CLASSICAL MECHANICS TFY4345 - Solution Exercise Set 4

(1a) *See figures in the Norwegian version of the solution.* Let z be the vertical coordinate for the mass of particle 2, in which case $s = r - z$ is constant. The Lagrange-function for the system is then given by $L = T_1 + T_2 - V_1 - V_2$:

$$T_1 = m(\dot{r}^2 + r^2\dot{\theta}^2)/2, V_1 = 0,$$
$$T_2 = m\dot{z}^2/2 = m\dot{r}^2/2, V_2 = mgz = mg(r-s). \tag{1}$$

(1b) The Lagrange equation for θ gives $l = mr^2\dot{\theta}$ as a constant since θ is a cyclic coordinate. The Lagrange equation for r, after inserting for l, then reads:

$$2m\ddot{r} - l^2/(mr^3) + mg = 0 \tag{2}$$

In the case of circular motion $r = r_0$ is a constant so that $\ddot{r} = 0$. This gives us $l^2/(mr_0^3) = mg$ so that $r = [l^2/(m^2g)]^{1/3}$.

(1c) Inserting $r = r_0 + x$ in the Lagrange equation for r gives us:

$$2m\ddot{x} - \frac{l^2}{mr_0^3(1+x/r_0)^3} + mg = 0. \tag{3}$$

If $x/r_0 \ll 1$, then to first order we have

$$\frac{1}{(1+x/r_0)^3} \simeq 1 - 3x/r_0. \tag{4}$$

Using this, and that $l^2/(mr_0)^3 = mg$, in the equation of motion above provides us with the desired form:

$$\ddot{x} + 3gx/(2r_0) = 0 \tag{5}$$

With the given conditions, the solution is $x = x_0 \cos\omega t$. Inserting this x into the equation of motion for x above provides the angular frequency $\omega = \sqrt{3g/(2r_0)}$.

(2) *See figures in the Norwegian version of the solution.* Since we have translational invariance in the y-direction, this component of the momentum is conserved. It must be the same on both sides of the barrier. Therefore, $mv_{0,y} = mv_y$ such that $v_0 \sin\alpha = v\sin\beta$. In order to relate this to the potential and energy, we need to express the ratio of v and $v0$ in terms of these quantities. Since energy is conserved (no explicit time-dependence in the system), we have $mv_0^2/2 = mv^2/2 + V_0 = E$. This gives us $v^2/v_0^2 = 1 - V_0/E$ which inserted back into the conservation of momentum in the y-direction gives

$$\sin\alpha/\sin\beta = \sqrt{1 - V_0/E} = n. \tag{6}$$

(3) *See figures in the Norwegian version of the solution.* Let us first identify the kinetic energy for the m_1 particles. There is a contribution from the motion in the plane where θ varies: $\frac{1}{2}m_1a^2\dot{\theta}^2$. There is also a contribution from the azimuthal motion associated with the angular frequency Ω: $\frac{1}{2}m_1(a\sin\theta)^2\Omega^2$. As for particle m_2, this one slides vertically. The distance from A is $2a\cos\theta$ so that the kinetic energy reads $T_2 = 2m_2a^2\dot{\theta}^2\sin^2\theta$. The potential energies, measured with reference level $V = 0$ when $\cos\theta = 0$, have the form:

$$V_1 = -2m_1ga\cos\theta, V_2 = -2m_2ga\cos\theta. \tag{7}$$

In total, we then have the following Lagrangian:

$$L = m_1a^2(\dot{\theta}^2 + \Omega^2\sin^2\theta) + 2m_2a^2\dot{\theta}^2\sin^2\theta$$
$$+ 2(m_1+m_2)ga\cos\theta. \tag{8}$$

If $\theta = 0$ at $t = 0$, then we should expect θ to increase as time passes by until an equilibrium configuration is obtained. For $\Omega \to \infty$, we should have $\theta \to \pi/2$.

Computing the objects $\partial L/\partial\theta$ and $\partial L/\partial\dot{\theta}$ gives us the following equation of motion:

$$2\ddot{\theta}(m_1 + 2m_2\sin^2\theta) + 4m_2\dot{\theta}^2\sin(2\theta) - (m_1\Omega^2$$
$$+ 2m_2\dot{\theta}^2)\sin(2\theta) + \omega_0^2(m_1+m_2)\sin\theta = 0, \tag{9}$$

where $\omega_0^2 = 2g/a$. In the equilibrium configuration, there should be no time-dependence on θ so that $\theta = \theta_0$ and $\ddot{\theta} = \dot{\theta} = 0$. In that case, the equation of motion simplifies greatly to:

$$-(m_1\Omega^2)\sin(2\theta_0) + \omega_0^2(m_1+m_2)\sin\theta_0. \tag{10}$$

From this, we identify that [after using a trigonometric identity to rewrite $\sin(2\theta_0)$]

$$\cos\theta_0 = \frac{m_1+m_2}{2m_1}\frac{\omega_0^2}{\Omega^2}. \tag{11}$$

If $m_1 = m_2 = m$, we can write the Lagrangian for this system as an effective one-particle problem where

$$L = ma^2(1 + 2\sin^2\theta)\dot{\theta}^2 - V'(\theta) \tag{12}$$

with an effective potential

$$V'(\theta) = -ma^2(\Omega^2\sin^2\theta + 2\omega_0^2\cos\theta) \tag{13}$$

Assume $\Omega > \omega_0$. In that case, one can verify that $dV'/d\theta = 0$ when $\cos\theta = \omega_0^2/\Omega^2$ just as required for the equilibrium solution. This is a minimum since

$$\frac{d^2V'(\theta)}{d\theta^2}\Big|_{\theta=\theta_0} > 0. \tag{14}$$