CLASSICAL MECHANICS TFY4345 - Solution Exercise Set 13

The starting point is

$$H = p_r^2/(2m) + p_\theta^2/(2mr^2) + p_z^2/(2m) + a(r) + b(z). \quad (1)$$

With $p_i = \partial_{q_i} S$ and $S = W - \alpha t$, along with the ansatz $W = p_\theta \theta + S_1(r) + S_2(z)$, we obtain

$$\frac{1}{2m}[S_1'(r)]^2 + p_\theta^2/(2mr^2) + a(r) - E = -b(z) - \frac{1}{2m}[S_2'(z)]^2. \quad (2)$$

Now, the left side is a function of r whereas the right side is a function of z, hence they must be constants:

$$\beta \equiv b(z) + \frac{1}{2m}[S_2'(z)]^2 = \text{ constant }. \quad (3)$$

This gives us $S_2(z) = \int \sqrt{2m[\beta - b(z)]}dz$ from the right hand side, whereas from the left hand side

$$S_1(r) = \int \sqrt{2m[E - \beta - a(r) - p_\theta^2/(2mr^2)]}dr \quad (4)$$

We have then found an expression for all terms in Hamilton's principal function S.