CLASSICAL MECHANICS TFY4345 - Solution Exercise Set 5

(1a) The position of particle 1 is $x_1 = l_1 \sin\theta_1$ and $y_1 = l_1 \cos\theta_1$ whereas for particle 2 we have $x_2 = l_1 \sin\theta_1 + l_2 \sin\theta_2$ and $y_2 = l_1 \cos\theta_1 + l_2 \cos\theta_2$. Since the general expression for kinetic energy is $T = m_i(\dot{x}_i^2 + \dot{y}_i^2)/2$, we then have $L = T_1 + T_2 - V_1 - V_2$:

$$\begin{aligned} L = &(m_1+m_2)l_1^2\dot{\theta}_1^2/2 + m_2 l_2^2\dot{\theta}_2^2/2 \\ &+ m_2 l_1 l_2 \dot{\theta}_1\dot{\theta}_2\cos(\theta_1-\theta_2) \\ &+ (m_1+m_2)g l_1 \cos\theta_1 + m_2 g l_2 \cos\theta_2. \end{aligned} \tag{1}$$

The Lagrange-equations for θ_1 and θ_2 then become:

$$\begin{aligned} &(m_1+m_2)l_1^2\ddot{\theta}_1 + m_2 l_1 l_2 \ddot{\theta}_2 \cos(\theta_1-\theta_2) \\ &\quad + m_2 l_1 l_2 \dot{\theta}_2^2 \sin(\theta_1-\theta_2) \\ &\quad + (m_1+m_2) g l_1 \sin\theta_1 = 0 \end{aligned} \tag{2}$$

and

$$\begin{aligned} &m_2 l_2 \ddot{\theta}_2 + m_2 l_1 l_2 \ddot{\theta}_1 \cos(\theta_1-\theta_2) \\ &\quad - m_2 l_1 l_2 \dot{\theta}_1^2 \sin(\theta_1-\theta_2) \\ &\quad + m_2 g l_2 \sin\theta_2 = 0 \end{aligned} \tag{3}$$

(1b) If the masses and lengths are the same, in effect $m_1 = m_2 = m$ and $l_1 = l_2 = l$, we obtain:

$$\begin{aligned} &2\ddot{\theta}_1 + \ddot{\theta}_2\cos(\theta_1-\theta_2) \\ &\quad + \dot{\theta}_2^2\sin(\theta_1-\theta_2) + 2(g/l)\sin\theta_1 = 0, \\ &\ddot{\theta}_2 + \ddot{\theta}_1\cos(\theta_1-\theta_2) \\ &\quad - \dot{\theta}_1^2\sin(\theta_1-\theta_2) + (g/l)\sin\theta_2 = 0. \end{aligned} \tag{4}$$

Assume that the amplitude of the angular displacements are small so that $\theta_i \ll 1$. Approximating then $\sin\theta_i \simeq \theta_i$ and $\cos\theta_i \simeq 1$, we find

$$\begin{aligned} 2\ddot{\theta}_1 + \ddot{\theta}_2 + 2\omega_0^2\theta_1 &= 0, \\ \ddot{\theta}_2 + \ddot{\theta}_1 + \omega_0^2\theta_2 &= 0. \end{aligned} \tag{5}$$

We defined $\omega_0 = \sqrt{g/l}$. Use the ansatz $\theta_i = A_i \cos\omega t$ to obtain the following equation system:

$$(2\omega_0^2 - 2\omega^2)A_1 - \omega^2 A_2 = 0 \quad -\omega^2 A_1 + (\omega_0^2 - \omega^2)A_2 = 0. \tag{6}$$

To have a non-trivial solution, the determinant of this equation system has to vanish which gives the condition $\omega^4 - 4\omega_0^2\omega^2 + 2\omega_0^4 = 0$, with solutions $\omega_1^2 = (2+\sqrt{2})\omega_0^2$ and $\omega_2^2 = (2-\sqrt{2})\omega_0^2$.