Volume of the Space of Positive Definite Sequences

JOHN MAKHOUL, FELLOW, IEEE

Abstract—**The coefficients of a normalized real positive definite sequence, such as a normalized autocorrelation sequence, are constrained to be less than or equal to unity in magnitude. It is well known that the coefficients are further constrained, but the constraints become complicated rather quickly as the sequence order increases. One measure of the additional constraints placed on positive definite sequences is the volume of the portion of real space they occupy. The ratio of that volume to the volume of the hypercube circumscribing the positive definite region can be interpreted as the probability that a random sequence restricted to less than unity in magnitude is in fact a positive definite sequence. We derive a closed-form expression for the volume of the space occupied by positive definite sequences of any order. The derivation provides a simple iterative procedure to determine the constraints placed on each autocorrelation coefficient in terms of lower order coefficients. As a byproduct of the derivation, we provide further insight into the high spectral sensitivity of autocorrelation coefficients to small perturbations near the positive definite boundary.**

I. Introduction

GIVEN a pth-order real sequence r_i, $i = 0, 1, \cdots, p$, where $r_0 = 1$, it is well known that a necessary condition for the sequence $\{r_i\}$ to be positive definite (abbreviated p.d.) is that

$$|r_i| \le 1, \qquad 1 \le i \le p. \tag{1}$$

While (1) is clearly a necessary condition for positive definiteness, it is not sufficient. It is well known that the coefficients r_i are further constrained, but because the expressions for the constraints become unwieldy for high orders, their constraining effect becomes more difficult to understand. There is also a general belief that the constraining effect of positive definiteness somehow increases as the order increases, but the nature and magnitude of that effect is not known. This paper provides numeric measures of the constraining effect of positive definiteness on sequences as a function of sequence order. One measure we provide is the ratio of the volume of the set S_p in p dimensions occupied by all pth-order positive definite sequences (referred to as "p.d. space" below) to the volume of the corresponding pth-order hypercube C_p defined by (1). A closed-form expression for that ratio is derived in this paper. The ratio can be interpreted as the probability that a random sequence $\{r_i\}$ which obeys (1) is in fact positive definite; the probability is shown to decrease dramatically as p increases. By defining the cube-equivalent length of a volume as the pth root of that volume, we show that the relative constraining effect of positive definiteness for different orders increases as the square root of p.

The main objective of this paper is simply to gain a better understanding of positive definiteness in terms of how it constrains p.d. sequences, such as autocorrelations. As a byproduct of the work leading to the expression for the volume of p.d. space, we also obtain a mathematical expression that sheds additional light on the observed spectral sensitivity of autocorrelation coefficients to perturbations near the positive definite boundary, i.e., for spectra having large dynamic range.

The main result in this paper requires finding the volume V_p of S_p, defined as

$$V_p = \int_{r \in S_p} dr \tag{2}$$

where $\boldsymbol{r} = [r_1, r_2, \cdots, r_p]$, $d\boldsymbol{r}$ is shorthand for the elemental volume $dr_1 dr_2 \cdots dr_p$ in p-dimensional space, and the integral represents a multiple integral over the p variables, where the limits of integration are restricted to those vectors $\boldsymbol{r}$ that are in p-dimensional p.d. space S_p. Since the volume of the hypercube C_p is 2^p, the probability we seek is equal to the normalized volume v_p, where

$$v_p = 2^{-p} V_p. \tag{3}$$

The main problem in computing the multiple integral in (2) is to determine the upper and lower bounds implied by $\boldsymbol{r} \in S_p$. Sections II and III provide alternate methods for determining the bounds. The method in Section III results in a simple iterative procedure for determining the constraints placed on each coefficient in terms of lower-order coefficients. However, neither method lends itself well to finding a general closed-form solution to the volume. Such a closed-form solution is given in Section IV, followed in Section V by a numeric examination of the constraining effect of positive definiteness. Section VI then contains a discussion of the spectral sensitivity of autocorrelation coefficients.

II. Positive Definite Sequences

A real sequence $\{r_i, 0 \le i \le p\}$ with $r_0 > 0$ and $r_{-i} = r_i$ is said to be *positive definite* (strictly speaking positive semidefinite or nonnegative definite) if and only if the Toeplitz forms [1], [2]

$$T_n = \sum_{j=0}^{n} \sum_{i=0}^{n} r_{i-j} u_i u_j \tag{4}$$

Manuscript received March 8, 1988; revised May 16, 1989.
The author is with BBN Laboratories, 10 Moulton Street, Cambridge, MA 02138.
IEEE Log Number 8933430.

0096-3518/90/0300-0506$01.00 © 1990 IEEE

obey the inequality

$$T_n \geq 0, \qquad \text{for } n = 0, 1, \cdots, p \tag{5}$$

for every choice of complex u_i, $0 \leq i \leq n$. Without loss of generality, we shall assume below that the sequence $\{r_i\}$ is normalized so that $r_0 = 1$. All sequences $\{r_i, 1 \leq i \leq p\}$ that obey (5) form a *convex* set S_p, which we shall call p.d. space. The *boundary* of the set S_p is formed by those sequences for which $T_n = 0$ for some n, $1 \leq n \leq p$; we shall call such sequences *singular*.

The definition of p.d. sequences given above cannot be used directly to specify the boundary of S_p, which we need to compute the volume integral in (2). We need a condition that relates the coefficients r_i to each other. Such a condition can be written using the determinants of Toeplitz matrices formed from $\{r_i\}$. Define the Toeplitz matrices $\boldsymbol{R}_n$ as

$$\boldsymbol{R}_n = \begin{bmatrix} 1 & r_1 & \cdots & r_n \\ r_1 & 1 & \cdots & r_{n-1} \\ \vdots & \vdots & & \vdots \\ r_n & r_{n-1} & \cdots & 1 \end{bmatrix}. \tag{6}$$

Then, a sequence $\{r_i, 1 \leq i \leq p\}$ is p.d. if and only if the determinants of all the Toeplitz matrices formed from the sequence are nonnegative [1], i.e.,

$$|\boldsymbol{R}_n| \geq 0, \qquad \text{for } n = 1, \cdots, p. \tag{7}$$

It is clear from (7) that as n increases, the conditions for positive definiteness for $p > n$ do not change for the lower-order coefficients $\{r_i, 1 \leq i \leq n\}$. In other words, each set S_n is completely defined by the values of $\{r_i, 1 \leq i \leq n\}$ and does not depend on values of r_i, $i > n$. Note also from (7) that $|\boldsymbol{R}_n|$ is quadratic in r_n, so that the conditions for positive definiteness on r_n, which form the bounds on r_n in (2), can be obtained from the roots of the quadratic equation.

For $n = 1$, for example, we have from (7) that $1 - r_1^2 \geq 0$, or

$$-1 \leq r_1 \leq 1. \tag{8}$$

So, the set S_1 consists of the real line segment between -1 and $+1$ and is identical to C_1 as defined by (1). For $n = 2$, one can show from (7) that we must have $(1 - r_2)(1 + r_2 - 2r_1^2) \geq 0$, or

$$2r_1^2 - 1 \leq r_2 \leq 1. \tag{9}$$

The set S_2 is enclosed by a truncated parabola, as shown in Fig. 1. The circumscribing square in Fig. 1 is defined by (1) and encloses C_2. Since C_2 encloses S_2 but is larger, we conclude that (1) is a necessary but not sufficient condition for positive definiteness. It is easy to see from (8) that $V_1 = 2$ and so $v_1 = 1$. To find V_2, we simply compute the area inside the truncated parabola in Fig. 1, which is defined by conditions (8) and (9). The answer can be shown to be $V_2 = 8/3$, and $v_2 = 2/3$ from (3). The latter value implies that, given a random sequence r_1, r_2, both less than 1 in magnitude, the probability that such a sequence will be p.d. is $2/3$.

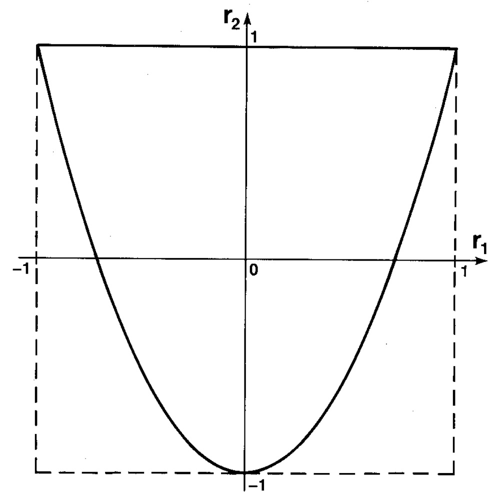


Fig. 1. Positive definite values of $\{r_1, r_2\}$ tuplets are enclosed in a truncated parabola. The ratio of the positive definite area to that of the circumscribing square is equal to $2/3$ in two dimensions.

Conditions such as (8) and (9) are very useful in determining the volume of the set S_p we seek, as we saw above for $p = 2$. Unfortunately, deriving such conditions for higher orders using (7) is very laborious. Below we use a different approach that simplifies the problem considerably and leads to the bounds for arbitrary orders.

III. Autocorrelation of All-Pole Models

Without loss of generality, we shall assume that our p.d. sequence is a p.d. *autocorrelation* sequence. As is well known [3], any p.d. autocorrelation sequence of order p can be considered as that of an autoregressive process of order p or as the autocorrelation corresponding to an all-pole model of the form

$$H_p(z) = \frac{\sqrt{G_p}}{1 + \sum_{k=1}^{p} a_p(k) z^{-k}} \tag{10}$$

where $\{a_p(k)\}$ are the predictor or autoregressive coefficients, and G_p is a gain term adjusted so that $r_0 = 1$. Given any p.d. sequence $\{r_i\}$, one can compute the parameters of the corresponding $H_p(z)$ from the following relations, known as the normal equations or the Yule-Walker equations [3]:

$$\sum_{k=1}^{p} a_p(k) r_{i-k} = -r_i, \qquad 1 \leq i \leq p \tag{11}$$

$$G_p = 1 + \sum_{k=1}^{p} a_p(k) r_k.$$

Relation (11) is a special set of p linear equations in p unknowns $\{a_p(k)\}$, which can be solved using the Levinson–Durbin recursion [3]:

$$G_0 = r_0 = 1$$

For $n = 1, 2, \cdots, p$, compute:

$$K_n = -\frac{1}{G_{n-1}}\left[r_n + \sum_{k=1}^{n-1} a_{n-1}(k)\, r_{n-k}\right] \quad (12)$$

$$a_n(n) = K_n$$

$$a_n(k) = a_{n-1}(k) + K_n a_{n-1}(n-i), \quad 1 \le i \le n-1 \quad (13)$$

$$G_n = (1 - K_n^2)\, G_{n-1}. \quad (14)$$

For p.d. $\{r_i\}$, the solution always exists, is unique, and results in a minimum-phase or stable $H_p(z)$. We note in the recursion above that in the process of computing the parameters of $H_p(z)$, we actually compute the parameters of all lower-order all-pole filters $H_n(z)$, $n = 1, \cdots, p$, whose autocorrelations match the sequence $\{r_i, i = 0, \cdots, n\}$ for each n. From (14), the gain G_n can be written as

$$G_n = \prod_{i=1}^{n} (1 - K_i^2). \quad (15)$$

The auxiliary parameters $\{K_n, 1 \le n \le p\}$ are the coefficients of a lattice realization of the all-pole model $H_p(z)$, and are known as the *reflection* coefficients; they are equal to the negative of the *partial correlation* coefficients of the corresponding autoregressive process. For a strictly positive definite $\{r_i\}$, the corresponding reflection coefficients are unique and are limited to $|K_n| < 1$, $1 \le n \le p$. For a singular sequence $\{r_i\}$ on the p.d. boundary, the magnitude of one or more of the K_n will equal 1, and the mapping is no longer unique. (However, for a singular autocorrelation sequence, there corresponds a unique line spectrum whose spectral lines are located at the poles of that $H_n(z)$ whose $K_i < 1$, $1 \le i \le n-1$, and $|K_n| = 1$, where $n \le p$.) Therefore, for a general autocorrelation sequence $\{r_i\}$, we have the condition

$$-1 \le K_n \le 1, \qquad 1 \le n \le p. \quad (16)$$

From (12) we can solve for r_n:

$$r_n = -K_n G_{n-1} - \sum_{k=1}^{n-1} a_{n-1}(k)\, r_{n-k}. \quad (17)$$

We can now write the lower and upper bounds on r_n for a p.d. sequence by substituting $K_n = 1$ and $K_n = -1$ in (17), respectively,

$$-G_{n-1} - \sum_{k=1}^{n-1} a_{n-1}(k)\, r_{n-k} \le r_n \le G_{n-1} - \sum_{k=1}^{n-1} a_{n-1}(k)\, r_{n-k}. \quad (18)$$

Since G_{n-1} and $a_{n-1}(k)$ can be expressed from (12)–(14) in terms of $\{r_i, 1 \le i \le n-1\}$, (18) gives the bounds for each r_n in terms of lower-order autocorrelation coefficients. For example, it is straightforward to show that (18) leads to (8) and (9) as the bounds on p.d. r_1 and r_2, respectively. Similarly, the bounds for r_3 can be shown to be

$$\frac{1}{1-r_1^2}\left[-(1 - 2r_1^2 + r_2)(1 - r_2) + r_1(2r_2 - r_2^2 - r_1^2)\right] \le r_3 \le \frac{1}{1-r_1^2}\left[(1 - 2r_1^2 + r_2)(1 - r_2) + r_1(2r_2 - r_2^2 - r_1^2)\right].$$

While this method for finding the autocorrelation bounds is simpler than the method of determinants given in Section II, it is clear from the bounds for r_3 that the expressions will become unwieldy as the order increases. Furthermore, the method does not lend itself to finding a closed-form solution for the volume. A different approach is presented next.

IV. Volume of Positive Definite Space

While the autocorrelation bounds in (18) are complicated, in general, the corresponding bounds on the reflection coefficients K_n in (16) are very simple. The approach, therefore, is to transform the volume integral in (2) to one over the coefficients K_n instead of r_n. Equation (17) gives the needed transformation; the differential volume $d\boldsymbol{r}$ in (2) is transformed to a differential volume $d\boldsymbol{K}$ in reflection coefficient space, via the relation

$$d\boldsymbol{r} = |\det \boldsymbol{J}|\, d\boldsymbol{K} \quad (19)$$

where $d\boldsymbol{K}$ is shorthand for $dK_1 dK_2 \cdots dK_p$, and $\boldsymbol{J}$ is the Jacobian matrix

$$\boldsymbol{J} = \begin{bmatrix} \frac{\partial r_1}{\partial K_1} & \cdots & \frac{\partial r_1}{\partial K_p} \\ \vdots & & \vdots \\ \frac{\partial r_p}{\partial K_1} & \cdots & \frac{\partial r_p}{\partial K_p} \end{bmatrix}. \quad (20)$$

It is clear from (17) and (12)–(14) that r_n can be written explicitly in terms of K_i, $1 \le i \le n$, but that r_n does not depend on values of K_i, $i > n$. Therefore

$$\frac{\partial r_n}{\partial K_i} = 0, \qquad i > n, \quad (21)$$

which implies that the Jacobian matrix in (20) is lower triangular. Therefore, the determinant is simply equal to the product of its diagonal elements

$$\det \boldsymbol{J} = \prod_{n=1}^{p} \frac{\partial r_n}{\partial K_n}. \quad (22)$$

But from (17), $\partial r_n / \partial K_n = -G_{n-1}$. Therefore, after taking the absolute value in (19), we have

$$dr = \prod_{n=1}^{p} G_{n-1}\, dK = \prod_{n=1}^{p-1} G_n\, dK \qquad (23a)$$

$$= \prod_{n=1}^{p-1} \prod_{i=1}^{n} (1 - K_i^2)\, dK$$

$$= \prod_{n=1}^{p-1} (1 - K_n^2)^{p-n}\, dK \qquad (23b)$$

where we substituted $G_0 = 1$ and for G_n from (15). Therefore, the volume is given by

$$V_p = \int_{-1}^{1} \cdots \int_{-1}^{1} \prod_{n=1}^{p-1} (1 - K_n^2)^{p-n}\, dK_1 \cdots dK_p.$$

But this is separable as a product of integrals

$$V_p = \int_{-1}^{1} (1 - K_1^2)^{p-1}\, dK_1 \cdots \int_{-1}^{1} \cdot (1 - K_{p-1}^2)\, dK_{p-1} \int_{-1}^{1} dK_p. \qquad (24)$$

After integrating out dK_p, we see that all the remaining integrals are even and similar in form; therefore, we can rewrite (24) as

$$V_p = 2^p \prod_{n=1}^{p-1} \int_0^1 (1 - x^2)^{p-n}\, dx. \qquad (25)$$

But [4]

$$\int_0^1 (1 - x^2)^m\, dx = \frac{\sqrt{\pi}}{2} \frac{\Gamma(m+1)}{\Gamma\left(m + \frac{3}{2}\right)}, \qquad m > 0 \qquad (26)$$

where $\Gamma(\cdot)$ is the Gamma (factorial) function, with [4]

$$\Gamma(n) = (n-1)! \qquad (27)$$

$$\Gamma\left(n + \frac{1}{2}\right) = \sqrt{\pi} \frac{1 \cdot 3 \cdot 5 \cdot 7 \cdot \cdots \cdot (2n-1)}{2^n}. \qquad (28)$$

From the above, one can show that

$$\int_0^1 (1 - x^2)^m\, dx = \prod_{i=1}^{m} \frac{2i}{2i+1}, \qquad m > 0, \qquad (29)$$

which has superior computational properties to (26), especially for large values of m. (Equation (29) can be derived directly using integration by parts, which leads to a recursive solution and (29).) Therefore

$$V_p = 2^p \prod_{i=1}^{p-1} \left(\frac{2i}{2i+1}\right)^{p-i}, \qquad p > 0 \qquad (30)$$

and from (3)

$$v_p = \prod_{i=1}^{p-1} \left(\frac{2i}{2i+1}\right)^{p-i}, \qquad p > 0. \qquad (31)$$

TABLE I

THE SECOND COLUMN GIVES VALUES OF v_p, THE VOLUME OF POSITIVE DEFINITE SPACE RELATIVE TO THAT OF THE CIRCUMSCRIBING CUBE, FOR DIFFERENT VALUES OF DIMENSION p. COLUMNS 3 AND 4 GIVE VALUES OF THE CUBE-EQUIVALENT LENGTHS l_p AND l_p' FOR POSITIVE DEFINITE SPACE AND THE SPHERE INSCRIBED IN THE SAME CUBE, RESPECTIVELY

	POSITIVE-DEFINITE SPACE		SPHERE
Dimension	Normalized Volume	Cube-Equivalent Length	Cube-Equivalent Length
1	1.000E+00	1.0000	1.0000
2	6.667E-01	0.8165	0.8862
3	3.556E-01	0.7084	0.8060
4	1.625E-01	0.6350	0.7452
5	6.605E-02	0.5807	0.6970
6	2.440E-02	0.5386	0.6574
7	8.320E-03	0.5045	0.6242
8	2.648E-03	0.4763	0.5957
9	7.931E-04	0.4524	0.5709
10	2.251E-04	0.4317	0.5491
20	8.931E-11	0.3144	0.4164
30	2.686E-18	0.2596	0.3497
40	1.500E-26	0.2262	0.3075
50	2.386E-35	0.2030	0.2778
100	1.119E-84	0.1447	0.2008

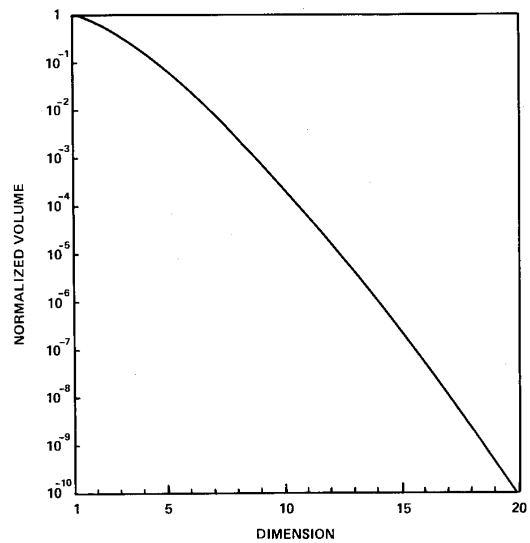


Fig. 2. Volume of positive definite space relative to that of the circumscribing cube as a function of the dimension of the space.

This is the desired closed-form expression for the volume of p-dimensional p.d. space relative to the circumscribing p-cube of length 2. For $p = 2$, for example, (31) gives $v_2 = 2/3$, etc. It is interesting and rewarding that a space whose bounds (18) are so complicated to specify, should have such a simple expression for its volume.

Table I and Fig. 2 show the behavior of the normalized volume v_p as p increases. For $p = 20$, for example, v_p is approximately equal to 10^{-10}! This means that if one chooses a random sequence $r_1, \cdots, r_{20}$ bounded by one in magnitude, the probability that the sequence is a p.d. (or autocorrelation) sequence is 10^{-10}. Equation (31), therefore, represents one aspect of the constraints placed on valid autocorrelation sequences.

Part of the reason that v_p diminishes rapidly as p increases is merely a byproduct of going to higher-dimen-

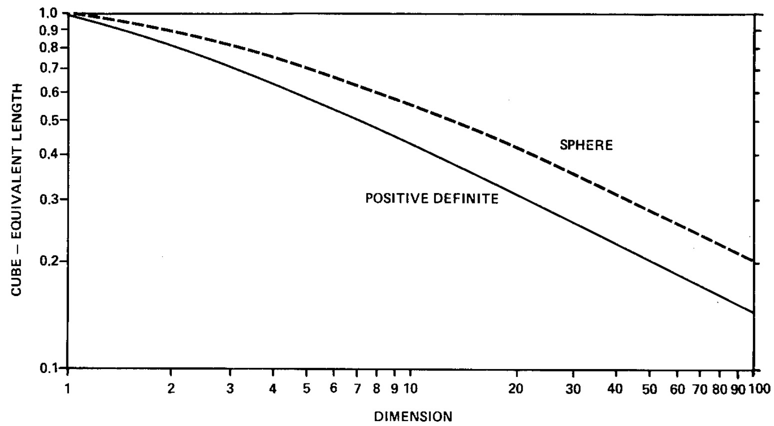


Fig. 3. The cube-equivalent lengths of positive definite space and the sphere, relative to the same circumscribing cube, as a function of dimension.

sional spaces. For example, if one has a p-cube measuring $1/3$ in each dimension, its volume will be 3^{-p}, which decreases very rapidly as p increases. To mitigate this effect in comparing volumes in different dimensions, we define below and compare the cube-equivalent length of each volume.

V. Cube-Equivalent Length

The *cube-equivalent length* of a p-dimensional volume is the length of one side of that p-cube whose volume is equal to the given volume. It is clear that the cube-equivalent length, or simply the length, of a region is equal to the pth root of its volume

$$\text{length} = (\text{volume})^{1/p}. \tag{32}$$

One can then compare the lengths for different dimensions p. For the problem at hand, we define

$$l_p = v_p^{1/p} \tag{33}$$

as the length of p-dimensional p.d. space, where v_p is given by (31). Table I gives values of l_p, and the solid curve in Fig. 3 shows a plot of l_p as a function of p on a log-log scale. A decrease of l_p with p can be interpreted as an increase in the constraining effect of positive definiteness with increased order. Fig. 3 shows graphically the magnitude of this increase in constraining effect as the order p increases.

For comparison purposes, the dashed plot in Fig. 3 shows the length of a p-dimensional sphere relative to its circumscribing cube, which can be shown to be equal to [5]:

$$l'_p = \frac{\sqrt{\pi}}{2\left[\frac{p}{2}\Gamma\left(\frac{p}{2}\right)\right]^{1/p}}. \qquad (\text{sphere}) \tag{34}$$

We see that p.d. space is consistently smaller than the sphere inscribed in the same cube.

The behavior of the volumes whose lengths are plotted in Fig. 3 can be examined further for large p by utilizing Stirling's approximation to the Gamma function. We first rewrite (31) in terms of Gamma functions as

$$v_p = \prod_{n=2}^{p} \frac{\sqrt{\pi}}{2} \frac{\Gamma(n)}{\Gamma\left(n + \frac{1}{2}\right)}. \tag{35}$$

From Stirling's approximation [4]

$$\Gamma(x) \approx \sqrt{2\pi}\, e^{-x} x^{x-1/2} \tag{36}$$

we can write

$$\frac{\Gamma(n)}{\Gamma\left(n + \frac{1}{2}\right)} \approx \sqrt{e}\, \frac{1}{\left(1 + \frac{1}{2n}\right)^n} \frac{1}{\sqrt{n}}. \tag{37}$$

But one can show that

$$\lim_{n\to\infty} \left(1 + \frac{1}{2n}\right)^n = \sqrt{e}. \tag{38}$$

Therefore, we have, for large n,

$$\frac{\Gamma(n)}{\Gamma\left(n + \frac{1}{2}\right)} \approx \frac{1}{\sqrt{n}} \qquad (\text{large } n) \tag{39}$$

and

$$v_p \approx \frac{(\sqrt{\pi}/2)^{p-1}}{\sqrt{p!}}. \qquad (\text{large } p) \tag{40}$$

Using Stirling's formula again for $p!$, taking the pth root, and allowing p to be large, we obtain

$$l_p \approx \frac{\sqrt{\pi e}}{2} \frac{1}{\sqrt{p}}. \qquad (\text{large } p) \tag{41}$$

In other words, the length of p.d. space varies inversely as the square root of the order p. (This is why the plot of l_p in Fig. 3 is a straight line for large p on the log-log scale.) Equation (41) gives one of the main results in this paper, namely, for large order p, the constraining effect of positive definiteness increases as the square root of p.

A similar approximation to the length of the sphere in

(34) can be shown to result in

$$l_p' \approx \frac{\sqrt{\pi e}}{\sqrt{2}} \frac{1}{\sqrt{p}}. \quad \text{(large } p\text{)} \tag{42}$$

By comparing (41) and (42), we see that p.d. space is smaller than the sphere inscribed in the same cube by a factor of $1/\sqrt{2} = 0.707$ for large p. It is interesting that both spaces exhibit the $1/\sqrt{p}$ behavior in high dimensions.

VI. Autocorrelation Coefficient Sensitivity

Equation (23) gives some insight into the relative coefficient sensitivity between reflection coefficients and autocorrelation coefficients. First, we note that (23a) can be written as [6]

$$d\boldsymbol{r} = |\boldsymbol{R}_{p-1}| \, d\boldsymbol{K} \tag{43}$$

where $\boldsymbol{R}_{p-1}$ is the $p \times p$ Toeplitz autocorrelation matrix defined in (6). The determinant of $\boldsymbol{R}$ is a measure of the elemental volume in autocorrelation space relative to $\boldsymbol{K}$ space. Of particular interest is the region near the p.d. boundary, which corresponds to spectra with high dynamic range (in the all-pole case, the poles approach the unit circle). This region is also characterized by one or more of the reflection coefficients approaching unit magnitude. We conclude from (23b) that as we approach the p.d. boundary, the elemental autocorrelation volume approaches zero. The region near $|K_n| = 1$ in $\boldsymbol{K}$ space, therefore, maps into a very thin region in $\boldsymbol{r}$ space, which means that spectra with large dynamic range occupy a relatively small part of $\boldsymbol{r}$ space. Another interpretation is that variations in the reflection coefficients near the p.d. boundary result in much smaller variations in the autocorrelation coefficients, or inversely, variations in the autocorrelation coefficients near the p.d. boundary result in much larger variations in the reflection coefficients. We had shown previously that the region near the p.d. boundary in $\boldsymbol{K}$ space is already very dense [7] (i.e., the spectral sensitivity of reflection coefficients is very high near the boundary), but evidently the corresponding region is much denser in $\boldsymbol{r}$ space and, therefore, the spectral sensitivity of autocorrelation coefficients is even much greater near the p.d. boundary. That is why the computation of autocorrelation coefficients has to be particularly accurate when we are near the p.d. boundary, i.e., when the spectra have large dynamic range.

VII. Conclusion

Treating a positive definite sequence as the autocorrelation of an autoregressive process allowed us to find a surprisingly simple closed-form solution for the volume of p.d. space as a function of the sequence order. The ratio of that volume to the volume of the circumscribing cube can be interpreted as the probability that a sequence bounded by unity magnitude is positive definite. The asymptotic behavior of the volume as the order increases was derived and compared to that of a sphere of the same order. By defining the cube-equivalent length of a volume as the pth root of that volume, we showed that the relative constraining effect of positive definiteness for different orders increases as the square root of p for large p. Finally, as a byproduct of the derivation of the volume, we showed that the coefficient sensitivity of autocorrelation coefficients is much higher than that of reflection coefficients near the p.d. boundary, which corresponds to spectra with high dynamic range.

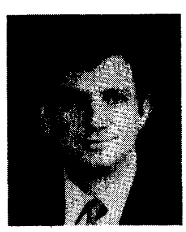
A problem related to the one posed in this paper is to find the probability that a sequence bounded by unity magnitude is the autocorrelation of a *finite-length* sequence. An approach similar to the one given here gives an interesting expression for the elemental volume but, unfortunately, does not lead to a closed-form solution for the volume of autocorrelation space for finite-length data. However, a computational method using Monte Carlo and importance sampling techniques has been developed which allows us to compute the desired volume to good accuracy up to order 50. An initial version of this work can be found in [8].

Acknowledgment

The author wishes to thank A. El-Jaroudi for computing the values in Table I; H. Gish for many interesting discussions and for providing the reference to the volume of the sphere; A. Steinhardt for his valuable comments on the paper; and one of the reviewers for his suggestions.

References

[1] U. Grenander and G. Szegö, *Toeplitz Forms and Their Applications.* Berkeley, CA: University of California Press, 1958.
[2] J. Stewart, "Positive definite functions and generalizations, An historical survey," *Rocky Mountain J. Math.*, vol. 6, no. 3, pp. 409–434, Summer 1976.
[3] J. Makhoul, "Linear prediction: A tutorial review," *Proc. IEEE*, pp. 561–580, Apr. 1975.
[4] H. B. Dwight, *Tables of Integrals and Other Mathematical Data.* New York: Macmillan, 1961, Ch. 12.
[5] W. Fleming, *Functions of Several Variables.* New York: Springer-Verlag, 1977, p. 220.
[6] G. K. C. Clarke and T. J. Ulrych, "Analysis of irregularities in the earth's rotation," in *Methods in Computational Physics*, vol. 13. New York: Academic, 1973, pp. 391–430.
[7] R. Viswanathan and J. Makhoul, "Quantization properties of transmission parameters in linear predictive systems," *IEEE Trans. Acoust., Speech, Signal Processing*, vol. ASSP-23, pp. 309–321, June 1975.
[8] J. Makhoul and A. El-Jaroudi, "Volume of the space of finite autocorrelations," in *Proc. IEEE ASSP Workshop on Spectrum Estimation and Modeling*, Minneapolis, MN, Aug. 3–5, 1988, pp. 283–288.



John Makhoul (S'64–M'70–SM'78–F'80) was born in Deirmimas, Lebanon, on September 19, 1942. He received the B.E. degree from the American University of Beirut, Beirut, Lebanon, in 1964, the M.Sc. degree from Ohio State University, Columbus, in 1965, and the Ph.D. degree from the Massachusetts Institute of Technology, Cambridge, in 1970, all in electrical engineering.

Since 1970 he has been with BBN Laboratories, Cambridge, MA, where he is a Chief Scientist, working on various aspects of speech communication and digital signal processing. He is also an Adjunct Professor at Northeastern University and a Research Affiliate at the M.I.T. Research Laboratory of Electronics.

Dr. Makhoul was a recipient of the 1978 Senior Award, the 1982 Technical Achievement Award, and the 1988 Society Award of the IEEE Acoustics, Speech, and Signal Processing Society. His 1975 IEEE Proceedings paper on linear prediction has been named a "Citation Classic" by the Institute for Scientific Information. He is a Fellow of the Acoustical Society of America.