e - c_A^e\gamma^5)u(1)\right]$$

with $q = p_1 + p_2 = p_3 + p_4$. Since we will consider energies near 90 GeV, we may safely neglect all masses.

From the second term, we find that q_μ contracts with γ^μ to give $\bar{u}(4)\,\not{q}\,(c_V - c_A\gamma^5)v(3)$.

Since $\not{q} = \not{p}_3 + \not{p}_4$ and $\bar{u}(4)\not{p}_4 = 0$ [Dirac eq. for $m=0$] in addition to:

$$\not{p}_3(c_V - c_A\gamma^5)v(3) = (c_V + c_A\gamma^5)\not{p}_3 v(3) = 0,$$

$q_\mu q_\nu$ then gives no contribution. Performing the spin summation (Casimir's trick):

$$\langle|\mathcal{M}|^2\rangle = \left[\frac{g_Z^2}{8(q^2-M_Z^2c^2)}\right]^2 \mathrm{Tr}\left\{\gamma^\mu(c_V^f - c_A^f\gamma^5)\not{p}_3\gamma^\nu(c_V^f - c_A^f\gamma^5)\not{p}_4\right\}$$

$$\times\,\mathrm{Tr}\left\{\gamma^\mu(c_V^e - c_A^e\gamma^5)\not{p}_1\gamma_\nu(c_V^e - c_A^e\gamma^5)\not{p}_2\right\}$$

Performing the traces (using rules for γ-matrices) and integrating over the scattering angle, we find:

$$\sigma = \frac{1}{3\pi}\left[\frac{\hbar c g_Z^2 E}{4[(2E)^2-(M_Z c^2)^2]}\right]^2 \left[(c_V^f)^2+(c_A^f)^2\right]\left[(c_V^e)^2+(c_A^e)^2\right]$$

in the CM frame.

When total energy $2E \to M_Z c^2$, σ diverges! To counter this (one would never observe a true mathematical divergence physically), take into account the finite lifetime τ_Z of Z^0:

$$\frac{1}{q^2-(M_Z c)^2} \longrightarrow \frac{1}{q^2-(M_Z c)^2+i\hbar M_Z \Gamma_Z} \qquad (\Gamma_Z = \tau_Z^{-1}).$$

This "smears" out the mass, and leads to:

$$\sigma = \frac{(\hbar c g_Z^2 E)^2}{48\pi}\,\frac{\left[(c_V^f)^2+(c_A^f)^2\right]\left[(c_V^e)^2+(c_A^e)^2\right]}{\left[(2E)^2-(M_Z c^2)^2\right]^2+(\hbar M_Z c^2 \Gamma_Z)^2}$$

Correction may be neglected except if $2E \sim M_Z c^2$.

Now, the same process mediated by a photon gives:

$$\sigma = \frac{(\hbar c g_e^2)^2 (Q^f)^2}{48\pi E^2} \qquad (Q^f = \text{charge of } f \text{ in units of } e)$$

Compare γ- and Z^0- mediated scattering directly:

$$\frac{\sigma(e^+e^- \to Z^0 \to \mu^+\mu^-)}{\sigma(e^+e^- \to \gamma \to \mu^+\mu^-)} \simeq \frac{2E^4}{[(2E)^2-(M_Zc^2)^2]^2+(\hbar\Gamma_Z M_Zc^2)^2}$$

when inserting for Θ_W in $c_{V,A}^f$ and $c_{V,A}^e$.

$$\lim_{2E \ll M_Zc^2} \frac{\sigma_Z}{\sigma_\gamma} \simeq 2\left(\frac{E}{M_Zc^2}\right)^4 \ll 1$$

$$\lim_{2E \to M_Zc^2} \frac{\sigma_Z}{\sigma_\gamma} \simeq \frac{1}{8}\left(\frac{M_Zc^2}{\hbar\Gamma_Z}\right)^2 \gg 1$$

Have used $\hbar\Gamma_Z = 2.5$ GeV. Hence, weak mechanism strongly favored near Z^0 pole.

ELECTROWEAK UNIFICATION

Chiral fermion states

We would now like to explore where the GWS parameters $[c_V, c_A = c_V(\Theta_W), c_A(\Theta_W)]$ and $g_W, g_Z, M_W = M_Z \cos\Theta_W$ all come from.

First, note that Glashow's original aim was to unify weak and EM interactions as manifestations of one fundamental "electroweak" interaction.

<u>Problem</u>: if it's the same underlying ~~mechanism~~ interaction, why is γ massless and $W^{\pm}$ & Z^0 so heavy?

<u>Solution</u>: Higgs mechanism (next chapter).

What about structural difference between weak and EM vertices: γ^μ vs. $\gamma^\mu(1-\gamma^5)$ for $W^{\pm}$ (maximal mixing of vector-axially)?

This can be fixed by <u>absorbing</u> $(1-\gamma^5)$ into the particle spinor itself:

$$u_L(p) \equiv \frac{(1-\gamma^5)}{2}\, u(p).$$

In general, u_L is not an eigenstate of helicity operator in spite of "L" representing "left-handed", i.e. helicity -1:

$$\gamma^5 u(p) = \begin{pmatrix} \frac{c(\vec{p}\cdot\vec{\sigma})}{E+mc^2} & \underline{0} \\ \underline{0} & \frac{c(\vec{p}\cdot\vec{\sigma})}{E-mc^2} \end{pmatrix} u(p) \qquad (\text{not } \propto u(p))$$

PROOF: $\gamma^5 u = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} u_A \\ u_B \end{pmatrix} = \begin{pmatrix} u_B \\ u_A \end{pmatrix}$. Now use that $(\not{p}-mc)u = 0 = \begin{pmatrix} (\frac{E}{c}-mc)u_A - \vec{p}\cdot\vec{\sigma}\, u_B \\ \vec{p}\cdot\vec{\sigma}\, u_A - (\frac{E}{c}+mc)u_B \end{pmatrix}$

$\Rightarrow u_A = \frac{c}{E-mc^2}\, \vec{p}\cdot\vec{\sigma}\, u_B$ & $u_B = \frac{c}{E+mc^2}(\vec{p}\cdot\vec{\sigma})u_A$. Insert above and get

$$\gamma^5 u = \begin{pmatrix} \frac{c}{E+mc^2}\vec{p}\cdot\vec{\sigma} & \\ & \frac{c}{E-mc^2}\vec{p}\cdot\vec{\sigma} \end{pmatrix} u. \quad \text{qed.}$$

If $m=0$, then $\gamma^5 u(p) = (\hat{p}\cdot\vec{\Sigma})u(p)$ where $\hat{p}\cdot\vec{\Sigma}$ is the helicity operator with eigenvalues ± 1.

$$\frac{1}{2}(1-\gamma^5)u(p) = \begin{cases} 0 & \text{if } u(p) \text{ has helicity } +1 \\ u(p) & \text{if } u(p) \text{ has helicity } -1. \end{cases} \quad (\text{for } m=0)$$

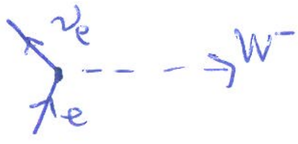
This holds exactly for $m=0$ only, but u_L is always called a left-handed particle: the projection operator $\frac{1}{2}(1-\gamma^5)$ picks out the -1 helicity component from $u(p)$. [for $m=0$]

For an antiparticle: $v_L(p) \equiv \frac{(1+\gamma^5)}{2} v(p)$.

For right-handed counterpart, let $\gamma^5 \to (-\gamma^5)$.

$\Rightarrow$ Chiral fermion states ("chiral" is Greek for hand)

Now: weak & EM interactions can be expressed in a more unified form.

Consider  which contributes

$j_\mu^- = \bar{\nu}\gamma_\mu\left(\frac{1-\gamma^5}{2}\right)e$ to $\mathcal{M}$ [$\bar{\nu}$ and e represent the spinors].

j_μ^-: weak current (analogue to electric current in QED)

Now, anticommutation of $\{\gamma^\mu, \gamma^5\} = 0$ gives:

$$\gamma_\mu\left(\frac{1-\gamma_5}{2}\right) = \left(\frac{1+\gamma^5}{2}\right)\gamma_\mu . \quad \text{Also} \quad \left(\frac{1-\gamma_5}{2}\right)^2 = \left(\frac{1-\gamma^5}{2}\right).$$

$$\Rightarrow \gamma_\mu\left(\frac{1-\gamma^5}{2}\right) = \left(\frac{1+\gamma^5}{2}\right)\gamma_\mu\left(\frac{1-\gamma^5}{2}\right).$$

We obtain $j_\mu^- = \bar{\nu}_L \gamma_\mu e_L$: purely vectorial vertex, but only couples left-hand ν's to left-hand e's.

May accomplish the same thing in QED.

$$u = \left(\frac{1-\gamma^5}{2}\right)u + \left(\frac{1+\gamma^5}{2}\right)u = u_L + u_R \quad , \text{ and then}$$

$$j_\mu^{QED} = -\bar{e}\gamma_\mu e = -\bar{e}_L \gamma_\mu e_L - \bar{e}_R \gamma_\mu e_R .$$

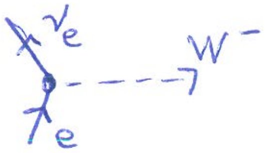
Cross-terms can be shown to vanish and we built in a factor (-1) due to negative electric charge.

The main virtue of the above formulation is then that the actual vertex may be turned into purely vectorial for both QED and weak interactions by allowing $(1-\gamma^5)$ to characterize the particles instead.

NB! The "L" and "R" notation only represents <u>true</u> handedness for $m=0$ (and approximately for $E \gg mc^2$), so it is in general just convenient notation.

Weak isospin and Hypercharge

The negatively charged weak current and positive equivalent:

$j_\mu^- = \bar{\nu}_L \gamma_\mu e_L$  $j_\mu^+ = \bar{e}_L \gamma_\mu \nu_L$ 

can be expressed compactly by:

$\chi_L = \begin{pmatrix} \nu_e \\ e \end{pmatrix}_L$, $\tau^{\pm} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ so that

$j_\mu^{\pm} = \bar{\chi}_L \gamma_\mu \tau^{\pm} \chi_L$. We see that $\tau^{\pm} = \frac{1}{2}(\tau^1 \pm i\tau^2)$

where τ^j are the first two Pauli spin matrices.

This is similar to our treatment of isospin (chapter 4), in fact we would have a full "weak isospin" symmetry [SU(2)] if a third current existed:*

$$j_\mu^3 = \bar{\chi}_L \gamma_\mu \tfrac{1}{2} \tau^3 \chi_L . \quad +$$

To incorporate the EM current (and R) into this framework, we introduce the "weak hypercharge" current:

$$j_\mu^Y = 2 j_\mu^{em} - 2 j_\mu^3$$

\+ That is precisely the neutral weak current! But what about right-hand couplings via Z^0?

~~[illegible]~~ the hyperchange $Y = 2Q - 2I^3$ (Q = electric charge in units of e, I^3 = third comp. of isospin).

Note that j_μ^Y is invariant with respect to rotations in weak isospin space (e.g. $e_L \leftrightarrow \nu_L$). ~~Note that~~ Such transformations do not influence right-handed states.

The total underlying symmetry group is $SU(2)_L \otimes U(1)$ *:
$U(1)$ refers to the weak hypercharge (which involves both chiralities, lower symmetry).

This formalism can be extended to other lepton/quark doublets: $\chi_L \to \begin{pmatrix} \nu_e \\ e \end{pmatrix}_L, \begin{pmatrix} \nu_\mu \\ \mu \end{pmatrix}_L, \begin{pmatrix} u \\ d' \end{pmatrix}_L, \begin{pmatrix} c \\ s' \end{pmatrix}_L$, etc.

We formally define three weak isospin currents $\vec{j}_\mu$ and a weak hypercharge current j_μ^Y:

$$\vec{j}_\mu = \frac{1}{2} \bar{\chi}_L \gamma_\mu \vec{\tau} \chi_L, \quad j_\mu^Y = 2 j_\mu^{QED} - 2 j_\mu^3,$$

where $j_\mu^{QED} = \sum_{i=1}^{2} Q_i (\bar{u}_{iL} \gamma_\mu u_{iL} + \bar{u}_{iR} \gamma_\mu u_{iR})$,

summed over particles i in the doublet.

---

* This means that the Lagrangian describing these interactions is invariant both under $SU(2)$ and $U(1)$ transformations of the ~~[illegible]~~ fermion-spinors.

ELECTRO-WEAK MIXING

The GWS model states that the weak isospin currents couple with strength g_W to a weak isotriplet of intermediate vector bosons $\vec{W}$ (vector in weak isospin space) while j_μ^Y couples with strength $\frac{g'}{2}$ to an isosinglet intermediate vector boson B :

$$-i\left[g_W \vec{j}_\mu \cdot \vec{W}^\mu + \frac{g'}{2} j_\mu^Y B^\mu\right] \qquad (*)$$

This structure contains all the electrodynamic and weak interactions. We may express the first term via the charged currents:

$$\vec{j}_\mu \cdot \vec{W}^\mu = \frac{1}{\sqrt{2}}\left(j_\mu^+ W^{\mu +} + j_\mu^- W^{\mu -}\right) + j_\mu^3 W^{\mu 3}$$

where $W^{\mu\pm} = \frac{1}{\sqrt{2}}\left(W^{\mu 1} \mp i W^{\mu 2}\right)$ are the wavefunctions for the $W^\pm$ particles.

The couplings to $W^\pm$ can now be read off from the coefficients of (*).

$\boxed{\text{EXAMPLE}}$ $e^- \to \nu_e + W^-$.

For this process, we have $j_\mu^- = \bar{\nu}_L \gamma_\mu e_L = \bar{\nu} \gamma_\mu \frac{(1-\gamma_5)}{2} e$.

Inserted into (*): $-ig_W \frac{1}{\sqrt{2}} j_\mu^- W^{\mu-} = -\frac{ig_W}{2\sqrt{2}} [\bar{\nu}\gamma_\mu (1-\gamma^5) e] W^{\mu-}$.

Thus, the vertex factor is $\frac{-ig_W}{2\sqrt{2}} \gamma_\mu (1-\gamma^5)$, as we knew.

Now, in GWS theory the $SU(2)_L \otimes U(1)$ is spontaneously broken in the sense that the two neutral states, W^3 and B, mix and produce one massless linear combination (the photon) and one massive combination (the Z^0).

$$A_\mu = B_\mu \cos\theta_W + W_\mu^3 \sin\theta_W$$
$$Z_\mu = -B_\mu \sin\theta_W + W_\mu^3 \cos\theta_W.$$

If we write the neutral part portion of the electroweak interaction with the physical A_μ and Z_μ states:

$$-i\left[g_W j_\mu^3 W^{\mu 3} + \frac{g'}{2} j_\mu^Y B^\mu\right] = -i\left\{\left[g_W \sin\theta_W j_\mu^3 + \frac{g'}{2}\cos\theta_W j_\mu^Y\right] A^\mu + \left[g_W \cos\theta_W j_\mu^3 - \frac{g'}{2}\sin\theta_W j_\mu^Y\right] Z^\mu\right\}$$

So that the previous symmetry in isospin space $[SU(2)]$ is gone.

By comparing this with the known EM coupling:

$-ig_e j^{QED}_\mu A^\mu$, and using $j^{QED}_\mu = j^3_\mu + \frac{1}{2} j^Y_\mu$,

we obtain consistency only if

$$g_W \sin\theta_W = g' \cos\theta_W = g_e .$$

$\Rightarrow$ Electroweak and EM coupling constants are not independent.

Similar procedure for Z^0: $g_Z = \dfrac{g_e}{\sin\theta_W \cos\theta_W}$.

Can also read out the vector and axial couplings $\{c_V, c_A\}$ for neutral weak processes.

$\boxed{\text{EXAMPLE}}$ $\nu_e \to \nu_e + Z^0$.

Can only have contribution from j^3_μ (not j^{QED}_μ)

The total coupling to Z^μ can be written as:

$$-ig_Z \left(j^3_\mu - \underbrace{\sin^2\theta_W \, j^{QED}_\mu}_{=0} \right) Z^\mu$$

Insert $j^3_\mu = \bar{\nu}_L \gamma_\mu \nu_L \Rightarrow -\frac{ig_Z}{2} \left[\bar{\nu} \gamma_\mu \left(\frac{1-\gamma_5}{2} \right) \nu \right] Z^\mu$

Thus, $c_V^\nu = c_A^\nu = \frac{1}{2}$.

Will see in next chapter precisely how massless gauge fields can gain mass by SSB.

CHAPTER 11 - GAUGE THEORIES

Assuming familiarity with the Lagrangian formulation of classical particle mechanics, we develop Lagrangian field theory. Then, we introduce the fundamental concepts of local gauge invariance, spontaneous symmetry breaking, and the Higgs mechanism.

LAGRANGIANS IN RELATIVISTIC FIELD THEORY

<u>Particle</u>: localized entity, typically want to find $x = x(t)$.

<u>Field</u>: occupies some region of space (and time), typically want to find $\phi_i = \phi_i(x,y,t)$.
(e.g. temperature at x,y,t, electric potential V).

In field theory, one starts with a Lagrangian density $\mathcal{L}$ which is a function of ϕ_i and its derivatives $\partial_\mu \phi_i \equiv \frac{\partial \phi_i}{\partial x^\mu}$

Euler-Lagrange equation: $\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} \right) = \frac{\partial \mathcal{L}}{\partial \phi_i}$ $(i = 1, 2, 3, \ldots)$

Space & time coordinates treated equally due to relativism.

$\boxed{\text{EXAMPLE}}$ Klein - Gordon Lagrangian.

Describes a scalar spin-0 field:

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \varphi)(\partial^\mu \varphi) - \frac{1}{2}\left(\frac{mc}{\hbar}\right)^2 \varphi^2.$$

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} = \partial^\mu \varphi,$$ To see this : $\mathcal{L} = \frac{1}{2}(\partial_0 \varphi \partial_0 \varphi - \partial_1 \varphi \partial_1 \varphi - \dots)$

$$\Rightarrow \frac{\partial}{\partial(\partial_0 \varphi)} = \partial_0 \varphi = \partial^0 \varphi, \quad \frac{\partial}{\partial(\partial_1 \varphi)} = -\partial_1 \varphi = \partial^1 \varphi, \dots$$

Moreover, $\frac{\partial \mathcal{L}}{\partial \varphi} = -\frac{mc^2}{\hbar}\varphi$. Total equation:

$$\underline{\partial_\mu \partial^\mu \varphi + \left(\frac{mc}{\hbar}\right)^2 \varphi = 0}$$

Similarly, we have:

- Dirac Lagrangian $\mathcal{L} = i(\hbar c)\bar{\Psi}\gamma^\mu \partial_\mu \Psi - (mc^2)\bar{\Psi}\Psi$

$$\Rightarrow i\gamma^\mu \partial_\mu \Psi - \left(\frac{mc}{\hbar}\right)\Psi = 0.$$

(Ψ and $\bar{\Psi}$ spinors treated as independent fields)

- Proca Lagrangian (vector spin-1 field)

$$\mathcal{L} = -\frac{1}{16\pi} F^{\mu\nu} F_{\mu\nu} + \frac{1}{8\pi}\left(\frac{mc^2}{\hbar}\right) A^\nu A_\nu \Rightarrow \partial_\mu F^{\mu\nu} + \left(\frac{mc^2}{\hbar}\right) A^\nu = 0$$

For $m=0$ (massless field), this gives Maxwell's equations for empty space. A source term (charge / current density) is added by writing:

$$\mathcal{L} = -\frac{1}{16\pi} F^{\mu\nu} F_{\mu\nu} - \frac{1}{c} J^{\mu} A_{\mu} \quad (\text{recall } F^{\mu\nu} = \partial^{\mu} A^{\nu} - \partial^{\nu} A^{\mu})$$

$$\Rightarrow \partial_{\mu} F^{\mu\nu} = \frac{4\pi}{c} J^{\nu}.$$

Note that it follows that $\partial_{\nu} J^{\nu} = 0$: the current inserted must satisfy the continuity equation.

All $\mathcal{L}$'s above were chosen to give the right field equations : in contrast to deriving $L = T - U$ (classical), $\mathcal{L}$ is axiomatic in relativistic field theory.

$\mathcal{L}$ is not unique : can add a simple constant or divergence $\partial_{\mu} M^{\mu}$, equations are unaffected.

LOCAL GAUGE INVARIANCE

Dirac Lagrangian is invariant under the transformation $\psi \to e^{i\theta}\psi$ (global transformation, θ is a real number).

Since $\bar{\psi} \to e^{-i\theta}\bar{\psi}$, all exp. factors cancel due to $\bar{\psi}\psi$.

The phase-factor is a convention (wavefunction phase unobservable, only magnitude or difference in phase between wavefunctions may be observed). Local gauge symmetry just means that we are free to change the convention/reference point of the phase at any space-time point without changing the physics (as opposed to only ever be allowed a global)

When $\theta \neq \theta(x)$, this is a global gauge transformation.

If $\theta = \theta(x)$, it is a local gauge transformation.

$\mathcal{L}$ is <u>not invariant</u> under $\psi \to \psi e^{i\theta(x)}$

$$\mathcal{L} \to \mathcal{L} - \hbar c\,(\partial_\mu \theta)\,\bar{\psi}\gamma^\mu\psi .$$

Defining $\lambda(x) \equiv -\frac{\hbar c}{q}\theta(x)$, we then have

$$\mathcal{L} \to \mathcal{L} + (q\bar{\psi}\gamma^\mu\psi)\partial_\mu\lambda \quad \text{when} \quad \psi \to e^{-iq\lambda(x)/\hbar c}\psi .$$

We can obtain local gauge invariance by adding an extra term to $\mathcal{L}$. Assume that

$$\mathcal{L} = [i\hbar c\,\bar{\psi}\gamma^\mu\partial_\mu\psi - mc^2\bar{\psi}\psi] - (q\bar{\psi}\gamma^\mu\psi)A_\mu .$$

where A_μ is the so-called gauge field. If the fields transform like:

$$\psi \to \psi e^{-iq\lambda(x)/\hbar c}\psi \quad \text{and} \quad A_\mu \to A_\mu + \partial_\mu\lambda ,$$

then $\mathcal{L}$ is invariant under such a local gauge transformation.

The term $q\bar{\psi}\gamma^\mu\psi A_\mu$ describes the coupling between the vector field A_μ and ψ.

If A_μ is present, there should also be a "free" term in $\mathcal{L}$. The massless Proca Lagrangian does the trick: $\mathcal{L} = -\frac{1}{16\pi} F^{\mu\nu} F_{\mu\nu}$, since $F^{\mu\nu}$ is invariant under the gauge transf. (while a mass term $A^\nu A_\nu$ is not).

Therefore: Imposing local gauge-invariance on the Dirac $\mathcal{L}$, we must introduce a massless vector field A_μ so that:

$$\mathcal{L} = [i\hbar c \bar{\psi} \gamma^\mu \partial_\mu \psi - mc^2 \bar{\psi}\psi] + [-\frac{1}{16\pi} F^{\mu\nu} F_{\mu\nu}] - [q \bar{\psi} \gamma^\mu \psi A_\mu].$$

We identify A_μ as the EM vector potential: $A_\mu \to A_\mu + \partial_\mu \lambda$ leaves the $\vec{B}$-field invariant and the two last terms above give the Maxwell Lagrangian with a source term: $J^\mu = cq(\bar{\psi}\gamma^\mu \psi) \Longrightarrow$ current produced by Dirac particles.

This describes electrodynamics with the coupling between Dirac fermions and photons, and in particular the current J^μ produced by the former.

Origin: local gauge invariance.

The main difference between global and local gauge transformations arise related to derivatives of the field.

Let $\partial_\mu \to D_\mu \equiv \partial_\mu + i\frac{q}{\hbar c}A_\mu$ to convert a global gauge invariance to a local one : "minimal coupling".
Note that the gauge field here must be massless.

In terms of symmetry, we have a local U(1) gauge symmetry since $e^{i\theta}$ belongs to U(1).

The mathematical structure here may be extended to a Lagrangian with two spin-$\frac{1}{2}$ fields ψ_1 and ψ_2, in which case we get a local SU(2) gauge symmetry (Yang-Mills theory). *

INTERPRETING $\mathcal{L}$: MASS TERM

In general: $\mathcal{L}$ consists of two kind of terms:

- Free Lagrangian of each field ($\mathcal{L}_0$)
- Interaction terms for the fields ($\mathcal{L}_{int}$)

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* Definition: A gauge theory is a field theory described by an $\mathcal{L}$ that is invariant under a group of continuous transf. Thus, there are ~~redundant~~ deg. of freed. in the $\mathcal{L}$ which are referred to as the gauge

INTERLUDE : What about linear term in $\mathcal{L}$?

Assume then that we have:

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\varphi)(\partial^\mu\varphi) - a\varphi^2 - b\varphi^3 - c\varphi.$$

Introduce $\eta = \varphi - \varphi_0$ and write

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\eta)(\partial^\mu\eta) - A\eta^2 - B\eta^3$$

Identify $\{A, B, \varphi_0\}$ such that

$$a\varphi^2 + b\varphi^3 + c\varphi = A\eta^2 + B\eta^3$$ (3 equations, one for each order of φ)

But: note that $\eta = 0$ is not necessarily the ground-state if φ was it (in general it is not). So even though we can drop the linear term without loss of generality when considering a generic form of $\mathcal{L}$, we must pay attention to how it modifies the constants for a specific theory.

$\mathcal{L}_{int}$ may be obtained e.g. by invoking local gauge invariance. To determine Feynman rules for a given $\mathcal{L}$:

$\mathcal{L}_0 \Longrightarrow$ propagator for a field/particle

$\mathcal{L}_{int} \Longrightarrow$ vertex factors.

Local gauge invariance then provides a way to find $\mathcal{L}_{int}$, and hence the couplings (vertices): works beautifully for strong and EM interactions.

<u>But</u>: in weak interactions, the gauge field is not massless!

It is still possible to create a gauge theory with massive gauge fields via SSB and the Higgs mechanism.

Before exploring this, consider how to identify a mass term in $\mathcal{L}$ in the first place.

$\boxed{\text{EXAMPLE}}$

What is the mass term in $\mathcal{L} = \frac{1}{2}(\partial_\mu \varphi)(\partial^\mu \varphi) + e^{-(\alpha\varphi)^2}$?

Expanding: $\mathcal{L} = \frac{1}{2}(\partial_\mu \varphi)(\partial^\mu \varphi) + 1 - \alpha^2\varphi^2 + \frac{1}{2}\alpha^4\varphi^4 - \ldots$

Discard constants (1) and identify $m = \frac{\sqrt{2}\alpha\hbar}{c}$.

Higher-order terms represent couplings , , ...

Mass term proportional to second order in the field (φ^2).

There are exceptions:

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \varphi)(\partial^\mu \varphi) + \frac{1}{2}\mu^2 \varphi^2 - \frac{1}{4}\lambda^2 \varphi^4$$

$\Rightarrow$ Imaginary mass? Sign is opposite to before.

<u>Resolution</u>: Feynman calculus with field Lagrangians is really a perturbation procedure starting from the ground-state. $\varphi^4, \varphi^6, \ldots$ then represent higher-order corrections to the ground-state.

So far, the ground-state - field config. that gives minimum energy - has been $\varphi = 0$. But above, this is not the case!

We thus have to rewrite it ($\mathcal{L}$) in terms of deviations from the ground-state.

$$\left.\begin{array}{l} (\partial_\mu \varphi)(\partial^\mu \varphi) : \text{kinetic energy} \\ -\frac{1}{2}\mu^2 \varphi^2 + \frac{1}{4}\lambda^2 \varphi^4 : \text{potential energy} \end{array}\right\} \mathcal{L} = T - U.$$

Minimum of $\mathcal{U}(\varphi)$ at $\varphi_m = \pm \mu/\lambda$. Introduce new field as deviation from minimum value:

$$\eta \equiv \varphi \mp \frac{\mu}{\lambda} \Rightarrow \mathcal{L} = \frac{1}{2}(\partial_\mu \eta)(\partial^\mu \eta) - \mu^2 \eta^2 \pm \mu\lambda\eta^3 - \frac{1}{4}\lambda^2\eta^4 + \frac{1}{4}\left(\frac{\mu^2}{\lambda}\right)^2$$

Now, the mass term is identified: $m = \frac{\sqrt{2}\mu\hbar}{c}$.

and the other terms are couplings  , ...

The $\mathcal{L}$'s expressed with φ and η represent the same physical system, have only changed notation. <u>However:</u> the φ-version is not suitable for Feynman calculus (perturbation series in φ would not converge).

<u>Conclusion</u>: to identify mass term, do the following

1) Locate ground-state
2) Reexpress $\mathcal{L}$ as a function of deviation η
3) Expand in powers of η: mass comes from η^2 term.

SPONTANEOUS SYMMETRY BREAKING

This phenomenon refers to a situation where the Lagrangian (or eqs. of motion) have a given symmetry, whereas the groundstate that is realized does not share that symmetry.

Consider e.g. a ferromagnet. The Lagrangian [or Hamiltonian (energy function) if you prefer] only depends on the magnitude of the magnetization $|\vec{M}|$.

However, once a material becomes magnetic, the magnetization has to be aligned along some direction: the ground-state has chosen <u>one particular</u> solution out of all the available ones (with equal energy), and thus lowered the symmetry.

Don't be fooled by "breaking", could be substituted by "choosing".

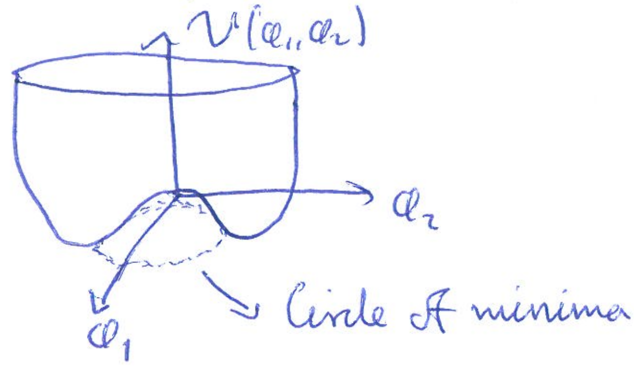
Spontaneous symmetry breaking then occurs when choosing one particular ground state.

EXAMPLE

Consider $\mathcal{L} = \frac{1}{2}(\partial_\mu \varphi_1)(\partial^\mu \varphi_1) + \frac{1}{2}(\partial_\mu \varphi_2)(\partial^\mu \varphi_2)$
$+ \frac{1}{2}\mu^2 (\varphi_1^2 + \varphi_2^2) - \frac{1}{4}\lambda^2 (\varphi_1^2 + \varphi_2^2)^2$

This is invariant under rotations in φ_1, φ_2-space
[$SO(2)$ symmetry, $\varphi_1 \to \varphi_1 c + \varphi_2 s$, $\varphi_2 \to -\varphi_1 s + \varphi_2 c$]

Minima given by $\varphi_{min,1}^2 + \varphi_{min,2}^2 = \mu^2/\lambda^2$



To use Feynman calculus, we expand around the ground-state : i.e, we choose one particular ground state (vacuum) $\varphi_{min,1} = \mu/\lambda$, $\varphi_{min,2} = 0$.

Fluctuation fields are then $\eta = \varphi_1 - \mu/\lambda$, $\xi = \varphi_2$.

Rewrite $\mathcal{L}$:

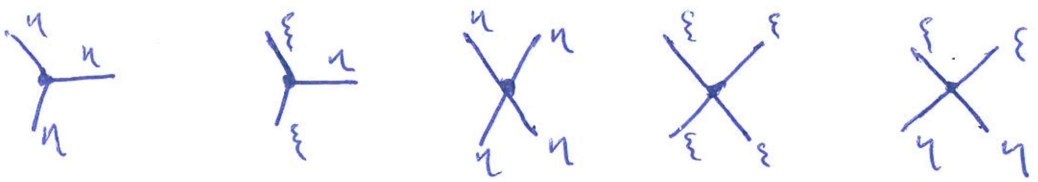
$$\mathcal{L} = \left[\frac{1}{2}(\partial_\mu \eta)(\partial^\mu \eta) - \mu^2\eta^2\right] + \left[\frac{1}{2}(\partial_\mu \xi)(\partial^\mu \xi)\right]$$

$$+ \left[\mu\lambda(\eta^3 + \eta\xi^2) - \frac{\lambda^2}{4}(\eta^4 + \xi^4 + 2\eta^2\xi^2)\right] + \frac{\mu^4}{4\lambda^2}$$

1st term: free K-G Lagrangian with mass $m_\eta = \frac{\sqrt{2}\mu\hbar}{c}$

2nd term: —"— with mass $m_\xi = 0$.

3rd term: five couplings



4th term: irrelevant constant.

The $\mathcal{L}$ is no longer symmetric in any of the fields since we've chosen to express it in terms of one particular solution.

Notice that ξ is massless automatically.

Goldstone's theorem: spontaneous breaking of a continuous global symmetry always generates one or more massless scalar (spin-0) particles, called Goldstone bosons.

We were looking for a massive gauge field, not massless! This occurs when we apply SSB to local gauge invariance.

THE HIGGS MECHANISM

Let us first make the notation a bit more convenient:

$$\varphi \equiv \varphi_1 + i\varphi_2 \rightarrow \varphi^* \varphi = \varphi_1^2 + \varphi_2^2$$

Then: $\mathcal{L} = \frac{1}{2}(\partial_\mu \varphi)^*(\partial^\mu \varphi) + \frac{1}{2}\mu^2(\varphi^*\varphi) - \frac{1}{4}\lambda^2(\varphi^*\varphi)^2$

$\mathcal{L}$ is now invariant under $U(1)$ phase transformations:

$\varphi \to e^{i\theta}\varphi$ ($SO(2)$ and $U(1)$ are isomorphic, so we haven't changed anything except notation).

We can make $\mathcal{L}$ invariant under local $U(1)$ gauge, by introducing a massless A^{μ} field and a minimal coupling $\partial_\mu \to D_\mu = \partial_\mu + i\frac{q}{\hbar c} A_\mu$

Then, the following $\mathcal{L}$ has a local gauge symmetry:

$$\mathcal{L} = \frac{1}{2}\left[\left(\partial_\mu - \frac{iq}{\hbar c} A_\mu\right)\varphi^* \right]\left[\left(\partial^\mu + \frac{iq}{\hbar c} A^\mu\right)\varphi\right]$$

$$+ \frac{1}{2}\mu\,(\varphi^*\varphi) - \frac{1}{4}\lambda^2 (\varphi^*\varphi)^2 - \frac{1}{16\pi} F^{\mu\nu}F_{\mu\nu}$$

Defining again $\eta = \varphi_1 - \mu/\lambda$, $\xi = \varphi_2$, we get:

$$\mathcal{L} = \left[\frac{1}{2}(\partial_\mu \eta)(\partial^\mu \eta) - \mu^2\eta^2\right] + \left[\frac{1}{2}(\partial_\mu \xi)(\partial^\mu \xi)\right]$$

$$+ \left[-\frac{1}{16\pi} F^{\mu\nu}F_{\mu\nu} + \frac{1}{2}\left(\frac{q}{\hbar c}\frac{\mu}{\lambda}\right)^2 A_\mu A^\mu\right] + \text{interaction terms.}$$

The gauge-field is now <u>massive</u>: $m_A = 2\sqrt{\pi}\left(\frac{q\mu}{\lambda c^2}\right)$.

We still have the massless Goldstone boson ξ, and also the interaction terms contain strange terms like $\sim(\partial_\mu \xi)A^\mu$

, suggesting a conversion between ξ and A_μ.

However: both of these problems are related to the $\xi = \varphi_2$ field. We can remove this field completely by exploiting the local gauge invariance:

$$\varphi \to \varphi' = (\varphi_1 c - \varphi_2 s) + i(\varphi_1 s + \varphi_2 c)$$

so that if $\theta = -\operatorname{atan}(\varphi_2/\varphi_1)$, then $\varphi_2' = 0$.

Since $\mathcal{L}$ has the same form expressed with (φ', A_μ') as with (φ, A_μ) (that's what gauge invariant means), we now have $\xi = 0$ in <u>this gauge</u>.

$$\mathcal{L} = \left[\frac{1}{2}(\partial_\mu \eta)(\partial^\mu \eta) - \mu^2\eta^2\right] + \left[-\frac{1}{16\pi} F^{\mu\nu}F_{\mu\nu} + \frac{1}{2}\left(\frac{q}{\hbar c}\frac{\mu}{\lambda}\right)^2 A_\mu A^\mu\right]$$
$$+ \frac{\mu}{\lambda}\left(\frac{q}{\hbar c}\right)^2 \eta (A_\mu A^\mu) + \frac{1}{2}\left(\frac{q}{\hbar c}\right)^2 \eta^2 A_\mu A^\mu - \lambda\mu\eta^3 - \frac{1}{4}\lambda^2\eta^4 + \left(\frac{\mu^2}{2\lambda}\right)^2$$

The $\mathcal{L}$'s in these two gauges describe <u>exactly</u> the same physics, but the new choice makes $\mathcal{L}$ easier to interpret.

We are left with a single massive scalar (the Higgs particle) and a massive gauge field A^μ.

A^μ "ate" the Goldstone boson ξ, and acquired mass (longitudinal polarization, new degree of freedom.

The Higgs mechanism then allows for the gauge bosons to become massive, and this is what happens in the standard model where a spontaneous symmetry breaking of Higgs field (which also demotes $SU(2)_L \times U(1)_Y$ to $U(1)_{EM}$) gives mass to $W^{\pm}$ and Z^0, while leaving the photon massless.

It is therefore good reason to believe that fundamental interactions (such as weak, strong, EM) may be described by local gauge theories.

The SSB of the Higgs field gives mass both to gauge fields and fermion fields, since the latter also couple to the Higgs field via a so-called Yukawa coupling. This reduces the $SU(2)_L \times U(1)_Y$ symmetry of $\mathcal{Y}$ to a $U(1)_{EM}$ symmetry after the SSB.

Note that in the electroweak Lagrangian, there can be no bare mass-terms for the fermions since these would break $SU(2)_L$ symmetry: $m\bar{\psi}\psi = m(\bar{\psi}_L\psi_R + \bar{\psi}_R\psi_L)$

YANG-MILLS THEORY

Before/instead of giving the full electroweak Lagrangian, we give an example of a simpler SU(2) invariant Lagrangian and its belonging gauge fields. In particular, we will recover the coupling structure used in GWS theory ($\vec{j}_\mu \cdot \vec{W}^\mu$) and see that there are thus 3 gauge fields required (unlike 1 for QED).

Y-M theory is an example of a non-Abelian gauge theory (SU(2) is non-Abelian since 2x2 matrices do not commute).

Suppose we then have two spin $\frac{1}{2}$ fields, ψ_1 and ψ_2 [just like in the weak isospin doublet $\binom{\nu_e}{e}_L$].

Free $\mathcal{L}$: $\mathcal{L} = [i\hbar c \bar{\psi}_1 \gamma^\mu \partial_\mu \psi_1 - m_1 c^2 \bar{\psi}_1 \psi_1] + (1 \to 2)$.

Consider equal masses $m_1 = m_2$ for simplicity (doesn't change the qualitative procedure):

$$\mathcal{L} = i\hbar c \bar{\psi}\gamma^\mu \partial_\mu \psi - c^2 m \bar{\psi}\psi \qquad \left(\psi = \begin{pmatrix}\psi_1 \\ \psi_2\end{pmatrix}, \bar{\psi} = (\bar{\psi}_1 \bar{\psi}_2)\right)$$

This $\mathcal{L}$ has a symmetry $\psi \to U\psi$, $\bar{\psi} \to \bar{\psi}U^\dagger$ where U is any 2×2 unitary matrix. General form of an SU(2) transform: $U = e^{i\vec{\tau}\cdot\vec{a}}$ where $\vec{\tau}$ are the Pauli mat.

$\Rightarrow$ $\mathcal{L}$ is invariant under global SU(2) transformations.

If we now promote this to a <u>local</u> SU(2) symmetry, we must introduce gauge fields. For a local transform, we let $\vec{a} = \vec{a}(x^\mu)$. Let us write this as:

$$\psi \to S\psi \text{ with } S = e^{-iq\vec{\tau}\cdot\vec{\lambda}(x)/\hbar c} \quad \text{(Local SU(2) transform)}$$

Our original $\mathcal{L}$ is not invariant, but we can fix this by (like before) replacing ∂_μ with the covariant derivative D_μ:

$$D_\mu = \partial_\mu + i\frac{q}{\hbar c}\vec{\tau}\cdot\vec{A}_\mu$$

We then need to identify a transformation rule for the gauge fields $\vec{A}_\mu$ so that

$D_\mu \psi \to S(D_\mu \psi)$, in which case we have an $\mathcal{L}$ that is invariant under local $SU(2)$ transformations.

One can show (see problem and references in book):

$$\tau \cdot A_\mu' = S(\vec{\tau}\cdot\vec{A}_\mu)S^{-1} + i\left(\frac{\hbar c}{q}\right)(\partial_\mu S)S^{-1} \qquad (*)$$

will do the job. It is instructive to consider the special case of infinitesimal transformations ($|\vec{\lambda}|$ small):

We obtain $S \simeq 1 - \frac{iq}{\hbar c}\vec{\tau}\cdot\vec{\lambda}$, $\partial_\mu S \simeq -\frac{iq}{\hbar c}\vec{\tau}\cdot\partial_\mu\vec{\lambda}$.

Inserted into (*):

$$A_\mu' \simeq A_\mu + \partial_\mu \lambda + \frac{2q}{\hbar c}(\vec{\lambda}\times\vec{A}_\mu) \qquad (\dagger)$$

With these transformation rules, the Lagrangian:

$$\mathcal{L} = i\hbar c\bar{\psi}\gamma^{\mu}D_{\mu}\psi - mc^{2}\bar{\psi}\psi$$

$$= i\hbar c\bar{\psi}\gamma^{\mu}\partial_{\mu}\psi - mc^{2}\bar{\psi}\psi - (q\bar{\psi}\gamma^{\mu}\vec{\tau}\psi)\cdot\vec{A}_{\mu}$$

is invariant under local SU(2) transformations.

We should include the free Lagrangian of the three new vector fields as well:

$$\mathcal{L}_{A} = -\frac{1}{16\pi}\sum_{j} F_{j}^{\mu\nu}F_{\mu\nu j} = -\frac{1}{16\pi}\vec{F}^{\mu\nu}\cdot\vec{F}_{\mu\nu}$$

These must be massless gauge fields; since a term $m_{A}^{2}\vec{A}^{\nu}\vec{A}_{\nu}$ would break SU(2) invariance

Also, we must revise the structure of $F^{\mu\nu}$ since the A-field now has an extra term in (†):

$$\vec{F}^{\mu\nu} = \partial^{\mu}\vec{A}^{\nu} - \partial^{\nu}\vec{A}^{\mu} - \frac{2q}{\hbar c}(\vec{A}^{\mu}\times\vec{A}^{\nu}).$$

<u>Conclusion</u>: full Y-M Lagrangian is then

$$\mathcal{L} = [i\hbar c\bar{\Psi}\gamma^\mu \partial_\mu \Psi - mc^2\bar{\Psi}\Psi] - \frac{1}{16\pi}\vec{F}^{\mu\nu}\vec{F}_{\mu\nu}$$

$$- (q\bar{\Psi}\gamma^\mu \vec{\tau}\Psi)\cdot\vec{A}_\mu .$$

and describes two equal mass Dirac fields interacting with three massless gauge fields.

Note the form of the coupling $\vec{J}^\mu\cdot\vec{A}_\mu$, exactly like in electroweak theory.