CLASSICAL MECHANICS TFY4345 - Solution Exercise Set 11

(1a) The starting equation for the amplitudes and frequencies of the oscillations is:

$$(V_i j - \omega^2 T_{ij})A_j = 0, \tag{1}$$

where $\eta_i = A_i e^{-i\omega t}$, $A_i = C_\alpha \Delta_{i\alpha}$. Inserting this amplitude into the above equation gives us:

$$(V_{ij} - \omega_\alpha^2 T_{ij})\Delta_{j\alpha} = 0, \tag{2}$$

where $\alpha = (1, s, a)$ labels the three different eigenfrequencies for this system. We know that:

$$V - \omega^2 T = \begin{pmatrix} k - \omega_\alpha^2 m & -k & 0 \\ -k & 2k - \omega_\alpha^2 M & -k \\ 0 & -k & k - \omega_\alpha^2 m \end{pmatrix} \tag{3}$$

We then have the following system of equations that needs to be solved

$$\begin{aligned} &(k - \omega_\alpha^2 m)\Delta_{1\alpha} - k\Delta_{2\alpha} = 0, \\ &-k\Delta_{1\alpha} + (2k - \omega_\alpha^2 M)\Delta_{2\alpha} - k\Delta_{3\alpha} = 0, \\ &-k\Delta_{2\alpha} + (k - \omega_\alpha^2 m)\Delta_{3\alpha} = 0 \end{aligned} \tag{4}$$

supplemented by the normalization condition:

$$T_i j \Delta_{i\alpha}\Delta_{j\beta} = \delta_{\alpha\beta}. \tag{5}$$

Note that summation over i, j is implied since those are repeated indices. The normalization condition reads explicitly for $\alpha = \beta$:

$$m\Delta_{1\alpha}^2 + M\Delta_{2\alpha}^2 + m\Delta_{3\alpha}^2 = 1. \tag{6}$$

Consider now the $\alpha = 1$ mode for which $\omega_1 = 0$ (derived in the lectures). Inserting this into the equation system gives the solution $\Delta_{11} = \Delta_{21} = \Delta_{31} = 1/\sqrt{\mu}$ with $\mu = 2m + M$. For the symmetric mode $\alpha = s$, we have $\omega_s = \sqrt{k/m}$. Inserting this into the equation system and using the normalization condition gives the solution

$$\Delta_{1,s} = 1/\sqrt{2m},\ \Delta_{2,s} = 0,\ \Delta_{3,s} = -1/\sqrt{2m}. \tag{7}$$

Physically, this corresponds to the C-atom being at rest whereas the O-atoms oscillate with opposite phase. Finally, we consider the antisymmetric mode $\alpha = a$ for which $\omega_a = \sqrt{k\mu/(Mm)}$. This provides the solution:

$$\Delta_{1,a} = \sqrt{M/(2m\mu)},\ \Delta_{2,a} = -2\sqrt{m/(2M\mu)},\ \Delta_{3,a} = \sqrt{M/(2m\mu)}. \tag{8}$$

In this case, the O-atoms have the same amplitude and phase, whereas the C-atom has opposite phase and a different amplitude.

(1b) We have that

$$\begin{aligned} u_x &= dx/dt = \frac{dx'}{\gamma(dt' + vdz'/c^2)} = \frac{u_x'}{\gamma(1 + vu_z'/c^2)}, \\ u_y &= dy/dt = \frac{u_y'}{\gamma(1 + vu_z'/c^2)}, \\ u_z &= dz/dt = \frac{\gamma(dz' + vdt')}{\gamma(dt' + vdz'/c^2)} = \frac{u_z' + v}{1 + vu_z'/c^2}. \end{aligned} \tag{9}$$

Differentiating:

$$du_x = \frac{du_x'}{\gamma(1 + vu_z'/c^2)} - \frac{u_x'}{\gamma(1 + vu_z'/c^2)^2}\frac{v}{c^2}du_z' \tag{10}$$

which when $\mathbf{u}' \to 0$ reduces to $du_x = \gamma^{-1}du_x'$. Similarly for du_y. For du_z, we find:

$$du_z = \frac{du_z'}{1 + vu_z'/c^2} - \frac{u_z' + v}{(1 + vu_z'/c^2)^2}\frac{v}{c^2}du_z'. \tag{11}$$

Using that $dt = \gamma dt'(1 + vu_z'/c^2)$ with $\mathbf{u}' \to 0$ gives $dt = \gamma dt'$ and finally:

$$\begin{aligned} a_x &= \frac{du_x}{dt} = (1 - \beta^2)a_x', \\ a_y &= \frac{du_y}{dt} = (1 - \beta^2)a_y', \\ a_z &= \frac{du_z}{dt} = (1 - \beta^2)^{3/2}a_z'. \end{aligned} \tag{12}$$

(1c) *See figure in the Norwegian version of the solution.* We have $\nu^0 = dN/dt^0$ where dN is the number of waves emitted in the time interval dt^0 in the instantaneous rest-system S'. We then have $dt^0 = d\tau$ (the eigentime of the system under consideration). If t is the time measured by the observer, we have $dt = \gamma d\tau$. The frequency measured in the center is then

$$\nu = dN/dt = dN/(\gamma d\tau) = \nu^0/\nu = \sqrt{1 - \beta^2}\nu^0. \tag{13}$$

This is the transversal Doppler effect. See figure in Norwegian solution.