CLASSICAL MECHANICS TFY4345 - Solution Exercise Set 9

(1) Energy-conservation combined with conservation of angular momentum gives us:

$$m\dot{r}^2/2 + (V + \frac{l^2}{2mr^2}) = E. \quad (1)$$

This is an effective 1D problem with $V' = V + \frac{l^2}{2mr^2}$. In order for the particle to reach the center it needs to have sufficiently high energy to overcome the potential barrier, i.e. $E > V'(r \to 0)$. This can be written as

$$Er^2 > r^2V + l^2/(2m). \quad (2)$$

For $r \to 0$, the l.h.s. goes to zero, so that the condition on the potential becomes:

$$(r^2V)|_{r\to 0} < -l^2/(2m). \quad (3)$$

This can be fulfilled either with $V(r) = -k/r^2$ where $k > l^2/2m$ or if $V(r) = -A/r^n$ with $n > 2$ where A is a positive constant.

(2) *See figure in the Norwegian version of the solution.* When the particle touches the surface at $r = a$, conservation of the total energy dictates that

$$E = mv_0^2/2 = mv^2/2 - k/a. \quad (4)$$

Also, conservation of angular momentum provides us with l infinitely far away being equal to l when the particle touches the surface, i.e.

$$l = mv_0 s_{\max} = mva \quad (5)$$

Combining these two equations allow us to identify $s_{\max}$:

$$s_{\max} = \sqrt{a^2 + 2ka/(mv_0^2)}. \quad (6)$$

All particles with impact parameter $s < s_{\max}$ will hit the surface, so that $\sigma_{\text{eff}} = \pi s_{\max}^2$.

(3) We have that

$$L = T - V = m(\dot{r}^2 + r^2\dot{\theta}^2 + \dot{z}^2)/2 - V(r,\theta,z) \quad (7)$$

The Lagrange-equations for the three generalized coordinates then read:

$$m\ddot{r} - mr\dot{\theta}^2 = -\partial_r V,\ d(mr^2\dot{\theta})/dt + \partial_\theta V = 0,\ m\ddot{z} + \partial_z V = 0. \quad (8)$$

Using Hamilton's equations, we identify the canonical momenta:

$$p_r = m\dot{r},\ p_\theta = mr^2\dot{\theta},\ p_z = m\dot{z}. \quad (9)$$

These equations can be used to replace the time derivatives of the generalized coordinates in terms of the momenta, so that the final Hamiltonian reads:

$$H = T + V = (p_r^2 + p_\theta^2/r^2 + p_z^2)/(2m) + V(r,\theta,z). \quad (10)$$