CLASSICAL MECHANICS TFY4345 - Solution Exercise Set 7

(1a) The position of m_1 is $x_1 = x$. For particle 2, we have $x_2 = x + l\sin\theta$ and $y_2 = l\cos\theta$. Only particle 2 has a potential energy (choosing $V = 0$ at $y = 0$) so that the total Lagrangian becomes:

$$L = L(\dot{x},\theta,\dot{\theta}) = \frac{m_1+m_2}{2}\dot{x}^2 + m_2(2l\dot{x}\dot{\theta}\cos\theta + l^2\dot{\theta}^2)/2 + m_2 gl\cos\theta. \quad (1)$$

Since x is a cyclic coordinate, the canonical momentum

$$p_x = \partial L/\partial\dot{x} = (m_1+m_2)\dot{x} + m_2 l\dot{\theta}\cos\theta \quad (2)$$

will be a constant. The center of mass of the system is stated to not move in the x-direction, which is equivalent to stating that $p_x = 0$. Integrating the above expression for p_x then gives us:

$$(m_1+m_2)x + m_2 l\sin\theta = \text{constant} \quad (3)$$

Since $p_x = 0$, we can eliminate $\dot{x}$ from the equations via:

$$\dot{x} = -\frac{m_2}{m_1+m_2} l\dot{\theta}\cos\theta, \quad (4)$$

resulting in the total energy

$$E = \frac{m_2 l\dot{\theta}^2}{2}(1 - m_2\cos^2\theta/(m_1+m_2)) - m_2 gl\cos\theta. \quad (5)$$

This equation can be solved with respect to $\dot{\theta}$:

$$\dot{\theta} = \frac{1}{l}\sqrt{\frac{2(E+m_2 gl\cos\theta)}{m_2(1-m_2\cos^2\theta/(m_1+m_2))}} \quad (6)$$

which after integration provides us with:

$$t = l\sqrt{\frac{m_2}{2(m_1+m_2)}}\int d\theta\sqrt{\frac{m_1+m_2\sin^2\theta}{E+m_2 gl\cos\theta}}. \quad (7)$$

The center of mass is at rest and the system oscillates like a physical pendulum around its center of mass.

(1b) We have that

$$H = (\mathbf{p}-q\mathbf{A})^2/(2m) + q\phi = (p_k - qA_k)(p_k - qA_k)/(2m) + q\phi. \quad (8)$$

Hamilton's equations then give:

$$\dot{q}_i = \partial H/\partial p_i \to v_i = (p_i - qA_i)/m$$
$$-\dot{p}_i = \partial H/\partial q_i \to -qv_k A_{k,i} + q\partial_i\phi. \quad (9)$$

The Lorentz-force is identified via Newtons 2nd law:

$$F_i 6 = m\dot{v}_i = \dot{p}_i - q\dot{A}_i = q\partial_i(\mathbf{v}\cdot\mathbf{A}) - q\partial_i\phi - q[\partial_t A_i + (\mathbf{v}\cdot\nabla)A_i]. \quad (10)$$

Re-expressing this term via $\nabla\times\mathbf{A}$ in component-form, we finally arrive at the desired result:

$$F_i = q[-\partial_i\phi - \partial_t A_i] + q[\mathbf{v}\times\mathbf{B}]_i. \quad (11)$$

The first term is precisely the electric field E_i.